DIFFERENTIALS AND METRICS ON RIEMANN SURFACES(1)

BY

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Introduction. In this paper we show how to finite-dimensional spaces of differentials on a Riemann surface can be associated a finite collection of Riemannian metrics whose curvatures are closely related to the zeros of the differentials on the surface. In particular, we show that the Weierstrass points of a compact Riemann surface can be characterized as points of zero curvature of certain "naturally defined" metrics on the surface. The main theorem and application are in Part II; in Part I we have collected all the preliminary details. In the conclusion we point out the origin of this study and a possible direction for further research.

PART I

Let X be a Riemann surface, A the family of admissible local coordinates on X. Each $\alpha \in A$ is a homeomorphism of an open set $U_\alpha \subset X$ onto an open set $D_\alpha \subset C$ and if also $\beta \in A$, β: $U_\beta \to D_\beta$, then $\beta \circ \alpha^{-1}$ is a complex analytic homeomorphism of $\alpha(U_\alpha \cap U_\beta)$ onto $\beta(U_\alpha \cap U_\beta)$. We use the notation $U_{\alpha\beta} = U_\alpha \cap U_\beta$, $D_{\alpha\beta} = \alpha(U_\alpha \cap U_\beta)$ and $t_{\beta\alpha} = \beta \circ \alpha^{-1}$ on $D_{\alpha\beta}$. The analytic homeomorphism $t_{\beta\alpha}$: $D_{\alpha\beta} \to D_{\beta\alpha}$ is the transition function from the α coordinate to the β coordinate. Note that $t'_{\beta\alpha}(z) \neq 0$ for each $z \in D_{\alpha\beta}$. Here, and in the sequel, $'$ denotes the derivative of an analytic function.

Let m be an arbitrary integer and N an integer $\geqq 1$.

DEFINITION. A differential ϕ of weight m and dimension N on X is a collection $\{\phi_\alpha\}_{\alpha \in A}$ where each ϕ_α is a (vector-valued) analytic function ϕ_α: $D_\alpha \to C^N$ such that for each $p \in U_{\alpha\beta}$,

(1) $\phi_\alpha(\alpha(p)) = \phi_\beta(\beta(p))(t'_{\beta\alpha}(\alpha(p)))^m$

or, more simply

(2) $\phi_\alpha = (\phi_\beta \circ t_{\beta\alpha})(t'_{\beta\alpha})^m$ on $D_{\alpha\beta}$.

The function ϕ_α is usually called the "local expression for ϕ" and (2) expressed by

(3) $\phi(z)\, dz^m$ is invariant.

If ϕ is a differential of weight m and dimension N it has N components $\phi_1, \ldots, \phi_N$ each of which is a differential of weight m and dimension 1. We denote the jth component of ϕ_α ($1 \leqq j \leqq N$) as $\phi_{\alpha,j}$ so that $\phi_{\alpha,j} = \phi_{j,\alpha}$ in our notation. If we refer to a differential without specifying the dimension it is understood that the dimension is 1.

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By (2) above, if ϕ is a differential of weight m and dimension 1, the order of the analytic function ϕ_α at $\alpha(p)$ is the same as the order of ϕ_β at $\beta(p)$. This common order is called the order of ϕ at p and is denoted $v_p(\phi)$. Thus $v_p(\phi)$ is always an integer $\geqq 0$ except in the case that $\phi=0$ (i.e. ϕ is identically zero) in which case $v_p(\phi)=\infty$ for all $p \in X$. We recall two properties of the order function v_p which are a direct consequence of the corresponding statements for analytic functions in the plane:

(4) $v_p(\phi_1+\phi_2) \geqq \min (v_p(\phi_1), v_p(\phi_2))$
with equality if $v_p(\phi_1) \neq v_p(\phi_2)$.

(5) If $v_p(\phi_1)=v_p(\phi_2)=n<\infty$, there is a complex number c such that $v_p(\phi_1-c\phi_2) > n$.

If $p \in U_\alpha$, the analytic function ϕ_α can be expanded in a convergent power series about $z_0=\alpha(p)$,

(6) $\phi_\alpha(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$,
and $a_n=0$ for $n<v_p(\phi)$ while $a_n \neq 0$ if $n=v_p(\phi)$.

Definition. ϕ is normalized for p, α if in (6) $a_n>0$ when $n=v_p(\phi)$.

