STRICT ERGODICITY IN ZERO DIMENSIONAL DYNAMICAL SYSTEMS AND THE KRONECKER-WEYL THEOREM MOD 2[1]

BY

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In memory of L. Frank Lovett

1. **Introduction.** Our principal results[2] are number-theoretic. Let $X=\{x \mid 0 \leqq x<1\}$ be the compact group of real numbers modulo 1, and let $\theta \in X$ be irrational. The numbers $j\theta, j=0, \pm 1, \ldots,$ (here and henceforth to be reduced modulo 1) comprise a dense subgroup of X. For each interval $I \subseteq X$ and $n>0$ define $S_n=S_n(\theta, I)$ to be the number of integers j, $1 \leqq j \leqq n$, such that $j\theta \in I$. By the Kronecker-Weyl theorem [12] $\lim_{n\to\infty} S_n/n=\nu(I)$, where ν is Lebesgue measure on X.

We will be interested in the behavior of the sequence $\{x_n\}$ of parities of $\{S_n\}$. That is x_n is 0 or 1 as S_n is even or odd. Our first result concerns the existence of the limit

$$\mu_\theta(I) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n. \tag{1}$$

It is

THEOREM 1. *A necessary and sufficient condition for $\mu_\theta(I)$ to exist for every interval $I \subseteq X$ is that θ have bounded partial quotients.*

We will draw freely upon the language and results of continued fraction theory, as it is developed in [2] or [5]. Recall that θ has *bounded partial quotients* if and only if there exists a constant $c>0$ such that $|q\theta-p|>c/q$ for all integers p and q with $q>0$. In terms of $\|\cdot\|$, the closest integer function, the condition is $\|q\theta\|>c/q$, $q>0$. We note that $\|\cdot\|$ defines a group invariant metric on X.

An interval $I \subseteq X$ may contain either or both of its endpoints. If the endpoints are given, say $0 \leqq t_1<t_2 \leqq 1$, there are essentially two possibilities for I: either $I_1 =\{t \mid t_1<t<t_2\}$ or $I_2=\{t \mid 0 \leqq t<t_1\} \cup \{t \mid t_2<t<1\}$. ($I_2$ may be empty.) If $t_2-t_1 =2k\theta$ for some k, we will see that $\mu_\theta(I)$ exists, but it may not be 1/2, the expected value. However, if θ has bounded partial quotients, then for all other values of $t=t_2-t_1$ it will be true that $\mu_\theta(I)=1/2$.

For arbitrary θ and $t=t_2-t_1$ we will see that at least one of $\mu_\theta(I_1)$ or $\mu_\theta(I_2)$ is 1/2, the limit existing. Also, we define below a measure 0 subgroup, $K_0(\theta)$, of X such that if $t \notin K_0(\theta)$, then $\mu_\theta(I_1)=\mu_\theta(I_2)=1/2$. On the other hand, when θ has unbounded

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[2] The results herein have been announced in [10].

partial quotients, there will exist an uncountable subset of $K_0(\theta)$, and for every t in that subset an uncountable number of pairs $t_1, t_2, t_2-t_1=t$, such that $\mu_\theta(I_1)$ or $\mu_\theta(I_2)$ does not exist.

Using an idea of Furstenberg's [1] we will see that the existence problem (1) is intimately related to the nonexistence problem for a certain functional equation. Let f be the function which is -1 on I and 1 on the complement of I, and consider the equation

$$g(x+\theta) = f(x)g(x). \tag{2}$$

If (2) has a measurable solution, then it will have a measurable ± 1-valued solution, so that in what follows we may assume $g(x)=\pm 1$, a.e. ν.

Define $f^{(n)}(x)=f(x)f(x+\theta)\cdots f(x+(n-1)\theta)$, $n>0$. In the presence of a solution to (2) we have

$$f^{(n)}(x) = g(x)g(x+n\theta)$$

with its obvious consequence

$$\lim_{\|n\theta\|\to 0}\int_X f^{(n)}(x)\nu(dx) = 1. \tag{3}$$

Motivated by (3) we let $\alpha_n=\int_X f^{(n)}(x)\nu(dx)$ and consider the evidently weaker condition

$$\lim_{\|n\theta\|\to 0}\alpha_n^2 = 1. \tag{4}$$

It will develop that (4) is equivalent with $\{\alpha_n\}$ being a sequence which is almost periodic in the sense of Bohr. In fact there will exist constants $\sigma_m\geqq 0$, $m=0, \pm 1,\dots$, with $\sum_{m=-\infty}^{\infty}\sigma_m=1$ such that $\alpha_n=\sum_{m=-\infty}^{\infty}\sigma_m e^{\pi i m n\theta}$.

Let $[a_0; a_1, a_2,\dots]$ be the continued fraction representation of θ, and let $m_k/n_k=[a_0; a_1,\dots, a_k]$ be the sequence of convergents. If $\{b_k\}$ is a sequence of integers, possibly negative, such that $|b_k|\leqq a_{k+1}$ for each k, then $\sum_{k=1}^{\infty}\|b_k n_k\theta\|<\infty$. It follows for any integer m that

$$\langle m; b_1, b_2,\dots\rangle_\theta = m\theta+\sum_{k=1}^{\infty} b_k n_k\theta$$

defines a point of X.

DEFINITION 1. If $\theta=[a_0; a_1, a_2,\dots]$ is an irrational real number, we define $K_0(\theta)$ to be the set of $t=\langle m; b_1, b_2,\dots\rangle_\theta$ for which all b_j's are eventually *even* and such that

$$\lim_{k\to\infty} b_k n_k\|n_k\theta\| = 0. \tag{5}$$

$K_1(\theta)$ is the subset of $K_0(\theta)$ consisting of those elements for which

$$\sum_{k=1}^{\infty}|b_k|n_k\|n_k\theta\| < \infty. \tag{5'}$$

REMARK. If θ has bounded partial quotients, then (5) implies $b_k=0$ for sufficiently large k. If θ has unbounded partial quotients, then $K_1(\theta)$ is uncountable.

It will be seen that (4) depends only upon $t=t_2-t_1$, and not upon t_1. Thus, we may define $K(\theta)$ to be the set of t for which (4) is true. Like K_0 and K_1, K is a subgroup of X.

THEOREM 2. *For any irrational real number θ the inclusions*

$$K_1(\theta) \subseteq K(\theta) \subseteq K_0(\theta) \tag{6}$$

are true.

One proves easily that K_0 has measure 0 in X. A companion equation to (2) is

$$g(x+\theta) = -f(x)g(x) \tag{2'}$$

and only one of (2) or (2′) can have a measurable solution. It is for this reason that at least one of $\mu_\theta(I)$, $\mu_\theta(I')$ exists, where I' is the complement of I.

THEOREM 3. *If $t \in K_1(2\theta)$, there exists a measurable ± 1-valued solution to* (2) *or to* (2′).

In proving Theorem 3 we will exhibit the solution, and so in theory one can determine in terms of θ, t whether it is (2) or (2′) which is solvable.

The paper is organized as follows. In §2 we discuss the relationships between (1) and (2) and between (2) and (3), (4) in the abstract. Propositions 1 and 2, which are basic to the approach, may be viewed as recastings of Lemma 2.1 and the remark, p. 582, of [1]. As an illustration of the technique we prove in §3 a theorem of Kakutani's, [3] and [4], asserting the strict ergodicity of a certain class of dynamical systems. §4 is the main section, and in it we prove Theorems 1–3.

I am grateful to W. W. Adams for illuminating discussions of the theory of continued fractions.

2. A class of zero-dimensional minimal flows.

Let X be an infinite compact topological group, and suppose the following additional data are given:

1. A dense cyclic subgroup of X with generator θ.
2. Normalized Haar measure ν on X.
3. A closed set $F \subseteq X$ with $\nu(F)=0$. F' will denote the complement of F in X.

Notice that 1 implies X is abelian. We use 0 for the identity of X and addition for the group operation.

Next, there is given a continuous ± 1-valued function f on F', and it will be assumed that f is essentially discontinuous at some point of F. That is,

4. There exists $z \in F$ such that $\lim_{y\in F', y\to z} f(y)$ does not exist.

It will be convenient to assume that $\{0\}$ is the only subgroup of X which has a coset contained in F. With this assumption suppose $y \in X$ is such that on a dense subset of $\{F'-y\}$(3) $\cap F'$ the equation $f(x+y)=f(x)$ is true. If z is as in 4, it is easily seen that $z\pm y, z\pm 2y, \ldots$ all have the property 4. Thus, the zth coset of $\{jy\}_{j=-\infty}^{\infty}$ is contained in F, and so $y=0$.

Let $A=\{x+n\theta \mid x\in F, n\in \boldsymbol{Z}\}$ ($\boldsymbol{Z}$=integers). If $\varphi x=x+\theta$, then $\varphi A=A$ and $\varphi A'=A'$, where A' is the complement of A. Notice that $\nu(A')=1$.

Λ will be the two element set $\{-1, 1\}$, and we set up the product space $\Lambda^\infty =\prod_{k=-\infty}^{\infty}\Lambda_k$. The *2-shift* $\sigma: \Lambda^\infty \to \Lambda^\infty$ is defined by $(\sigma m)(n)=m(n+1)$, $m\in\Lambda^\infty$. Λ^∞ is compact with the product topology, and σ is a homeomorphism. We say (σ, Λ^∞), (φ, X), etc. are *flows*.

If $x\in A'$, define $m_x(n)=f(x+n\theta)$. This mapping of A' into Λ^∞ is continuous, and it is also equivariant in the sense that $m_{\varphi x}=\sigma m_x$, $x\in A'$. Let M be the closure of $M_0=\{m_x \mid x\in A'\}$. M is σ-invariant because M_0 is, and therefore (σ, M) is a flow. If $x\in A'$, and if m_k is a sequence of integers such that $\lim_{k\to\infty} f(x+(n+m_k)\theta)=g(n)$ exists for every $n\in\boldsymbol{Z}$, the continuity of f at $x+n\theta$ implies $\lim_{k\to\infty} g(n-m_k)= f(x+n\theta)$. Therefore, if $\lim_{k\to\infty}\sigma^{m_k}m_x=m$ exists (whence $m(n)=g(n)$), then also $\lim_{k\to\infty}\sigma^{-m_k}m=m_x$. If $x\in A'$ the set $\{\sigma^n m_x\}_{n\in \boldsymbol{Z}}$ is dense in M_0 and *a fortiori* in M. Therefore, by what we have just seen m_x belongs to the orbit closure of any $m\in M$, and being σ-invariant that orbit closure is all of M; (σ, M) is a *minimal flow*.

Suppose $\{x_n\}$ is a sequence in A' such that $\lim_{n\to\infty} m_{x_n}=m$ exists in M, and let α, β be cluster points of $\{x_n\}$ in X. We claim $\alpha=\beta$. To see this, note that $f(\alpha+n\theta) =f(\beta+n\theta)$, where defined, and so by the remark following 4, $\alpha=\beta$. We have a natural map $\pi: M\to X$ defined on M_0 by $\pi m_x=x$. It is easily checked that π is continuous and that $\varphi\pi=\pi\sigma$. Therefore $\pi: (\sigma, M)\to(\varphi, X)$ is a *homomorphism of flows*.