Clearly, given any ϕ, p, α there is a number c, $|c|=1$ so that $c\phi$ is normalized for p, α.

If ϕ is a differential of weight m and dimension N then the "derivatives of ϕ" in general are not differentials. In fact, by (2) we have

$$\phi'_\alpha = (\phi'_\beta \circ t_{\beta\alpha})(t'_{\beta\alpha})^{m+1}+(\phi_\beta \circ t_{\beta\alpha})m(t'_{\beta\alpha})^{m-1}t''_{\beta\alpha} \quad \text{on } D_{\alpha\beta}. \tag{7}$$

By ϕ'_α we understand the function obtained by differentiating each of the N components of ϕ_α. More generally, by the chain rule, if $k \geqq 1$,

$$\phi_\alpha^{(k)} = (\phi_\beta^{(k)} \circ t_{\beta\alpha})(t'_{\beta\alpha})^{m+k}+\sum_{j=0}^{k-1}(\phi_\beta^{(j)} \circ t_{\beta\alpha})L_j, \tag{8}$$

where L_j is a polynomial in $t'_{\beta\alpha}, \ldots, t_{\beta\alpha}^{(j+2)}$ with integer coefficients.

However, the derivatives of differentials can be brought into this framework to a certain extent by the following construction. Let $e_1, \ldots, e_N$ be the standard orthonormal basis of C^N, e_i has 1 in the ith place and 0 elsewhere. Let $\bigwedge$ denote exterior product. Now $\bigwedge^2 C^N$ can, and will, be identified with $C^{N(N-1)/2}$ and the standard orthonormal basis is then $e_{ij}=e_i \wedge e_j$, $1 \leqq i<j \leqq N$.

Consider $\phi_\alpha \wedge \phi'_\alpha=\sum_{i<j} M_{ij,\alpha}e_{ij}$ where

$$M_{ij,\alpha} = \det \begin{pmatrix} \phi_{i,\alpha} & \phi_{j,\alpha} \\ \phi'_{i,\alpha} & \phi'_{j,\alpha} \end{pmatrix}$$

and similarly $\phi_\beta \wedge \phi'_\beta=\sum_{i<j} M_{ij,\beta}e_{ij}$. By (8) with $k=1$, on $D_{\alpha\beta}$,

$$M_{ij,\alpha} = \det \begin{pmatrix} (\phi_{i,\beta} \circ t_{\beta\alpha})(t'_{\beta\alpha})^m, & (\phi_{j,\beta} \circ t_{\beta\alpha})(t'_{\beta\alpha})^m \\ (\phi'_{i,\beta} \circ t_{\beta\alpha})(t'_{\beta\alpha})^{m+1}+(\phi_{i,\beta} \circ t_{\beta\alpha})L_0, & (\phi'_{j,\beta} \circ t_{\beta\alpha})^{m+1}+(\phi_{j,\beta} \circ t_{\beta\alpha})L_0 \end{pmatrix} \tag{9}$$

or

$$M_{ij,\alpha} = (M_{ij,\beta} \circ t_{\beta\alpha})(t'_{\beta\alpha})^{m+(m+1)}. \tag{10}$$

In other words, we can define a differential $\phi \wedge \phi'$ of weight $2m+1$ and dimension $N(N-1)/2$ by $(\phi \wedge \phi')_{ij,\alpha} = M_{ij,\alpha}$. The same type of argument shows that if s is an integer, $1 \leqq s \leqq N-1$ then $\Phi = \phi \wedge \phi' \wedge \cdots \wedge \phi^{(s)}$ is a differential of weight $m+(m+1)+\cdots+(m+s) = (s+1)(2m+s)/2$ and dimension

$$C_{N,s+1} = N(N-1)\cdots(N-s)/(s+1)!.$$

The components of Φ are indexed by $(s+1)$-tuples $(i) = i_1 i_2 \cdots i_{s+1}$ where $1 \leqq i_1 < i_2 < \cdots < i_{s+1} \leqq N$ and in a local coordinate α, $\Phi_{(i),\alpha} = M_{(i),\alpha}$ where

(11) $$M_{(i),\alpha} = \det(\phi_{i_j,\alpha}^{(k-1)})_{1 \leqq k \leqq s+1; 1 \leqq j \leqq s+1}.$$