Let μ be a σ-invariant probability measure on M. One such exists, and we claim there is only one. Let g be an integrable Borel function on X, and define $G(m) =g(\pi m)$. Because ν is the unique φ-invariant measure on X, one has

$$\int_M G(m)\mu(dm) = \int_X g(x)\nu(dx).$$

Let $h\in C(M)$ be real-valued but otherwise arbitrary. We associate Borel (in fact lower and upper semicontinuous) functions on X with h as follows:

$$h_*(x) = \min_{\pi m=x} h(m),$$
$$h^*(x) = \max_{\pi m=x} h(m).$$

Certainly $H_*\leqq h\leqq H^*$ and $h_*=h^*$ a.e. ν. It follows that $\int_M h(m)\mu(dm)=\int_X h_*(x)\nu(dx)$, which proves μ is unique.

A flow (σ, M) is *strictly ergodic* if there exists on M a unique σ-invariant probability measure. Thus far, we have that (σ, M) is minimal and strictly ergodic.

(3) $F'-y=\{x-y \mid x\in F'\}$.

Next, set up the product space $N=M\times\Lambda$, and define $T: N\to N$ by

$$T(m,\varepsilon) = (\sigma m, m(0)\varepsilon) \qquad ((m,\varepsilon)\in N). \tag{7}$$

The powers of T are

$$\begin{aligned} T^n(m,\varepsilon) &= (\sigma^n m, m(n-1)m(n-2)\cdots m(0)\varepsilon) && (n>0)\\ &= (m,\varepsilon) && (n=0)\\ &= (\sigma^n m, m(n)\cdots m(-1)\varepsilon) && (n<0). \end{aligned} \tag{8}$$

Define π_2 on N by $\pi_2(m,\varepsilon)=\varepsilon$. Fixing (m,ε), a sequence $y=\{y_n\}\in\Lambda^\infty$ is defined by $y_n=\pi_2(T^n(m,\varepsilon))$. Y will be the orbit closure of y and $\mathcal{O}((m,\varepsilon))$ will be the orbit closure of (m,ε). If $(m',\varepsilon')\in\mathcal{O}((m,\varepsilon))$ let y' be defined in a similar fashion. The assignment $p: (m',\varepsilon')\to y'$ is continuous, and working backwards in (8) one sees that y' determines (m',ε') uniquely. That is, p is a homeomorphism onto its image in Λ^∞. Finally, $\sigma p=pT$, and therefore (a) p is onto Y, and (b) $p: (T,\mathcal{O}((m,\varepsilon)))\to(\sigma, Y)$ is an isomorphism of flows.

In the applications $y=\{y_n\}$ is the initial datum, and y will be expressible in terms of some (m,ε) in (8). Then any theorem which we can prove about all the points in $\mathcal{O}((m,\varepsilon))$ will contain a theorem about y as a special case. This idea is central in [1].

We will now investigate the minimality or nonminimality of (T,N). Following that we investigate strict ergodicity. As mentioned earlier, Propositions 1 and 2 appear in [1] in a different setting.

PROPOSITION 1. *The flow (T,N) fails to be minimal precisely when there exists a nontrivial continuous solution to the equation*

$$c(\sigma m) = m(0)c(m). \tag{9}$$

Proof. Suppose (9) has a nontrivial solution. Since $|m(0)|=1$, $m\in M$, $|c(m)|$ is σ-invariant. Therefore, by the continuity of c and minimality of (σ,M) $|c(m)|$ is constant. Since c is nontrivial, the constant is not 0, and we may write $m(0)=c(\sigma m)/c(m)$. In general, $\pi_2T^n(m,\varepsilon)=c(\sigma^n m)\varepsilon/c(m)$(4). If $\{n_k\}$ is any sequence such that $\sigma^{n_k}m\to m$, then $c(\sigma^{n_k}m)\to c(m)$. Therefore, $(m,-\varepsilon)$ cannot be in the orbit closure of (m,ε), and (T,N) is not minimal.

Conversely, suppose (T,N) is not minimal. For any $m_0, m_1\in M$ at least one of $(m_1, 1)$ and $(m_1,-1)$ is in the orbit closure of $(m_0,1)$. This is because (σ,M) is minimal. Supposing (m_1,ε) is in the orbit closure, it follows easily that $(m_1,-\varepsilon)$ is in the orbit closure of $(m_0,-1)$. Therefore, if both $(m_0,1)$ and $(m_0,-1)$ belong to the orbit closure of a point (m_2,ε), then that orbit closure is all of N. Therefore, because (T,N) is not minimal, (m_2,ε) can be chosen so that for each $m\in M$ there is a unique $c(m)\in\Lambda$ such that $(m,c(m))$ is in the orbit closure of (m_2,ε). The orbit closure is closed, and therefore c is continuous. By definition of T, c satisfies (9), and the proposition is proved.

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(4) Even for $n<0$ because $c(\sigma^n m)/c(m)=\pm 1=c(m)/c(\sigma^n m)$.

REMARK. If c is a solution to (9), then so are $\operatorname{Re} c$ and $\operatorname{Im} c$, the real and imaginary parts. If c is nontrivial, at least one of the latter will be nontrivial, and so replacing c by one of them we obtain a nontrivial real-valued solution to (9). Since $|c(m)|$ is constant, a suitable normalization yields a ± 1-valued solution to (9). When working with solutions to (9) we will generally take them to be ± 1-valued.

COROLLARY 1. *Suppose $\varphi^j F \cap F = \varnothing$, $j \neq 0$. Then (T, N) is a minimal flow.*

Proof. Suppose c is a ± 1-valued solution to (9), and let $U = \{x \in X \mid c(m)$ assumes two values on $\pi^{-1}x\}$. U is closed and certainly $A' \cap U = \varnothing$, because $\pi^{-1}x = m_x$, $x \in A'$. If z is as in 4, then $m(0)$ assumes two values on $\pi^{-1}z$. It follows from (9) that either $z \in U$ or $\varphi z \in U$. Suppose $\varphi z \in U$. Using (9) together with $\varphi z \notin F$, we have $\varphi^2 z \in U$. Similarly, $\varphi^n z \in U$, $n \geqq 3$. The latter set is dense, and we have a contradiction. If it is $z \in U$, a similar argument shows $\varphi^n z \in U$, $n < 0$, and again we get a contradiction. The corollary is proved.

DEFINITION 2. When X is understood, we define $\mathscr{S}(f, \theta, F)$ to be the flow (T, N).

We add another assumption to 1–4. It is

5. 2θ generates a dense subgroup of X.

With 5 we have that (σ^2, M) is also minimal.

Suppose (9) and

$$c(\sigma m) = -m(0)c(m) \tag{9'}$$

have continuous solutions c_0 and c_1. If $c(m) = c_0(m)c_1(m)$, then $c(\sigma m) = -c(m)$, because $m^2(0) = 1$. From $c(\sigma^2 m) = c(m)$ and the minimality of (σ^2, M), we conclude that $c(m)$ is constant. From $c(\sigma m) = -c(m)$ we conclude the constant is 0. Therefore, at least one of c_0 and c_1 is identically 0. *In the presence of assumption* 5 *at least one of* $\mathscr{S}(f, \theta, F)$ *and* $\mathscr{S}(-f, \theta, F)$ *is minimal.*

We now take up the problem of strict ergodicity for $\mathscr{S}(f, \theta, F)$. Let μ_0 be normalized counting measure on Λ, and define $\zeta = \mu \times \mu_0$ on N. (Recall that μ is the unique σ-invariant probability measure on M.) Also, let $\psi: N \to N$ be defined by $\psi(m, \varepsilon) = (m, -\varepsilon)$. Then ζ is both T- and ψ-invariant.

Let ζ_0 be a T-invariant probability measure on N. Then because $T\psi = \psi T$, $\psi\zeta_0$ is also T-invariant. By a direct computation $\zeta = (\zeta_0 + \psi\zeta_0)/2$. Let $\mathscr{P}$ be the compact, convex set of T-invariant probability measures on N. If ζ is an extreme point of $\mathscr{P}$, then $\zeta_0 = \psi\zeta_0 = \zeta$, and ζ is unique. If ζ is not an extreme point, then of course ζ cannot be unique. It is well known that ζ is extreme if and only if the process (T, N, ζ) is ergodic. That is, if and only if every measurable solution to $Th = h$ is constant a.e. ζ.

PROPOSITION 2. *$\mathscr{S}(f, \theta, F)$ fails to be strictly ergodic precisely when there exists a nontrivial measurable solution to* (9).

Proof. Suppose c is a solution to (9). Then $c(m)$ is not constant a.e. μ, nor is

$h(m, \varepsilon)=c(m)\varepsilon$ constant a.e. ζ. Using (9), $Th=h$, and (T, N, ζ) is not ergodic. Thus, $\mathscr{S}(f, \theta, F)$ is not strictly ergodic.

Conversely, suppose $\mathscr{S}(f, \theta, F)$ is not strictly ergodic, in which case (T, N, ζ) is not ergodic. Let h be a nonconstant measurable solution to $Th=h$, and write

$$\begin{aligned} h(m, \varepsilon) &= \frac{h(m, 1)+h(m, -1)}{2}+\varepsilon\,\frac{h(m, 1)-h(m, -1)}{2}, \\ h(\sigma m, m(0)\varepsilon) &= \frac{h(\sigma m, 1)+h(\sigma m, -1)}{2}+m(0)\varepsilon\,\frac{h(\sigma m, 1)-h(\sigma m, -1)}{2}. \end{aligned} \tag{10}$$

Let $c(m)=(h(m, 1)-h(m, -1))/2$. Setting $\varepsilon=1$, then -1, in (10) and using $Th=h$, we obtain $c(\sigma m)=m(0)c(m)$, a.e. μ. If $c=0$ a.e. μ, then $h(m, 1)=h(m, -1)$ a.e. μ, and $h_0(m)=h(m, 1)$ is σ-invariant. Then h_0 is constant a.e. μ, and *a fortiori* h is constant a.e. ζ. This is assumed not to be so, and c is a nontrivial solution to (9). The proposition is proved.

The ergodicity of (σ, M, μ) implies that a solution to (9) has constant modulus a.e. μ. Passing as before to the real or imaginary part, and then normalizing, we can arrange that c be ± 1 a.e. μ.

Suppose c_0 and c_1 are solutions to (9) and (9′) respectively. Assuming 5, the process (σ^2, M, μ) is ergodic, and if $c(m)=c_0(m)c_1(m)$, we have $c(\sigma m)=-c(m)$, $c(\sigma^2 m)=c(m)$, a.e. μ. It follows that $c=0$ a.e. μ, and therefore one of c_0 and c_1 is 0 a.e. μ.

COROLLARY 2. *In the presence of* 5, *at least one of the systems* $\mathscr{S}(f, \theta, F)$ *and* $\mathscr{S}(-f, \theta, F)$ *is (minimal and) strictly ergodic.*

LEMMA 1. *Let M be a compact metric space, and suppose (σ, M) is a minimal flow. Assume $h \in C(M)$ is a ± 1-valued function for which the equation $g(\sigma m)=g(m)h(m)$ has no nontrivial continuous solution. If $h^{(n)}(m)=h(\sigma^{n-1}m)\cdots h(m)$, $n>0$, and if*

$$g(m) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} h^{(j)}(m) \tag{11}$$

exists for every $m \in M$, then $g\equiv 0$, and the limit exists uniformly.

Proof. That $g(\sigma m)=g(m)h(m)$ is obvious from (11). The proof that $g\equiv 0$ follows [1, p. 584]. The pointwise everywhere limit of a sequence of continuous functions, g is of the first Baire class on M, and so both g and $|g|$ have points of continuity. Since $|g(\sigma m)|=|g(m)|$, $|g|$ can have a point of continuity only if it is constant. Say $|g|=\alpha$. At any discontinuity of g the variation is 2α, and once g has a discontinuity at m it has one at σ^j, $m, j>0$. In the latter case g has no points of continuity, and this is a contradiction. Therefore, g is continuous, and so by our assumption on h, $g\equiv 0$. We must prove uniformity in (11).

Let $N=M\times\Lambda$ and $T(m, \varepsilon)=(\sigma m, h(m)\varepsilon)$, $(m, \varepsilon) \in N$. If (11) does not exist uniformly, there will exist sequences $\{m_k\}\subseteq M$ and $\{n_k\}\subseteq Z$, $n_k \to\infty$, and a $\beta>0$

such that

$$\left|\frac{1}{n_k}\sum_{j=1}^{n_k} h^{(j)}(m_k)\right| \geqq \beta.$$

Let $\zeta_k=(1/n_k)\sum_{j=1}^{n_k}\delta_{T^j(m_k,1)}$. ($\delta_x$=point mass at x.) By definition, if $H((m,\varepsilon))=\pi_2(m,\varepsilon)$, then

$$\left|\int_N H((m,\varepsilon))\zeta_k(d(m,\varepsilon))\right| \geqq \varepsilon.$$

If ζ_0 is a weak-* cluster point of $\{\zeta_k\}$, then $T\zeta_0=\zeta_0$, and

$$\left|\int_N H((m,\varepsilon))\zeta_0(d(m,\varepsilon))\right| \geqq \varepsilon. \tag{12}$$

By the dominated convergence theorem

$$\begin{aligned}\int_N \varepsilon g(m)\zeta_0(d(m,\varepsilon)) &= \lim_{n\to\infty}\int_N \frac{1}{n}\sum_{j=1}^{n} H(T^j(m,\varepsilon))\zeta_0(d(m,\varepsilon))\\ &= \int_N H((m,\varepsilon))\zeta_0(d(m,\varepsilon)).\end{aligned}$$

Thus $g\not\equiv 0$, a contradiction. The lemma is proved.

REMARK. The conclusion of our lemma remains in force if (11) is assumed to exist for all but at most a countable number of points. For then by passing to a subsequence (11) can be made to exist for all m, and the remainder of the argument is unchanged.

We apply our lemma as follows[5]. Suppose $\mathscr{S}(f,\theta,F)$ is minimal but not strictly ergodic. Thus, (9) has a measurable ± 1-valued solution but not a continuous solution. Assume also that $\int_M c(m)\mu(dm)=\gamma\neq 0$. By the individual ergodic theorem we have

$$\lim_{n\to\infty}\frac{c(m)}{n}\sum_{j=1}^{n} c(\sigma^j m) = \gamma c(m) \qquad (\text{a.e. } \mu).$$

Using (9) it follows that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} m(j-1)\cdots m(0) = \gamma c(m) \qquad (\text{a.e. } \mu). \tag{13}$$

Since $\gamma c(m)\neq 0$ a.e. μ, it follows from our lemma that (13) must fail to exist for an uncountable number of $m\in M$. If F (for convenience) is countable, A will be countable, and there will exist $x\in A'$ such that (13) fails to exist for $m_x(n)=f(x+n\theta)$. We have proved

[5] As in [1], Theorem 3.1.

PROPOSITION 3. *Suppose f, θ, F are as in* 1–4, *and assume F is countable. If $\mathscr{S}(f, \theta, F)$ is minimal, and if*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f^{(j)}(x) \tag{14}$$

exists for each $x \in A'$, then it exists uniformly, and the limit is 0.

REMARK. Suppose $\varphi^j F \cap F = \varnothing$, $j \neq 0$. Let $f = 1$ on F. Then (14) is 0 uniformly in all x once it is uniform on A'.

Because π is essentially one-to-one, a nontrivial measurable solution to (9) exists if and only if one exists to

$$g(x+\theta) = f(x)g(x) \qquad (\text{a.e. } x). \tag{15}$$

As before, we may assume $g(x) = \pm 1$, a.e. ν. Define

$$\alpha_n = \int_X f^{(n)}(x)\nu(dx).$$

PROPOSITION 4. *A necessary and sufficient condition for the existence of a nontrivial solution to* (15) *is*

$$\lim_{n\theta\to 0} \alpha_n = 1. \tag{16}$$

Proof. The necessity follows from $g(x+n\theta)g(x) = f^{(n)}(x)$ and the continuity of translation in $\mathscr{L}^2(X, \nu)$. For sufficiency define a mapping $\{n\theta\} \to \mathscr{L}^2(X, \nu)$ by

$$h(n\theta, \cdot) = f^{(n)}(\cdot).$$

We will use the relation

$$h((m+n)\theta, x) = h(m\theta, x+n\theta)h(n\theta, x). \tag{17}$$

Given $\varepsilon > 0$, let W be a neighborhood of 0 in X such that if $m\theta \in W$, then $\alpha_m > 1-\varepsilon$. Clearly, the set where $h(m\theta, x) = -1$ has measure at most $\varepsilon/2$, and therefore $\|1-h(m\theta, \cdot)\|_2 \leqq (2\varepsilon)^{1/2}$. (Here $\|\cdot\|_2$ denotes $\mathscr{L}^2$ norm.) It follows from this inequality and (17) that $\{n\theta\} \to \mathscr{L}^2(X, \nu)$ is uniformly continuous, and therefore it extends to be continuous on X. That is, $y \to h(y, \cdot)$ is continuous, and (17) becomes

$$h(y+w, x) = h(y, x+w)h(w, x). \tag{18}$$

In (18) the identity holds a.e. x for fixed y, w. Set $y = \theta$. By the Fubini theorem it is true for almost all x that (18) holds a.e. in w. Fixing such an x, we have $h(w+\theta, x) = f(x+w)h(w, x)$, a.e. w. Let $g(w) = h(w-x, x)$. Then (15) holds, and for almost all x, g is nontrivial. The proposition is proved.

Let $\langle \cdot, \cdot \rangle$ be the inner product on $\mathscr{L}^2(X, \nu)$. R_θ will denote translation by θ, and we associate with f and θ a unitary operator $U = U(f, \theta)$,

$$Uh = f \cdot R_\theta h.$$

Given a nontrivial ± 1-valued solution to (15), define V, unitary on $\mathscr{L}^2(X, \nu)$, by

$$Vh = g \cdot h.$$

By (15) $R_\theta Vh = VUh$, and therefore U and R_θ are unitarily equivalent.

Of course U has pure point spectrum when it is equivalent to R_θ. By the same token, $-U$ has pure point spectrum, but 1 does not belong to this spectrum. (Assuming 5.) The difference between strict ergodicity and nonstrict ergodicity is determined for $\mathscr{S}(f, \theta, F)$ and $\mathscr{S}(-f, \theta, F)$ by whether $U(f, \theta)$ or $-U(f, \theta)$ has 1 in its point spectrum.

Since $\alpha_n = \langle 1, U^n 1\rangle$, $\{\alpha_n\}$ is a positive definite sequence. (For $n \leqq 0$, U^n is defined. $f^{(n)} = 1$, $n=0$, and for $n<0$ $f^{(n)}(x) = f(x+n\theta)f(x+(n+1)\theta)\cdots f(x-\theta)$.)

Proposition 5. *With notations as above, the following statements are equivalent:*

(i) $\{\alpha_n\}$ *is an almost periodic sequence,*

(ii) $\lim_{n\theta\to 0} \alpha_n^2 = 1$.

Proof. Let Γ be the group of §1. By Bochner's theorem there exists a probability measure γ on Γ such that

$$\alpha_n = \int_\Gamma e^{2\pi i n x}\gamma(dx). \tag{19}$$

$\{\alpha_n\}$ is almost periodic if and only if γ is atomic. Condition (ii) implies that the set $\{n \mid \alpha_n^2 > 1-\varepsilon\}$ is relatively dense for every $\varepsilon > 0$. This implies γ is atomic, and (i) follows.

Suppose (i) holds. Then γ is purely atomic. By a standard identification multiplication by $e^{2\pi i x}$ on $\mathscr{L}^2(\Gamma, \gamma)$ is unitarily equivalent with the restriction of U to the space spanned by $\{f^{(n)}\}_{n\in Z}$. Thus, U has an eigenvalue, say $e^{2\pi i\lambda}$. If $Uh = e^{2\pi i\lambda}h$, $\|h\|_2 = 1$, then $|h(x)| = 1$, a.e. ν, and $h(x+\theta)[h(x)]^- = e^{2\pi i\lambda}f$. In general, $h(x+n\theta)\bar{h}(x) = e^{2\pi i n\lambda}f^{(n)}(x)$ and it follows that (a) $\lim_{n\theta\to 0} e^{2\pi i n 2\lambda} = 1$, (b) $\lim_{n\theta\to 0} \alpha_n^2 = 1$. From (b) we have (ii) and from (a) we have $e^{2\pi i 2\lambda} = \chi(\theta)$ for some character χ. The proposition is proved, together with the additional statement

$$\alpha_n = \sum_\lambda \sigma_\lambda \chi_\lambda(n\theta) \qquad (\sigma_\lambda \geqq 0)$$

when (i) or (ii) holds.

To the equivalences (i) and (ii) we add a third.

Proposition 6. *Each of* (i) *and* (ii) *is equivalent to*

(iii) *There exists a continuous function $h(y, \cdot)$ from X to $\mathscr{L}^2(X, \nu)$ such that $h(\theta, \cdot) = f^{(2)}$, and such that*

$$h(y+w, x) = h(y, x+2w)h(w, x) \qquad (a.e.\ x). \tag{20}$$

Proof. If h exists, then $\alpha_{2n} = \int_X h(n\theta, x)\nu(dx)$. By the continuity $\lim_{n\theta\to 0} \alpha_{2n} = 1$, and as in (ii) $\Rightarrow$ (i) we have (iii) $\Rightarrow$ (i). For (ii) $\Rightarrow$ (iii) we argue as in Proposition 4, this time by setting $h(n\theta, x) = f^{(2n)}(x)$. We omit the details.