With a differential of dimension N we can associate a Riemannian metric on the surface X. If $u, v \in C^N$ we denote, as usual, $(u, v) = \sum u_i \bar{v}_i$ and $|u|^2 = (u, u)$. If f, g are functions on a plane open set D with values in C^N, (f, g), $|f|^2$ are, respectively, given by $(f, g)(z) = (f(z), g(z))$, $|f|^2(z) = |f(z)|^2$ for each $z \in D$. Let ϕ be a differential of weight m and dimension N, and $m \neq 0$. For each $\alpha \in A$ define $\rho_\alpha = |\phi_\alpha|^{2/m} = (\phi_\alpha, \phi_\alpha)^{1/m}$. Thus $\rho_\alpha \geqq 0$ for each $z \in D_\alpha$ and by (2),

(12) $$\rho_\alpha = (\rho_\beta \circ t_{\beta\alpha})|t'_{\beta\alpha}|^2$$

or "$\rho(z)|dz|^2$ is invariant." Thus $\rho = \{\rho_\alpha\}_{\alpha \in A}$ defines a Riemannian metric on X. More precisely, ρ defines a Riemannian metric on $X - Z(\phi)$ where

$$Z(\phi) = \{p \in X \mid \phi_\alpha(\alpha(p)) = 0 \text{ if } p \in U_\alpha\},$$

since, for every $p \in X - Z(\phi)$, $\rho_\alpha(p) > 0$ if $p \in U_\alpha$. Since each ϕ_α is analytic, as long as ϕ is not the differential which is identically 0, $Z(\phi)$ is a discrete subset of X and $X - Z(\phi)$ is a surface. We now compute the Gaussian curvature K of the metric ρ associated with ϕ. By a standard formula of differential geometry, if $p \in X - Z(\phi)$, $p \in U_\alpha$,

(13) $$K(p) = -(1/2\rho_\alpha)\Delta \log \rho_\alpha|_{\text{evaluated at } \alpha(p)}$$

where Δ is the Laplacian $\partial^2/\partial x^2 + \partial^2/\partial y^2$, $z = x+iy$ being the local variable in the plane open set D_α. Introducing

$$\partial/\partial z = \tfrac{1}{2}(\partial/\partial x - i(\partial/\partial y)), \qquad \partial/\partial \bar{z} = \tfrac{1}{2}(\partial/\partial x + i(\partial/\partial y)),$$

we have $\Delta = 4(\partial^2/\partial z\, \partial \bar{z})$. Now, if f, g are analytic functions on a plane open set D with values in C^N, then $(\partial/\partial z)(f, g) = (f', g)$ and $(\partial/\partial \bar{z})(f, g) = (f, g')$. From this, and the definition of $\rho_\alpha = (\phi_\alpha, \phi_\alpha)^{1/m}$ we obtain from (13) that

(14) $$K \circ \alpha^{-1} = -(2/m)(\phi_\alpha, \phi_\alpha)^{-(2+1/m)}\{(\phi_\alpha, \phi_\alpha)(\phi'_\alpha, \phi'_\alpha) - |(\phi_\alpha, \phi'_\alpha)|^2\} \quad \text{on } \alpha(U_\alpha - Z(\phi)).$$

The quantity inside the braces is just $|\phi_\alpha \wedge \phi'_\alpha|^2$ so that we can symbolically write the curvature as

(15) $$K = -(2/m)|\phi|^{-(4+2/m)}|\phi \wedge \phi'|^2.$$

In particular we note the following: If $m > 0$ then $K \leqq 0$ while if $m < 0$, $K \geqq 0$, and in either case $K(p) = 0$ if and only if $\phi \wedge \phi'$ has a zero at p.

PART II

Let H be a (nonzero) finite-dimensional vector space (over C) of differentials of weight m (and dimension 1) on X and set $N = \dim H$. For each $p \in X$ consider the set of integers

$$R_p = \{r \mid r \geqq 0, r = v_p(\phi) \text{ for some } \phi \in H\}. \tag{16}$$

PROPOSITION. *For each $p \in X$, R_p is a finite sequence of N integers $R_p = \{r_1 < r_2 < \cdots < r_N\}$. If $\phi_1, \ldots, \phi_N$ are chosen in H such that $v_p(\phi_i) = r_i$, then $\{\phi_1, \ldots, \phi_N\}$ is a basis of H.*

Proof. First we remark that if $\{\psi_j\}$ is an indexed family of nonzero elements of H and $v_p(\psi_j) \neq v_p(\psi_k)$ for any two indices $j \neq k$ and some fixed $p \in X$, then $\{\psi_j\}$ is linearly independent. For, if $\sum c_j\psi_j$ is a finite linear combination and $c_k \neq 0$ for some index k, then, by the property (4) of the order function, $v_p(\sum c_j\psi_j) \leqq v_p(\psi_k) < \infty$ so that $\sum c_j\psi_j \neq 0$. In particular, since H has finite dimension N, R_p is a finite sequence, $r_1 < r_2 < \cdots < r_M$, $M \leqq N$. Choose $\phi_i \in H$ such that $v_p(\phi_i) = r_i$. We will show that every $\phi \in H$ is a linear combination of $\phi_1, \ldots, \phi_M$ which will prove the proposition. Given $\phi \in H$, $\phi \neq 0$, $v_p(\phi) = r_j$ for some j, $1 \leqq j \leqq M$. For a suitable constant c_j we have $v_p(\phi - c_j\phi_j) > r_j$. Either $\phi - c_j\phi_j = 0$ or we can apply the same procedure to $\phi - c_j\phi_j$. Eventually we obtain for suitable numbers c_i,

$$v_p\left(\phi - \sum c_i\phi_i\right) > r_M$$

which gives $\phi - \sum c_i\phi_i = 0$, so that every element of H is a linear combination of $\phi_1, \ldots, \phi_M$.

In addition to our original assumptions concerning H, we now further assume that H is endowed with an inner product giving it the structure of a Hilbert space. In this case the above proposition can be sharpened.

PROPOSITION. *Let $p \in X$, α a local coordinate, $p \in U_\alpha$, $R_p = \{r_1 < r_2 < \cdots < r_N\}$. There is a unique basis $\{\phi_1, \ldots, \phi_N\}$ of H which is*

(a) *an orthonormal basis of H,*

(b) *each ϕ_i is normalized for p, α,*

(c) $v_p(\phi_i) = r_i$, $1 \leqq i \leqq N$.

Proof. Let $\psi_1, \ldots, \psi_N$ be a basis of H such that $v_p(\psi_i) = r_i$. Apply the Gram-Schmidt process to the sequence $\psi_N, \ldots, \psi_1$ to obtain an orthonormal basis $\theta_N, \ldots, \theta_1$ of H. The θ's are given by

$$\begin{aligned} \theta_N &= c_{NN}\psi_N \\ &\vdots \\ \theta_i &= c_{iN}\psi_N + \cdots + c_{ii}\psi_i \\ &\vdots \end{aligned}$$

and for each i, $c_{ii} \neq 0$. Hence $v_p(\theta_i) = v_p(\psi_i) = r_i$. Now choose numbers d_i, $|d_i| = 1$ so that $\phi_i = d_i\theta_i$ is normalized for p, α. Then $\{\phi_1, \ldots, \phi_N\}$ is a basis of H satisfying properties (a), (b), (c). To show uniqueness, suppose $\lambda_1, \ldots, \lambda_N$ is another basis with these properties. Then $\lambda_i = \sum_j u_{ij}\phi_j$ where (u_{ij}) is a unitary matrix. By (c) $u_{ij}=0$ if $j<i$ and $u_{ii} \neq 0$, while by (b) $u_{ii}>0$. But such a unitary matrix is necessarily the identity matrix so that each $\lambda_i = \phi_i$.

A basis with the properties (a), (b), (c) will be called a normalized p, α basis of H.

To study the sequence $R_p = \{r_i(p)\}_{1 \leqq i \leqq N}$ we introduce, for $1 \leqq j \leqq N$,

$$w_j(p) = \sum_{i=1}^{j} (r_i(p) - (i-1)) \quad \text{and} \quad W_j = \{p \in X \mid w_j(p) > 0\}.$$

Clearly, $W_1 \subset W_2 \subset \cdots \subset W_N$. Let $W = W_N$, $w(p) = w_N(p)$. The set W can be called the Weierstrass points of H and $w(p)$ the multiplicity of p as a Weierstrass point. Indeed, if X is a compact Riemann surface and H the space of abelian differentials of the first kind (differentials of weight 1) then the set W of H is precisely the set of classical Weierstrass points as defined by Hurwitz (see [2, in particular pp. 395–402]).