Suppose (20) holds, and suppose θ is divisible to the extent that there are $\theta_0, \theta_1 \in X$ with $2\theta_0 = \theta_1$ and $2\theta_1 = \theta$. Setting $y = w = \theta_1$ in (20), we get

$$f(x)f(x+\theta) = h(\theta_1, x+\theta)h(\theta_1, x) \qquad (\text{a.e. } x).$$

It follows that $f(x)h(\theta_1, x)$ is invariant under R_θ and hence must be constant. Thus for $\varepsilon=1$ or -1, $h(\theta_1, x)=\varepsilon f(x)$. Use (20) again, this time with $y=w=\theta_0$. We get

$$h(\theta_0, x+\theta_1)h(\theta_0, x) = \varepsilon f(x).$$

Therefore, $\mathscr{S}(\varepsilon f, \theta_1, F)$ is not strictly ergodic. It will however be minimal if $h(\theta_0, \cdot)$ is not continuous when lifted to M.

REMARK. Suppose f is such that (4) does not hold. Then $\mathscr{S}(f, \theta, F)$ is strictly ergodic. If $h \in C(N)$, a theorem of Oxtoby [9] asserts that uniformly in $w \in N$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} h(T^j w) = \int_N h(w)\zeta(dw).$$

If χ is a character on X, but viewed as a continuous function on M, then N, and if we take $h(w)=\chi(w)\pi_2(w)$, then the integral is 0. We conclude that uniformly in $x \in A'$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f^{(j)}(x)e^{2\pi i j\lambda} = 0 \tag{21}$$

for those λ such that $\chi(\theta)=e^{2\pi i\lambda}$ generates a continuous character on X. If for some other λ (21) does not exist, or if it exists and is not zero, then there exists $\zeta_0 \in \mathscr{P}$ such that (T, N, ζ_0) has $e^{2\pi i\lambda}$ in its point spectrum (cf. [11]). Since ζ is unique, $\zeta=\zeta_0$, and (T, N, ζ) has point spectrum. Then U must also, and this implies (4). This is a contradiction, and (21) holds uniformly in $x \in A'$ for all real λ.

3. **Some binary systems.** X will be the set of infinite sequences $x=(x_1, x_2, \dots)$ of integers such that $0 \leqq x_j < 2^j$ and $x_j \equiv x_{j+1} \pmod{2^j}$, $j=1, 2, \dots$. With coordinatewise convergence and addition (mod 2^j) X is a compact group. Let $\theta=(1, 1, \dots)$, and notice that $-\theta=(1, 3, 7, \dots)$.

If $x \in X$, then for all j, $x_{j+1}=x_j+\varepsilon_j 2^j$, where $\varepsilon_j=0$ or 1. We set $x_0=0$, so that the formula also holds when $j=0$. Since any (0, 1)-valued sequence $\varepsilon_0, \varepsilon_1, \dots$ uniquely determines a point of X by the same formula, X may be viewed as the set of formal 2-adic expansions

$$x = \sum_{j=0}^{\infty} \varepsilon_j 2^j \qquad (\varepsilon_j = 0 \text{ or } 1). \tag{22}$$

If $n>0$, the expansion for $n\theta$ is precisely the binary expansion of n, from which it is evident that $\{n\theta\}$ is dense in X.

If $x \in X$, $x \neq -\theta$, we define $\tau(x)$ to be the first index j such that $x_j=x_{j+1}$. Thus, $\tau(0)=0$, $\tau(\theta)=1$, $\tau(2\theta)=0, \dots$. We will prove

$$\tau(-x-2\theta) = \tau(x) \qquad (x \neq -\theta). \tag{23}$$

In the first place, if $\tau(x_1)=0=\tau(x_2)$, then $\tau(x_1-x_2)=0$, and (23) holds when

$\tau(x)=0$. If $\tau(x)\neq 0$, then $x=(1,3,\ldots,2^{\tau(x)}-1,2^{\tau(x)}-1,x_{\tau(x)+2},\ldots)$, and $-x=(1,1,\ldots,1,2^{\tau(x)}+1,2^{\tau(x)+2}-x_{\tau(x)+2},\ldots)$. Now $-2\theta=(0,2,6,\ldots,2^{l}-2,\ldots)$, and therefore

$$-x-2\theta = (1,3,\ldots,2^{\tau(x)}-1,2^{\tau(x)}-1,\ldots)$$

From this equation (23) follows.

Let $F=\{-\theta=(1,3,7,\ldots)\}$.

Suppose $\psi(n)$, $n=0,1,\ldots$, is a ± 1-valued sequence *which is not eventually constant*. On F' we define $f(x)=\psi\circ\tau(x)$. Certainly f is continuous, and our condition on ψ ensures that $\lim_{n\to\infty} f((2^n-1)\theta)=\lim_{n\to\infty}\psi(n)$ does not exist. Since $\lim_{n\to\infty}(2^n-1)\theta=-\theta$, our axioms 1 and 4 are true. Since $\varphi^j F\cap F=\varnothing$, $\mathscr{S}(f,\theta,F)$ is minimal by Corollary 1.

We will prove $\mathscr{S}(f,\theta,F)$ is strictly ergodic. To this end it will be sufficient to establish that $\lim_{n\theta\to 0}\alpha_n\neq 1$, where $\alpha_n=\int_X f^{(n)}(x)\nu(dx)$. We will show that

$$\alpha_{2^n} = \psi(n-1)\sum_{j=n}^{\infty}\psi(j)2^{n-j-1} \qquad (n>0). \tag{24}$$

Since $2^n\theta\to 0$ and $\psi(n-1)\psi(n)=-\psi(n-1)\psi(n+1)$ infinitely often, it will not even be true that (4) holds.

Define $f(-\theta)=1$. Since f is continuous on F', we have

$$\alpha_{2^n} = \lim_{N\to\infty}\frac{1}{N}\sum_{j=0}^{N-1} f^{(2^n)}(x+j\theta)$$

for all x. Set $x=0$. We have

$$\alpha_{2^n} = \lim_{N\to\infty}\frac{1}{N}\sum_{j=0}^{N-1} f(j\theta)f((j+1)\theta)\cdots f((j+2^n-1)\theta). \tag{25}$$

In any sequence $j\theta,(j+1)\theta,\ldots,(j+2^n-1)\theta$ there will be 2^{n-1} elements with $\tau(l\theta)=0$, 2^{n-2} with $\tau(l\theta)=1,\ldots,2$ with $\tau(l\theta)=n-2$, and 1 with $\tau(l\theta)=n-1$. The remaining value is $\tau(l\theta)=n,n+1,\ldots$, the possibilities occurring with frequencies $1/2,1/4,\ldots$. Since the numbers 2^j are even $1\leqq j\leqq n-1$, a typical term in (25) is $\psi(n-1)\psi(n+j)$, $j\geqq 0$. Because $\sum_{j=1}^{\infty}1/2^j=1$, it is legitimate to pass from (25) to (24) using the frequencies indicated above.

Let (σ,M) and $\mathscr{S}(f,\theta,F)$ be as discussed in §2. Certainly there exists $m\in M$ with $m(n)=f(n\theta)$, $n\neq -1$, and $m(-1)=1$. As in the discussion following (8) we define y_n by

$$\begin{aligned} y_n &= m(n-1)\cdots m(0) && n>0\\ &= 1 && n=0\\ &= m(n)\cdots m(-1) && n<0. \end{aligned}$$

Using (23) we have for $n>0$

$$\begin{aligned} y_{-n} &= m(-n)\cdots m(-1) \\ &= m(-n)\cdots m(-2) \\ &= m(n-2)\cdots m(0) \\ &= y_{n-1}. \end{aligned}$$

Let Y be the orbit closure of $y=\{y_n\}$ in Λ^∞ under σ.

THEOREM 4. *Let ψ be a ± 1-valued sequence defined for $n=0, 1, \ldots$, and suppose ψ is not eventually constant. If y_n, $n>0$, is defined by $y_n=\prod_{k=0}^{n-1} \psi \circ \tau(k\theta)$, and if $y_0=1$, $y_{-n}=y_{n-1}$, $n\geqq 0$, then (σ, Y) is minimal and strictly ergodic, where Y is the orbit closure of y under σ.*

REMARK. We have seen that not even (4) holds for α_n. Therefore, if y is as in the theorem, and if λ is a real number, then by the remark at the end of §2

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} y_j e^{2\pi i j\lambda} = 0.$$

The systems defined above are equivalent to systems discussed by Kakutani in [3] and [4], and Theorem 4 is merely a restatement of his theorem regarding these systems. The equivalence is exhibited below.

Let α, $0<\alpha\leqq 1$, be a real number with binary expansion $\alpha=\sum_{j=0}^{\infty} \delta_j 2^{-(j+1)}$, $\delta_j=0$ or 1. There is no ambiguity if we require δ_j to be 1 infinitely often. Define $\psi(n)$, $n\geqq 0$, by

$$\psi(n) = (-1)^{\sum_{j=0}^{n} \delta_j}$$

and note that ψ is not eventually constant. Conversely, given such an ψ, define $\delta_0=(1-\psi(0))/2$, $\delta_{n+1}=(1-\psi(n)\psi(n+1))/2$, $n>0$. Then $\psi(n)=(-1)^{\sum_{j=0}^{n}\delta_j}$. In our setting y_n, $n>0$, is defined by

$$\begin{aligned} y_n &= \prod_{j=0}^{n-1} \psi \circ \tau(j\theta) \\ &= \prod_{j=0}^{n-1} (-1)^{\sum_{i=0}^{\tau(j\theta)} \delta_i}. \end{aligned} \tag{26}$$

Let $j\theta=\sum_{s=0}^{\infty} \varepsilon_s^{(j)} 2^s$ in (22). (The sum is finite.) We notice that

$$\sum_{i=0}^{\tau(j\theta)} \delta_i = 2\delta_{\tau(j\theta)} + \sum_{i=0}^{\infty} \delta_i \varepsilon_i^{(j)} - \sum_{i=0}^{\infty} \delta_i \varepsilon_i^{(j+1)}$$

and when this is substituted in (26), we get

$$\begin{aligned} y_n &= (-1)^{-\sum_{i=0}^{\infty} \delta_i \varepsilon_i^{(n)}} \\ &= (-1)^{\sum_{i=0}^{\infty} \delta_i \varepsilon_i^{(n)}}. \end{aligned}$$

Then with $y_0=1$, $y_{-n}=y_{n-1}$, $n\geqq 0$, the system (σ, Y) is that defined by Kakutani in [4]. When $\alpha=1$, he observes that (σ, Y) is the system of Morse [7].

4. **The Kronecker-Weyl problem mod 2.** X and θ are as in the introduction. If $0 \leqq t_1 < t_2 < 1$, let f be one of the ± 1-valued functions on X with discontinuities at t_1 and t_2. We will study equation (4), and since that equation depends only upon θ and $t = t_2 - t_1$, we will assume $t_1 = 0$, $t_2 = t$. As before $f^{(n)}(x) = f(x)f(x+\theta)\cdots f(x+(n-1)\theta)$, $x \in X$.

Formally, $f^{(n)}$ has discontinuities at the points $0, -\theta, \ldots, (1-n)\theta, t, t-\theta, \ldots, t+(1-n)\theta$, and these will be the actual discontinuities provided $t \neq k\theta$ for some k. First we dispose of the case $t = k\theta$.