Let $\{\phi_1, \ldots, \phi_N\}$ be any orthonormal basis of H. We now make the assumption that the weight m of the differentials in H is $\neq 0$. We associate to the orthonormal basis the N-dimensional differential $(\phi_1, \ldots, \phi_N)$ which in turn determines the metric ρ, as in Part I. But this metric ρ depends only on H, not on the orthonormal basis chosen. For if $\{\psi_1, \ldots, \psi_N\}$ is another orthonormal basis determining the metric σ, then in a local coordinate α we have for $z \in D_\alpha$,

$$\rho_\alpha^m(z) = \sum_{j=1}^{N} |\phi_{j,\alpha}(z)|^2 \quad \text{and} \quad \sigma_\alpha^m(z) = \sum_{i=1}^{N} |\psi_{i,\alpha}(z)|^2.$$

But $\psi_i = \sum_{j=1}^{N} u_{ij}\phi_j$ for a suitable unitary (u_{ij}) hence

$$\begin{aligned} \sigma_\alpha^m(z) &= \sum_i \Big(\sum_j u_{ij}\phi_{j,\alpha}(z)\Big)\Big(\sum_k \bar{u}_{ik}\bar{\phi}_{k,\alpha}(z)\Big) \\ &= \sum_{j,k}\sum_i u_{ij}\bar{u}_{ik}\phi_{j,\alpha}(z)\bar{\phi}_{k,\alpha}(z) \\ &= \rho_\alpha^m(z) \quad \text{and so } \sigma_\alpha(z) = \rho_\alpha(z). \end{aligned}$$

This metric depending only on H will be called $\rho_0 = \rho_0(H)$ and the corresponding curvature function will be denoted K_0. We obtain further metrics ρ_s, $0 \leqq s \leqq N-1$, with corresponding curvatures K_s as follows. Let again $\phi = (\phi_1, \ldots, \phi_N)$ be the N-dimensional differential obtained from an orthonormal basis. Then for each s, $1 \leqq s \leqq N-1$ we can form the $C_{N,s+1}$-dimensional differential $\phi \wedge \phi' \wedge \cdots \wedge \phi^{(s)}$ of weight $(s+1)(2m+s)/2$ which again gives rise to a metric ρ_s, provided $2m+s \neq 0$, depending only on H, not on the particular orthonormal basis—for the exterior powers of a unitary matrix are still unitary. We still have to make precise where the metrics are defined, as they are only defined in Part I on $X - Z(\phi)$, $Z(\phi)$ being the discrete set of zeros of the N dimensional differential ϕ. We summarize the main results in a theorem.

THEOREM. *Let H be a finite-dimensional Hilbert space of differentials of weight m on a Riemann surface X. Let $N=\dim H$, $1\leqq N<\infty$. Suppose $2m+s\neq 0$ for $0\leqq s\leqq N-1$. Then to each integer s, $0\leqq s\leqq N-1$, there is defined a Riemannian metric ρ_s on $X-W_{s+1}$ where $W_1,\ldots,W_N$ are the Weierstrass points of H on X. The curvature K_s of ρ_s satisfies $K_s\leqq 0$ if $m>0$ and $K_s\geqq 0$ if $m<0$ and furthermore $K_s(p)=0$ if and only if $p\in W_{s+2}-W_{s+1}$.*

Proof. Let $p\in X$, α a local coordinate, $p\in U_\alpha$. Choose $\phi_1,\ldots,\phi_N$ as the normalized p, α basis of H and set $\phi=(\phi_1,\ldots,\phi_N)$. Thus, for $1\leqq i\leqq N$,

$$\phi_{i,\alpha}(z) = \sum_{n=0}^{\infty} a_{in}(z-z_0)^n \tag{17}$$

the power series all converging for z sufficiently near $z_0=\alpha(p)$ in D_α. By the normalization,

$$a_{in} = 0 \quad \text{if } n < r_i(p) \quad \text{and} \quad a_{in} > 0 \quad \text{if } n = r_i(p). \tag{18}$$