Suppose $t = k\theta$ with, say, $k > 0$. Only the points $k\theta, \ldots, \theta, (k-n)\theta, \ldots, (1-n)\theta$ are actual discontinuities of $f^{(n)}$. Let the points $\theta, \ldots, k\theta$ be ordered as real numbers in $[0, 1)$; i.e. as $0 < l_1\theta < \cdots < l_k\theta < 1$. Recalling that $\|\cdot\|$ denotes distance to closest integer, we note that when $\|n\theta\| < \min_{1 \leqq j \leqq k} \|l_j\theta\|$, then for each j $(l_j - n)\theta$ is the discontinuity of $f^{(n)}$ closest to $l_j\theta$. Thus, every other interval between discontinuities has length $\|n\theta\|$, and since $f^{(n)}$ is of alternating sign on these intervals, $|\alpha_n| = |1 - 2k\|n\theta\||$. Equation (4) follows, and a similar argument proves (4) when $k < 0$.

In all that follows it is assumed that $t \neq k\theta$. Let the $2n$ discontinuities of $f^{(n)}$ be ordered as $0 = x_1 < x_2 < \cdots < x_{2n} < x_{2n+1} = 1$. Of course in X, $x_1 = x_{2n+1}$, but it will often be convenient to differentiate between approaches to "0". To fix notations we assume of f that

$$\begin{aligned} f(x) &= 1 \qquad && 0 \leqq x < t \\ &= -1 && t \leqq x < 1. \end{aligned}$$

We have then that

$$f^{(n)}(x) = f^{(n)}(0)(-1)^{l-1} \qquad (x_l \leqq x < x_{l+1};\ 1 \leqq l \leqq 2n).$$

Let $\varepsilon_n = \min_{1 \leqq l \leqq 2n} x_{l+1} - x_l$. Since $f^{(n)}$ is 1 on n intervals and -1 on n intervals, an inequality $|\alpha_n| > 1 - \varepsilon$ implies $n\varepsilon_n < \varepsilon/2$. If (4) holds, then

$$\lim_{\|n\theta\| \to 0} n\varepsilon_n = 0. \tag{27}$$

For each $n > 0$, ε_n has one of two forms. Either

(a) $\varepsilon_n = \min_{1 \leqq k < n} \|k\theta\|$, or

(b) $\varepsilon_n = \min_{|k| < n} \|k\theta - t\|$.

We emphasize that in (b) k is allowed to be negative. Our technique consists in taking advantage of certain simplifications which can be made when ε_n is (b).

As in §1 we let $\{m_k/n_k\}_{k=1}^{\infty}$ be the sequence of convergents to θ. For the moment we need only the properties

(i) $\|n_k\theta\| = \min_{1 \leqq n < n_{k+1}} \|n\theta\|$ and

(ii) $n_{k+1}\|n_k\theta\| > 1/2$.

For these and other facts concerning continued fractions, see [2] or [5].

Suppose θ, t are such that (4) holds. For any $T > 0$ there exists a $\delta > 0$ such that if $\|n\theta\| < \delta$, then $n\varepsilon_n < 1/T$ or $\varepsilon_n < 1/Tn$. If $n = n_{k+1}$, property (ii) implies, for $T > 2$, that ε_n is (b). In particular, t lies in one of the $2n_{k+1} - 1$ intervals of length $2/Tn_{k+1}$

centered at the points $p\theta$, $|p|<n_{\zeta+1}$. The total measure of these intervals is bounded by $4/T$, and letting $T\to\infty$ it follows readily that, for fixed θ, *the set of t for which* (4) *holds has measure* 0.

A second representation for $f^{(n)}$ is useful. For each s, $0<s<1$, define φ_s on X by

$$\begin{aligned}\varphi_s(x) &= 1 \qquad 0 \leqq x < s\\ &= -1 \qquad s \leqq x < 1.\end{aligned}$$

We set $\varphi_0\equiv -1$, $\varphi_1\equiv 1$. Of course $\varphi_t=f$. Define $\varphi^{(n)}$, $n>0$, by

$$\varphi^{(n)}(x) = \varphi_{-\theta}(x)\varphi_{-2\theta}(x)\cdots\varphi_{(1-n)\theta}(x)\varphi_t(x)\cdots\varphi_{t+(1-n)\theta}(x). \tag{28}$$

There are $2n-1$ factors in $\varphi^{(n)}$, each of which is discontinuous at 0, and therefore $\varphi^{(n)}$ is discontinuous at 0. The remaining discontinuities of $\varphi^{(n)}$ coincide with the remaining discontinuities of $f^{(n)}$, and so since $\varphi^{(n)}(0)=1$, we have

$$f^{(n)}(x) = f^{(n)}(0)\varphi^{(n)}(x). \tag{29}$$

Define $\beta_n=\int_X \varphi^{(n)}(x)\nu(dx)$. If (4) holds, then also

$$\lim_{\|n\theta\|\to 0} \beta_n^2 = 1. \tag{30}$$

Suppose p and n are such that $|p|<n$ and $\|p\theta-t\|=\min_{|q|<n}\|q\theta-t\|$. Let us observe that if $0\leqq k<n$ and $k\neq p$, then

$$\|\varphi_{t-k\theta}-\varphi_{(p-k)\theta}\|_1 = 2\|p\theta-t\| \tag{31}$$

where $\|\cdot\|_1$ is the norm in $\mathscr{L}^1(X,\nu)$. For if $\alpha=t-k\theta$, $\beta=(p-k)\theta$, the functions in question differ by 2 on an interval of length $|\alpha-\beta|$. Either $\|\alpha-\beta\|=\|\alpha\|+\|\beta\|$ or $\|\alpha-\beta\|=|\alpha-\beta|$. The former cannot be true by assumption, since $\|\alpha-\beta\|=\|p\theta-t\|$. Thus, $|\alpha-\beta|=\|p\theta-t\|$, and (31) follows.

Assume p and n are as above with $p>0$. Arguing as in (31) one verifies that $\|\varphi_{t-p\theta}-\varphi_t(p\theta)\varphi_0\|_1=2\|p\theta-t\|$. Therefore, if we set $t=p\theta$ in (28) and define

$$\begin{aligned}\varphi_*^{(n)} &= \varphi_t(p\theta)\{\varphi_{-\theta}\cdots\varphi_{(1-n)\theta}\varphi_{p\theta}\cdots\varphi_{\{p+(1-n)\}\theta}\}\\ &= -\varphi_t(p\theta)\{\varphi_{p\theta}\cdots\varphi_\theta\varphi_{(p-n)\theta}\cdots\varphi_{(1-n)\theta}\},\end{aligned}$$

(31) and the triangle inequality yield

$$\|\varphi^{(n)}-\varphi_*^{(n)}\|_1 \leqq 2n\|p\theta-t\|. \tag{32}$$

Next, let p and n be as above with $p<0$. Generally, $\|n\theta\|<\|p\theta\|$ will be true, and when it is $\|1-\varphi_{p\theta}\varphi_{(p-n)\theta}\|_1=2\|n\theta\|$. Also, $\|\varphi_{-n\theta}-\varphi_{1/2}(n\theta)\varphi_1\|_1=2\|n\theta\|$. Define

$$\begin{aligned}\varphi_*^{(n)} &= \varphi_{1/2}(n\theta)\varphi_{-n\theta}\varphi_{p\theta}\varphi_{(p-n)\theta}\{\varphi_{-\theta}\cdots\varphi_{(1-n)\theta}\varphi_{p\theta}\cdots\varphi_{(p+(1-n))\theta}\}\\ &= \varphi_{1/2}(n\theta)\{\varphi_{p\theta}\cdots\varphi_{-\theta}\varphi_{(p-n)\theta}\cdots\varphi_{(-1-n)\theta}\}\end{aligned}$$

and this time we get

$$\|\varphi^{(n)}-\varphi_*^{(n)}\|_1 \leqq 2n\|p\theta-t\|+4\|n\theta\|. \tag{33}$$

We can unify the notations in (32) and (33) by setting $\mu=\operatorname{sgn} p$, writing φ_j for $\varphi_{j\theta}$, and defining

(34) $$\psi^{(n)} = \varphi_p\varphi_{p-\mu}\cdots\varphi_\mu\varphi_{p-n}\cdots\varphi_{\mu-n}.$$

Using (32) and (33) there is a choice of $\varepsilon=1$ or -1, (specifically, $\varepsilon=-\varphi_t(p\theta)$ or $\varepsilon=\varphi_{1/2}(n\theta)$) such that

(35) $$\|\varphi^{(n)}-\varepsilon\psi^{(n)}\|_1 \leqq 2n\|p\theta-t\|+4\|n\theta\|.$$

Definition 3. With θ and t as above, we define $S(\theta, t)=S$ to be the set of $n>0$ for which there is a p, $|p|<n$, with $\varepsilon_n=\|p\theta-t\|$.

A consequence of (35) is that if (4) or (30) holds, then

(36) $$\lim_{n\in S, \|n\theta\|\to 0} \gamma_n^2 = 1$$

where $\gamma_n=\int_X \psi^{(n)}(x)\nu(dx)$.

For a second consequence of (35) suppose $\|n\theta\|<\min_{1\leqq j\leqq \mu p}\|j\theta\|$. We pair terms in (34) as $\varphi_p\varphi_{p-n}, \varphi_{p-\mu}\varphi_{p-\mu-n}, \ldots, \varphi_\mu\varphi_{\mu-n}$, obtaining $|p|$ functions, each one being -1 on an interval of length $\|n\theta\|$. Were two of these intervals to overlap, there would exist j_1, j_2 with $1\leqq j_1<j_2\leqq \mu p$, and such that $\|(j_2-j_1)\theta\|<\|n\theta\|$. By assumption this cannot happen, and so the intervals are disjoint. We obtain that

(37) $$\int_X \psi^{(n)}(x)\nu(dx) = \{1-2\mu p\|n\theta\|\}$$

when $n\in S$[6] and $\|n\theta\|<\min_{1\leqq j\leqq \mu p}\|j\theta\|$. In particular, an inequality $|\gamma_n|>1-\varepsilon$ implies in turn

(38) $$\mu p\|n\theta\| < \varepsilon/2 \quad \text{or} \quad \mu p\|n\theta\| > 1-\varepsilon/2.$$

We will now establish part of the sufficiency statement in Theorem 1. We will show that if θ has bounded partial quotients, and if $t\neq k\theta$, then (4) cannot hold. It will follow that $\mathscr{S}(f, \theta, \{0, t\})$ is strictly ergodic, and the existence of (1) for all translates of I follows from the uniformity in (21). ((21) holds even for $x\in F$, provided $\varphi^j x\notin F$ for all j beyond a certain point.)

Proposition 7. *If θ has bounded partial quotients, and if $t\neq k\theta$, then* (4) *cannot hold.*

Proof. Let $c>0$ be such that $\|n\theta\|>c/n$ for $n>0$. If $\{n_k\}$ is the sequence of denominators of convergents to θ, then $\|n_k\theta\|<1/n_{k+1}$, and so $n_{k+1}<n_k/c$. We will also need the fact $n_k\|n_{k+1}\theta\|<1/2$.

Let $\varepsilon=1/2c^3<c^3<c^2<c$. If k is sufficiently large, then $n_{k+1}, n_{k+2}\in S(\theta, t)$, $n_{k+1}\varepsilon_{n_{k+1}}<\varepsilon/2$, and $|\gamma_{n_{k+j}}|>1-\varepsilon$, $j=1, 2$. Infinitely often there will exist p, $n_k<|p|\leqq n_{k+1}$, with $\varepsilon_{n_{k+1}}=\|p\theta-t\|$. (For *all* large k, $\varepsilon_{n_{k+1}}$ has the required form;

[6] $\psi^{(n)}$ is defined for $n\notin S$ if $\|n\theta\|<\|p\theta\|$.

the restriction on $|p|$ is what is not always true.) Suppose $\varepsilon_{n_{k+2}} = \|p\theta - t\|$. Since $|p|\ \|n_{k+2}\theta\| \leqq n_{k+1}\|n_{k+2}\theta\| < 1/2$, it follows from (38) that $|p|\ \|n_{k+2}\theta\| < \varepsilon/2$. On the other hand, $|p|\ \|n_{k+2}\theta\| > n_k\|n_{k+2}\theta\| > c^3 > \varepsilon$. We conclude that $\varepsilon_{n_{k+2}} = \|q\theta - t\|$ for some $|q| < n_{k+2}$ with $p \neq q$. By the triangle inequality $\|(p-q)\theta\| < \varepsilon_{n_{k+1}} + \varepsilon_{n_{k+2}}$, and therefore $n_{k+1}\|(p-q)\theta\| < \varepsilon$. Also, $0 < |p-q| < n_{k+1} + n_{k+2} \leqq n_{k+3}$, which means $\|(p-q)\theta\| > \|n_{k+3}\theta\|$. Collecting results, we have $\varepsilon > n_{k+1}\|n_{k+3}\theta\| > c^3$, a contradiction. We conclude that (4) cannot hold, and the proposition is proved.

Standing assumption: Unless stated otherwise θ and t will be assumed to be such that $t \neq k\theta$ and (4) holds. We have seen already that θ necessarily has unbounded partial quotients. Our task is to determine t.

LEMMA 2. *Given $\delta > 0$ and α, β with $0 < \alpha < \beta < 1$, there exists an index k_0 such that if $k \geqq 0$, and if*

$$\alpha n_{k+1} < n < \beta n_{k+1}$$

then $n \in S(\theta, t)$ $(\varepsilon_n = \|p\theta - t\|)$ and

$$|p|\ \|n_k\theta\| < \delta. \tag{39}$$

Proof. We may suppose $\delta < \min(\alpha, 1-\beta)$. Using (4) and (36) choose $\varepsilon > 0$ so that if $n \in S(\theta, t)$ and $\|n\theta\| < \varepsilon$, then $n\varepsilon_n < \delta/2$ and $|\gamma_n| > 1-\delta$. Choose k_0 large enough that if $k \geqq k_0$, the interval $\alpha n_{k+1} < n < \beta n_{k+1}$ contains an element with $\|n\theta\| < \varepsilon$ and also $\|n_k\theta\| < (1-\beta-\delta)/2$. For such k and n, $n\|n_k\theta\| > \alpha n_{k+1}\|n_k\theta\| > \alpha/2 > \delta/2$, and so $n \in S(\theta, t)$. If $\|n_k\theta\|$ is less than or equal to the length of the shortest interval of constancy of $\psi^{(n)}$ (i.e. the shortest distance between discontinuities of $\psi^{(n)}$), then (39) follows. Noticing that in the separate factors $\varphi_p \cdots \varphi_\mu$ and $\varphi_{p-n} \cdots \varphi_{\mu-n}$ in (34) the shortest interval is at least as great as $\|n_k\theta\|$, we find that if $\psi^{(n)}$ has a shorter distance between discontinuities, there must exist i, j such that $\varphi_i\varphi_{j-n}$ has a distance $\|(i-j+n)\theta\| < \|n_k\theta\|$ between discontinuities. It must be that $|i-j+n| \geqq n_{k+1}$, or

$$p \geqq i \geqq n_{k+1} - n + j \geqq (1-\beta)n_{k+1} \qquad (p > 0)$$
$$-p \geqq -j \geqq n_{k+1} - n - i \geqq (1-\beta)n_{k+1} \qquad (p < 0).$$

In either case there are fewer than $|p| - [(1-\beta)n_{k+1}]$ such pairs, and for each pair the interval $[i\theta, (j-n)\theta)$ (or $[(j-n)\theta, i\theta)$) contains no other discontinuity of $\psi^{(n)}$. Thus, of the $|p|$ shortest intervals for $\psi^{(n)}$ at least $[(1-\beta)n_{k+1}]$ have lengths $\|n_k\theta\|$ or greater. We conclude that $[(1-\beta)n_{k+1}]\|n_k\theta\| < \delta/2$, which is in contradiction to $[(1-\beta)n_{k+1}]\|n_k\theta\| > ((1-\beta)n_{k+1} - 1)\|n_k\theta\| > (1-\beta)/2 - \|n_k\theta\| > \delta/2$ for $k \geqq k_0$. The lemma is proved.

REMARK. Using $\|n_k\theta\| > 1/2n_{k+1}$ equation (39) implies

$$|p| < 2\delta n_{k+1}. \tag{39'}$$

REMARK. We are assuming $t \neq j\theta$. One consequence of this assumption is that given $M>0$ there exists n_0 such that if $n \geqq n_0$, and if $n \in S(\theta, t)$ with $\varepsilon_n = \|p\theta - t\|$, then

$$\min_k |n_k - |p|| > M. \tag{40}$$

Let $2\theta_0 = \theta$. If also $t \neq j\theta + \theta_0$, and if $\{p_s\}$ is a sequence such that $\lim_{s\to\infty} p_s\theta = t$, there exists s_0 such that if $s \geqq s_0$, then

$$||p_{s+1}| - |p_s|| > M. \tag{41}$$

The proofs are straightforward and will be omitted.

REMARK. In any interval of the form $n_k \leqq n < 2n_k$, $k>1$, the smallest values of $\|n\theta\|$ are known to be $\|n_k\theta\|$ and $\|(n_k+n_{k-1})\theta\| = \|n_{k-1}\theta\| - \|n_k\theta\|$. If it is known somehow that $n_k\|n_k\theta\| < 1/4$ and $n_k\|n\theta\| < 1/4$, then because $n_k\|n_{k-1}\theta\| > 1/2$, it must be that $n = n_k$.

We use our remarks first for proving

LEMMA 3. *Let $2\theta_0 = \theta$. If $t = m\theta + \theta_0$, then* (4) *cannot hold.*

Proof. For sufficiently large k, $\varepsilon_{n_k} = \|p_k\theta - t\|$, and we define $l_k = p_k - m$. From $p_k\theta - t = l_k\theta - \theta_0$ we obtain $\|(2l_k - 1)\theta\| = 2\varepsilon_{n_k}$. Since $2n_k\varepsilon_{n_k} < 1/2$ for large k it follows [5, Theorem 19], that

$$|2l_k - 1| \geqq n_k.$$

Thus $\liminf |p_k|/n_k \geqq 1/2$, and so by Lemma 2 $\lim_{k\to\infty} |p_k|/n_k = 1$. On the other hand θ has unbounded partial quotients, meaning that $n_k\|n_k\theta\| < 1/4$ infinitely often. For these values of k the remark preceding the lemma implies $|2l_k - 1| = n_k$. Therefore $\liminf |p_k|/n_k = 1/2$, a contradiction. The lemma is proved.

From our lemma we obtain that if (4) is true with $t \neq j\theta$, and if $\|p\theta - t\| = \|q\theta - t\|$, then $p = q$. For otherwise $t = p\theta + \theta_1 = q\theta - \theta_1$ and $-2\theta_1 = (p-q)\theta$. It follows that $t = q\theta + ((p-q)/2)\theta$, which is impossible.

We suppose $[a_0; a_1, a_2, \ldots]$ is the continued fraction expansion for θ. The a_k's generate the n_k's by the recursion formula $n_{k+1} = a_{k+1}n_k + n_{k-1}$. Moreover, $n_{k-1}\theta$ and $n_k\theta$ have opposite "senses"; that is in the representation $X = \{x \mid 0 \leqq x < 1\}$ $n_{k-1}\theta$ lies in the interval $0 \leqq x < 1/2$ (resp. $1/2 \leqq x < 1$) and $n_k\theta$ lies in the interval $1/2 \leqq x < 1$ (resp. $0 \leqq x < 1/2$). If $1 \leqq s \leqq a_{k+1}$, then the formulas

$$\begin{aligned} \|(n_{k-1} + sn_k)\theta\| &= \|n_{k-1}\theta\| - s\|n_k\theta\| \\ \|sn_k\theta\| &= s\|n_k\theta\| \end{aligned} \tag{42}$$

are true. In particular, the points $sn_k\theta$, $(n_{k-1}+sn_k)\theta$, $0 \leqq s \leqq a_{k+1}$, are $\|n_k\theta\|$ dense in a neighborhood of 0 in X bounded by $a_{k+1}n_k\theta$ and $n_{k-1}\theta$. If $0 < n < n_{k+1}$, and if $n\theta$ belongs to this neighborhood, there exists s such that $\|(sn_k - n)\| < \|n_k\theta\|$ (or $\|((n_{k-1}+sn_k) - n)\theta\| < \|n_k\theta\|$). Since $|sn_k - n| < n_{k+1}$ (or $|(n_{k-1}+sn_k - n)| < n_{k+1}$), it must be that $n=sn_k$ or $n_{k-1}+sn_k$. A special case of this remark has been used in the proof of Lemma 3. We have proved

LEMMA 4. *If* $0<|m|<n_{k+1}$, *and if* $\|m\theta\|<\|n_{k-1}\theta\|$, *then* $|m|=sn_k$ *or* $n_{k-1}+sn_k$ *for some* s *with* $1\leqq s\leqq a_{k+1}$.

In all that follows $\{p_s\}_{s=1}^{\infty}$ will be the sequence of integers, positive or negative, such that

$$\|p_s\theta-t\| = \min_{|p|<|p_{s+1}|} \|p\theta-t\|.$$

From Lemma 3 we have that if (4) holds, then $\|p_{s+1}\theta-t\|<\|p_s\theta-t\|$.

LEMMA 5. *Given* α, $0<\alpha<1$, *and* $\delta>0$ *there exists an index* k_1 *such that if* $k\geqq k_1$, *and if* $\alpha n_{k+1}\leqq|p_j|\leqq n_{k+1}$, *then*

(i) $a_{k+2}\geqq 2$ *implies* $a_{k+2}\geqq(1-3\delta)/2\delta$, *and*

(ii) $a_{k+2}=1$ *implies* $a_{k+3}\geqq(1-3\delta)/2\delta$.