From now on the coordinate α will be fixed and for greater clarity we will write simply ϕ_i instead of $\phi_{i,\alpha}$. Recall that $0\leqq r_1(p)<r_2(p)<\cdots<r_N(p)$ so that certainly $a_{in}=0$ if $n<i-1$ and $a_{i,i-1}\geqq 0$. Fix s, $0\leqq s\leqq N-1$ and let $\phi=\phi\wedge\phi'\wedge\cdots\wedge\phi^{(s)}$. By (11) the ith component of Φ at $z_0=\alpha(p)$ is

$$M_{(i)}(z_0) = \det\left((k-1)!\,a_{i_j,k-1}\right) \tag{19}$$

where $1\leqq k\leqq s+1$, $1\leqq j\leqq s+1$, $(i)=i_1\cdots i_{s+1}$, $1\leqq i_1<i_2<\cdots<i_{s+1}\leqq N$. If $i_{s+1}>s+1$ then for all k, $1\leqq k\leqq s+1$, $k-1<i_{s+1}-1$ so that $a_{i_{s+1},k-1}=0$ and $M_{(i)}(z_0)=0$. Thus the only possibly nonvanishing component of Φ at z_0 is $M_{1,2,\ldots,s+1}(z_0)=\det((k-1)!\,a_{j,k-1})$ which by the normalization is a triangular determinant with value $M_{1,2,\ldots,s+1}(z_0)=\prod_{k=1}^{s+1}(k-1)!\,a_{k,k-1}$ which is $\geqq 0$ and is $=0$ if and only if $a_{k,k-1}=0$ for some k, $1\leqq k\leqq s+1$, but this is if and only if $r_{s+1}(p)>s$ or $p\in W_{s+1}$. Thus the metric ρ_s fails to be positive at p if and only if $p\in W_{s+1}$, hence ρ_s defines a Riemannian metric on $X-W_{s+1}$.

This proves the first part of the theorem. The statement about the sign of the curvature was already observed in Part I. All that remains is to prove the statement about the vanishing of K_s. Keeping the notation introduced so far in the course of the proof, and by the formulas of Part I, we have, setting $b=(s+1)(2m+s)/2$, since this is the weight of Φ,

$$K_s(p) = -\frac{2}{b}(\Phi,\Phi)^{-(2+1/b)}\{(\Phi,\Phi)(\Phi',\Phi')-(\Phi,\Phi')(\Phi',\Phi)\} \tag{20}$$

where all the quantities on the right are evaluated at $z_0=\alpha(p)$.

We have already found

$$\begin{aligned}(\Phi,\Phi)(z_0) &= |M_{1,2,\ldots,s+1}(z_0)|^2\\ &= \left(\prod_{k=1}^{s+1}(k-1)!\,a_{k,k-1}\right)^2.\end{aligned}$$

Now we compute the components of Φ' at z_0. These are just $M'_{(i)}(z_0)$ which, taking into account the rule for differentiating a determinant and the fact that if two rows are identical the determinant vanishes, we obtain

$$M'_{(i)}(z_0) = \det \begin{pmatrix} \phi_{i_1} \cdots \phi_{i_{s+1}} \\ \phi'_{i_1} \cdots \phi'_{i_{s+1}} \\ \vdots \quad\quad \vdots \\ \phi_{i_1}^{(s-1)} \cdots \phi_{i_{s+1}}^{(s-1)} \\ \phi_{i_1}^{(s+1)} \cdots \phi_{i_{s+1}}^{(s+1)} \end{pmatrix} \text{ evaluated at } z_0. \tag{21}$$

Again, because of the normalization, in each of these determinants all terms above the diagonal vanish (at z_0) and the last column vanishes if $i_{s+1} > s+2$. In case $i_{s+1} = s+2$ then $(i) = 1, \ldots, \hat{j}, \ldots, s+2$ where the jth index is deleted, $1 \leqq j \leqq s+1$. But if $j \neq s+1$, the (jj) entry along the diagonal is $\phi_{j+1}^{(j-1)}(z_0) = (j-1)!\,a_{j+1,j-1} = 0$ so that again the determinant vanishes. Thus the only possibly nonzero coordinates in $\Phi'(z_0)$ are

$$M'_{1,\ldots,s+1}(z_0) = \left(\prod_{k=1}^{s} (k-1)!\,a_{k,k-1}\right)(s+1)!\,a_{s+1,s+1}$$

and

$$M'_{1,\ldots,s,s+2}(z_0) = \left(\prod_{k=1}^{s} (k-1)!\,a_{k,k-1}\right)(s+1)!\,a_{s+2,s+1}.$$