Proof. Suppose $a_{k+2}\geqq 2$, in which case $n_{k+1}\|n_{k+1}\theta\|<1/2$. We may suppose $\alpha<1-\delta$, and we choose k_0 by Lemma 2 for δ, α, $\beta=1-\delta$. Using (38) let $k_1\geqq k_0$ be large enough that if $k\geqq k_1$, then $\varepsilon_{n_{k+1}}=\|p\theta-t\|$ and either $|p|\ \|n_{k+1}\theta\|<\delta$ or $|p|\ \|n_{k+1}\theta\|>1-\delta$. If p_j is as in our lemma, then certainly $|p_j|\leqq|p|$, and therefore $|p_j|\ \|n_{k+1}\theta\|\leqq|p|\ \|n_{k+1}\theta\|<n_{k+1}\|n_{k+1}\theta\|<1/2$. Assuming $\delta<1/2$, $|p_j|\ \|n_{k+1}\theta\|<\delta$. Also, since $k\geqq k_0$, Lemma 2 tells us that $|p_j|>(1-\delta)_{n_{k+1}}$. Collecting results, we have

$$n_{k+1}\|n_{k+1}\theta\| < \frac{|p_j|}{1-\delta}\,\|n_{k+1}\theta\| < \frac{\delta}{1-\delta}.$$

This inequality combined with

$$\|n_{k+1}\theta\| > \frac{1}{2n_{k+2}} > \frac{1}{2(a_{k+2}+1)n_{k+1}}$$

yields the desired result (i).

The proof of (ii) is similar since $n_{k+1}\|n_{k+2}\theta\|<1/2$ is automatic. We omit the details, except to mention that $\alpha n_{k+1}\leqq|p_j|\leqq n_{k+1}$ implies $\alpha/2n_{k+2}\leqq|p_j|\leqq n_{k+2}$ when $a_{k+2}=1$. Thus, Lemma 2 is invoked for δ, $\alpha/2$, and $\beta=1-\delta$.

We continue with the notation of Lemma 5. Suppose we are in case (i), and define $m_s=n_k+sn_{k+1}$, $1\leqq s\leqq a_{k+2}$. Equation (42) gives

$$(a_{k+2}-s)\|n_{k+1}\theta\| < \|m_s\theta\| < (a_{k+2}+1-s)\|n_{k+1}\theta\|. \tag{43}$$

Since $(1-\delta)n_{k+1}\leqq|p_j|\leqq n_{k+1}$, it follows that

$$(1-\delta)(a_{k+2}-s)n_{k+1}\|n_{k+1}\theta\| < |p_j|\ \|m_s\theta\| < (a_{k+2}+1-s)n_{k+1}\|n_{k+1}\theta\|.$$

Using

$$\frac{1}{2(a_{k+2}+1)n_{k+1}} < \|n_{k+1}\theta\| < \frac{1}{a_{k+2}n_{k+1}}$$

we find

$$(44)\qquad \frac{1-\delta}{2}\left\{1-\frac{1+s}{1+a_{k+2}}\right\} < |p_j|\,\|m_s\theta\| < 1+\frac{1-s}{a_{k+2}}.$$

LEMMA 6. *Let the notations and assumptions be those of Lemma 5. There k_1 can also be chosen so that if* (i) *obtains, then $|p_{j+1}|<n_{k+2}$ and $|p_{j+1}-p_j|<n_{k+2}$.*

Proof. If δ is sufficiently small and k sufficiently large it will be true by Lemma 5 that if $\frac{1}{4}a_{k+2}\leqq s\leqq\frac{1}{3}a_{k+2}$ then the left side of (44) is at least 2δ and the right side is at most $1-2\delta$. If k is also large enough that (38) holds, then because $\|m_s\theta\|<\|n_k\theta\| =\min_{0<k<|p_j|}\|k\theta\|$, it cannot be that $\varepsilon_{m_s}=\|p_j\theta-t\|$. Also, $m_s\geqq\frac{1}{5}n_{k+2}$, large k, and therefore $m_s\|n_{k+1}\theta\|\geqq 1/10$. It cannot be that $\varepsilon_{m_s}=\|n_{k+1}\theta\|$, large k. We conclude that k_1 exists so that if $k\geqq k_1$, and if n_k, p_j, s are as above, then $\varepsilon_{m_s}=\|p\theta-t\|$, $|p|\geqq|p_{j+1}|$, and $|p_{j+1}-p_j|$, $|p_{j+1}|<n_{k+2}$. The lemma is proved.

By a similar argument using $m_s=n_{k+1}+sn_{k+2}$ one can prove

LEMMA 7. *Let the notations and assumptions be those of Lemma 5. Then k_1 can be chosen so that if* (ii) *obtains, then $|p_{j+1}|<n_{k+3}$ and $|p_{j+1}-p_j|<n_{k+3}$.*

As a corollary to Lemmas 6 and 7 we have

LEMMA 8. *Given α, $0<\alpha<1$, there exists an index k_2 such that if $k\geqq k_2$, and if $\alpha n_{k+1}\leqq|p_j|\leqq n_{k+1}$, then*

(i′) *$a_{k+2}\geqq 2$ implies $\|p_j\theta-t\|>\frac{1}{2}\|n_{k+1}\theta\|$, and*

(ii′) *$a_{k+2}=1$ implies $\frac{1}{2}\|n_{k+1}\theta\|>\|p_j\theta-t\|>\frac{1}{2}\|n_{k+2}\theta\|$.*

Proof. For (i′) let $k_2=k_1$ in Lemma 6. If $\|p_j\theta-t\|\leqq\frac{1}{2}\|n_{k+1}\theta\|$, then $\|(p_{j+1}-p_j)\theta\| <\|n_{k+1}\theta\|$. This implies $|p_{j+1}-p_j|\geqq n_{k+2}$, which is impossible. Thus (i′) is true for $k\geqq k_2$.

Suppose $a_{k+1}=1$. We have $\|p_j\theta-t\|>\frac{1}{2}\|n_{k+2}\theta\|$ by letting $k_2=k_1$ in Lemma 7 and arguing as above. We have left to produce k_2 so that the left inequality in (ii′) is true. Because $a_{k+2}=1$, we have $\|n_k\theta\|<2\|n_{k+1}\theta\|$, or $\frac{1}{4}\|n_k\theta\|<\frac{1}{2}\|n_{k+1}\theta\|$. If $\|(p_j\theta-t)\|\geqq\frac{1}{2}\|n_{k+1}\theta\|$, then also $\|(p_j\theta-t)\|\geqq\frac{1}{4}\|n_k\theta\|$. Choose $\delta<\alpha$, $1-\alpha$, and let $\beta=1-\delta$ in Lemma 2. If $k\geqq k_1$ there, our assumption on $|p_j|$ implies $|p_j| \geqq(1-\delta)n_{k+1}$, and therefore $|p_j|\;\|p_j\theta-t\|\geqq(1-\delta)/8$. If $|p_j|<n<\min(n_{k+1},|p_{j+1}|)$, then $n\in S(\theta,t)$, and $\varepsilon_n=\|p_j\theta-t\|$. If $\varepsilon>0$ is such that $\|n\theta\|<\varepsilon$ implies $n\varepsilon_n<(1-\delta)/8$, then for large k we have a contradiction. Indeed, for large k (hence j) the interval $|p_j|<n<\min(n_{k+1},|p_{j+1}|)$ contains elements with $\|n\theta\|<\varepsilon$. (See (40) and (41).) The lemma is proved.

NOTATION. Let $\varepsilon_j=\operatorname{sgn}p_j$, $j=1,2,\ldots.$

LEMMA 9. *If $0<\alpha<1$, there exists an index k_3 such that if $k\geqq k_3$, and if $\alpha n_{k+1} \leqq|p_j|\leqq n_{k+1}$, then either*

(i″) *$a_{k+2}\geqq 2$ and $p_{j+1}=p_j+\varepsilon_j n_{k+1}$, or*

(ii″) *$a_{k+2}=1$ and $p_{j+1}=p_j+\varepsilon_j n_{k+2}$.*

Also, we have either

(iii) $a_{k+2} \geqq 2$ *and* $p_{j-1} = p_j - \varepsilon_j n_{k+1}$, *or*

(iv) $a_{k+2} = 1$ *and* $p_{j-1} = p_j - \varepsilon_j n_{k+2}$.

Proof. We prove (i″) first. It may be assumed that $|p_j| \, \|p_j\theta - t\| < \alpha/4$, and $|p_j| \, \|n_k\theta\| > \alpha/2$ is guaranteed because $\alpha n_{k+1} \leqq |p_j|$. Thus, $\|p_j\theta - t\| < \|n_k\theta\|/2$, and therefore $\|(p_{j+1} - p_j)\theta\| < \|n_k\theta\|$. Thus, $|p_{j+1} - p_j| \geqq n_{k+1}$. We know for large k ($k \geqq k_1$) that $|p_{j+1} - p_j| < n_{k+2}$.

Since $\|p_j\theta - t\| > \frac{1}{2}\|n_{k+1}\theta\|$ for $k \geqq k_2$, one of the points $(p_j \pm \varepsilon_j n_{k+1})\theta$ is closer to t than is $p_j\theta$. Since $|p_j - \varepsilon_j n_{k+1}| < (1-\alpha)n_{k+1} < |p_j|$ for large k, it is eventually always $(p_j + \varepsilon_j n_{k+1})\theta$ that is closer. Since $|p_{j+1} - p_j| \geqq n_{k+1}$, we have $p_{j+1} = p_j + \varepsilon_j n_{k+1}$, provided $\varepsilon_{j+1} = \varepsilon_j$.

Suppose $\varepsilon_{j+1} \neq \varepsilon_j$, so that $p_{j+1} = p_j - \varepsilon_j q$, with $2|p_j| < q$. It must also be that $|p_{j+1}| \leqq |p_j| + n_{k+1}$, and therefore $2|p_j| < q \leqq 2|p_j| + n_{k+1} < 3n_{k+1}$. Since $\|q\theta\| < \|n_k\theta\|$, and since $n_{k+1} < 2|p_j|$, we have from (42) as our possibilities for q only $2n_{k+1}$, $n_k + n_{k+1}$, and $n_k + 2n_{k+1}$. Also, $q\theta$ and $n_{k+1}\theta$ must have opposing senses. Therefore $\|q\theta\| \geqq \|(n_k + 2n_{k+1})\theta\| = \|n_k\theta\| - 2\|n_{k+1}\theta\|$. If k is sufficiently large, the latter expression is at least $\frac{1}{2}\|n_k\theta\|$, while, also for large k, $\|(p_{j+1} - p_j)\theta\| < \frac{1}{2}\|n_k\theta\|$. From the contradiction, (i″) of our lemma obtains.

We next prove (iii). If $(\alpha/2)n_{k+1} < n < \alpha n_{k+1}$, then $n\|n_k\theta\| \geqq \alpha/4$. On the other hand, if k is sufficiently large there is an n in this interval with $n\varepsilon_n < \alpha/32$, and therefore $n \in S(\theta, t)$ and $\varepsilon_n > \|p_j\theta - t\|$. It follows that $\|p_j\theta - t\| < \|n_k\theta\|/8$. In fact, also $\|p_{j-1}\theta - t\| < \|n_k\theta\|/8$, and therefore $\|(p_j - p_{j-1})\theta\| < \|n_k\theta\|/4$.