We have at hand now all the ingredients to substitute in (20) to obtain the final formulas for the curvature

$$K_s(p) = -\frac{2}{b_s}\left(\prod_{k=1}^{s} (k-1)!\,a_{k,k-1}\right)^{-2/b_s} (s!\,a_{s+1,s})^{-2(1+1/b_s)}((s+1)!\,a_{s+2,s+1})^2 \tag{22}$$

where $b_s = (s+1)(2m+s)/2$ and in the case $s=0$ the empty product $\prod_{k=1}^{s}$ is to be taken to be 1. Since we are assuming $p \notin W_{s+1}$ all terms occurring are different from 0 except $a_{s+2,s+1}$ which is $\geqq 0$ and equals 0 if and only if $r_{s+2}(p) > s+1$, i.e., if and only if $p \in W_{s+2}$. This completes the proof of the theorem.

As an application of theorem we take X to be a compact Riemann surface of genus $g \geqq 2$ and H the g dimensional space of all abelian differentials of the first kind on X. H has a "natural" Hilbert space structure (see [1, Chapter 5]). In this case, $m=1$ so $2m+s>0$ for $0 \leqq s \leqq g-1$. The metric ρ_0 is everywhere defined on X, for $W_1 = \varnothing$, as it is well known that not all abelian differentials have a common zero on X, hence $r_1(p) = 0$ for every $p \in X$. It seems natural to call ρ_0 the Bergman metric on X. We can now state as a consequence of our main theorem:

THEOREM. *Let X be a compact Riemann surface of genus $g \geqq 2$, ρ_0 the Bergman metric on X, and K_0 the corresponding curvature. Then $K_0(p) \leqq 0$ for every $p \in X$ and $K_0(p) = 0$ if and only if X is hyperelliptic and p is one of the $2g+2$ classical Weierstrass points on X.*

Proof. Since $m=1>0$ we know $K_0(p)\leqq 0$ and $K_0(p)=0$ if and only if $p\in W_2$, that is, there is an abelian differential with a simple zero at p which by the Riemann-Roch theorem is equivalent to saying that there is a meromorphic function on X, analytic on $X-\{p\}$ and having a double pole at p, implying X hyperelliptic and p a Weierstrass point—and this argument is reversible. This proves the theorem.

At the opposite extreme to hyperelliptic surfaces are those "general surfaces" with g^3-g Weierstrass points each with $R_p=1,2,\ldots,g-1,g+1$. In this case the metrics $\rho_0,\ldots,\rho_{g-2}$ are everywhere defined on X with everywhere negative curvature while ρ_{g-1} is also defined on all X and the curvature vanishes precisely at the Weierstrass points. In other cases though, the situation is not so simple and we can only assert that each ρ_s, $0\leqq s\leqq g-1$, is defined on $X-W_{s+1}$ and the curvature vanishes at $W_{s+2}-W_{s+1}$.

Conclusion. We conclude by remarking on the original motivation for the study of these metrics. If X is a compact Riemann surface of genus $g\geqq 2$ then "in general" X has no automorphisms other than the identity and in any case there are at most $84(g-1)$, see [2]. The group of automorphisms of X is studied by considering the linear unitary representation afforded by the space H of abelian differentials of the first kind, [3]. Since this representation is unitary, all automorphisms induce isometries on X with respect to the metrics ρ_s, $0\leqq s\leqq g-1$ and so the curvatures K_s are invariant.

In particular, the maxima, minima and other critical points of K_s are invariant so that if certain information were available concerning the critical points, e.g. are they nondegenerate, isolated, etc., this would perhaps shed new light on the group of automorphisms. However, the study of the critical points of K_s seems to be difficult and at present no definitive statements can be made.

Bibliography

1. L. Ahlfors and L. Sario, *Riemann surfaces*, Princeton Univ. Press, Princeton, N. J., 1960.

2. A. Hurwitz, *Über algebraische Gebilde mit eindeutigen Transformationen in sich*, Mathematische Werke, Bd. I, Birkhäuser, Basel, 1932.

3. J. Lewittes, *Automorphisms of compact riemann surfaces*, Amer. J. Math. **85** (1963), 734–752.

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