Let $p = p_j - \varepsilon_j n_{k+1}$. If $p_{j-1} \neq p$, then at least $\operatorname{sgn} p_{j-1} \neq \operatorname{sgn} p_j$. For otherwise $|p_{j-1} - p_j| < n_{k+1}$, an impossibility. Let $p_{j-1} = p_j - \varepsilon_j q$, $q < 2|p_j| < 2n_{k+1}$. Again $q\theta$ and $n_{k+1}\theta$ must have opposing senses because $p_{j-1}\theta$ cannot lie between $p_j\theta$ and $p\theta$. (Then $\|(p_{j-1} - p_j)\theta\| < \|n_{k+1}\theta\|$.) The possibilities for q are limited by (42), and we find $\|q\theta\| \geqq \|n_k\theta\| - \|n_{k+1}\theta\| > \frac{1}{2}\|n_k\theta\|$. This is a contradiction, and (iii) follows.

We will omit the proofs of (ii″) and (iv) because they are so similar to (i″) and (iii). The lemma is finished.

LEMMA 10. *Suppose* $0 < \alpha < 1$, $n_k < |p_j| < \alpha n_{k+1}$, *and* $n_l < |p_{j+1}| \leqq \alpha n_{l+1}$. *There exists an index* k_4 *such that if* $k \geqq k_4$, *then* $p_{j+1} = p_j + \varepsilon_j n_l$.

Proof. Using Lemma 2 we assume $\alpha < 1/2$. From this we obtain that $\|p_j\theta - t\| > \frac{1}{2}\|n_l\theta\|$ since otherwise $\|(p_{j+1} - p_j)\theta\| < \|n_l\theta\|$, implying the impossibility $|p_{j+1} - p_j| \geqq n_{l+1}$. Since $|p_j - \varepsilon_j n_l| < \max(|p_j|, n_l)$, it must be $(p_j + \varepsilon_j n_l)\theta$ that is closer than $p_j\theta$ to t. Also, we have for large enough k that $\varepsilon_{n_l} = \|p_j\theta - t\|$, and therefore, again for large k, $\|p_j\theta - t\| < \frac{1}{8}\|n_{l-1}\theta\|$. It follows that $\|(p_{j+1} - p_j)\theta\| < \frac{1}{4}\|n_{l-1}\theta\|$, and $|p_{j+1} - p_j| \geqq n_l$. If $\varepsilon_{j+1} = \varepsilon_j$, then $p_{j+1} = p_j + \varepsilon_j n_l$. If $\varepsilon_{j+1} \neq \varepsilon_j$, then $p_{j+1} = p_j - \varepsilon_j q$, where $q\theta$ has sense opposing that of $n_l\theta$, and $\max(2|p_j|, n_l) \leqq q \leqq |p_j| + n_l$. Using Lemma 2 we can assume $|p_j| \, \|n_k\theta\|$ and $|p_{j+1}| \, \|n_l\theta\|$ are as small as we please; in particular small enough that $|p_j| + n_l < \frac{1}{8}n_{l+1}$. In this range the values of $\|q\theta\|$ with

$q\theta$ opposing $n_l\theta$ in sense are by (42) bounded below by $\frac{3}{4}\|n_{l-1}\theta\|$. Since $q=|p_{j+1}-p_j|$, we have reached a contradiction, and the lemma obtains.

LEMMA 11. *There exists an index j_0 such that if $j \geqq j_0$, then p_{j+1} arises from p_j as follows:*

(1) $n_k<|p_j|<n_{k+1}/2$, $n_l<|p_{j+1}|<n_{l+1}/2$, *and* $p_{j+1}=p_j+\varepsilon_j n_l$.

(2) $n_k<|p_j|<n_{k+1}/2$, $n_{l+1}/2<|p_{j+1}|<n_{l+1}$. *If* $a_{l+2}=1$, *then* $p_{j+1}=p_j-\varepsilon_j n_{l+2}$. *If* $a_{l+2}>1$, *then* $p_{j+1}=p_j-\varepsilon_j n_{l+1}$.

(3) $n_{k+1}/2<|p_j|<n_{k+1}$, $n_l<|p_{j+1}|<n_{l+1}$. *If* $a_{k+2}=1$, *then* $l=k+2$ *and* $p_{j+1}=p_j+\varepsilon_j n_{k+2}$. *If* $a_{k+2}>1$, *then* $l=k+1$ *and* $p_{j+1}=p_j+\varepsilon_j n_{k+1}$.

Proof. Let k_0 be such that Lemmas 8, 9, and 10 are true for $\alpha=1/2$ and $k \geqq k_0$. Let j_0 be such that $n_{k_0}<|p_{j_0}|$. Then (1) is a consequence of Lemma 10, (2) is a consequence of (iii) and (iv) of Lemma 9, and (3) is a consequence of Lemmas 8 and 9. Notice that it never happens that $\frac{1}{2}n_{k+1} \leqq |p_j| \leqq n_{k+1}$ and $\frac{1}{2}n_{l+1} \leqq |p_{j+1}| \leqq n_{l+1}$.

Using Lemma 1 we write t as

$$t = \lim_{j\to\infty} p_j\theta$$
$$= p_{j_0}\theta + \sum_{j_0+1}^{\infty} \varepsilon_j n_{l_j}\theta.$$

The terms for which $n_{l_j}=n_l$, l fixed, carry the same signature $\varepsilon_l=\varepsilon_{l_j}$ by our construction, and we may group them, setting $m\theta=p_{j_0}\theta$ and $b_l=0$, $l<l_{j_0+1}$, as

$$t = m\theta + \sum_{l=1}^{\infty} \varepsilon_l b_l n_l \theta \qquad (b_l \geqq 0) \quad \text{if } b_k \neq 0.$$

Define $q_k=(m+\sum_{l=1}^{k} \varepsilon_l b_l n_l)$. For large k we have sgn $q_k=\varepsilon_k$. If $b_k \neq 0$, we know that $n_{k-1}<|q_{k-1}|<n_k/2$ for large k, and therefore if δ is preassigned, $n_{k-1}<|q_{k-1}|<\delta n_k$. Let $j \geqq k$ be the next index for which $b_l \neq 0$. Then also $n_j<|q_j|<\delta n_{j+1}$, and $|q_j|/b_j n_j = |1+q_{k-1}/b_j n_j|$, where $|q_{k-1}/b_j n_j|<\delta(n_k/n_j) \leqq \delta$. Thus, $\lim_{j\to\infty; b_j\neq 0} |q_j/b_j n_j|=1$. We also know that $\lim_{j\to\infty} q_j\|n_j\theta\|=0$, and therefore

$$\lim_{j\to\infty} b_j n_j \|n_j\theta\| = 0.$$

If we show b_j is even for large j, the second inclusion in (6) will have been established. Since if $b_j \neq 0$, $\varepsilon_j b_j$ is the unique integer for which $|\varepsilon_j b_j n_j - q_j| < \frac{1}{2}n_j$, large j, that b_j is even will follow from

LEMMA 12. *If $0<\alpha<1$, and if $\delta>0$, there exists an index k_5 such that if $k \geqq k_5$, if $\|p_j\theta - t\| = \min_{|p| \leqq \alpha n_{k+1}} \|p\theta - t\|$, and if $n_k<|p_j|<\alpha n_{k+1}$, then*

$$|p_j - dn_k| < 12\delta n_k \tag{45}$$

for some even integer d.

Proof. Using Lemma 2 with our α playing the role of β there and δ that of both α and δ, we may assume k so large that $n_k<|p_j|<\delta n_{k+1}$. Let $\varepsilon>0$ be such that if $\|n\theta\|<\varepsilon$, then $n\varepsilon_n<\delta$, and if also $n \in S(\theta, t)$, then $|\gamma_n|>1-\delta$. Assume $\delta<1/2$. If k

is sufficiently large, the interval $\delta n_{k+1} < n < \frac{1}{2} n_{k+1}$ contains an n with $\|n\theta\| < \varepsilon$. Again using Lemma 2, $n \in S(\theta, t)$ and certainly $\varepsilon_n = \|p_j\theta - t\|$. Since $n + |p_j| < n_{k+1}$, we see in (34) that $\|n_k\theta\|$ is the length of the shortest interval of constancy for $\psi^{(n)}$. Define $a \geqq 1$ to be that integer such that $an_k \leqq |p_j| < (a+1)n_k$. Clearly $a = b_k$ or $b_k - 1$ if k is large. We consider separately the cases a even and a odd.

CASE 1. a even. If $1 \leqq s \leqq n_k$, the a points $\varepsilon_j s\theta, \varepsilon_j(s+n_k)\theta, \ldots, \varepsilon_j(s+(a-1)n_k)\theta$ are consecutive discontinuities of $\psi^{(n)}$. If we pair them, say as $\{\varepsilon_j s\theta, \varepsilon_j(s+n_k)\theta\}$, $\{\varepsilon_j(s+2n_k)\theta, \varepsilon_j(s+3n_k)\theta\}, \ldots, \{\varepsilon_j(s+(a-2)n_k)\theta, \varepsilon_j(s+(a-1)n_k)\theta\}$, the corresponding $\frac{1}{2}a$ functions in (34) are each -1 on an interval of length $\|n_k\theta\|$. A similar situation occurs among the factors φ_{i-n}. It is only necessary to know that 0 cannot separate discontinuities which are $\|n_k\theta\|$ apart, and this is a consequence of the final remark of the preceding paragraph. If we delete the functions in question from (34), we obtain a new function, $\psi_0^{(n)}$, given by

$$\psi_0^{(n)} = \varphi_{\varepsilon_j(an_k+1)} \cdots \varphi_{p_j} \varphi_{\varepsilon_j(an_k+1)-n} \cdots \varphi_{p_j-n} \tag{46}$$

such that $\|\psi^{(n)} - \psi_0^{(n)}\|_1 < 2an_k\|n_k\theta\| \leqq 2|p_j|\ \|n_k\theta\| < 2\delta$. It follows that if

$$\delta_0^{(n)} = \int_X \psi_0^{(n)}(x)\nu(dx),$$

then $|\delta_0^{(n)}| > 1 - 3\delta$.

CASE 2. a odd. As before we pair off points $\varepsilon_j s\theta, \varepsilon_j(s+n_k)\theta, \ldots, \varepsilon_j(s+(a-2)n_k)\theta$, $1 \leqq s \leqq n_k$, if $a \geqq 3$, and also we pair off $\varepsilon_j s\theta, \varepsilon_j(s-n_k)\theta$, $an_k + 1 \leqq s \leqq \varepsilon_j p_j$. The same is done with the φ_{i-n} terms, and we obtain

$$\psi_1^{(n)} = \varphi_{p_j+\varepsilon_j(1-n_k)} \cdots \varphi_{\varepsilon_j an_k} \varphi_{p_j+\varepsilon_j(1-n_k)-n} \cdots \varphi_{\varepsilon_j an_k - n} \tag{47}$$

and

$$\delta_1^{(n)} = \int_X \psi_1^{(n)}(x)\nu(dx)$$

with $|\delta_1^{(n)}| > 1 - 3\delta$.

The number of factors in $\psi_i^{(n)}$, $i = 0, 1$, is computed to be $2|p_j - \varepsilon_j(a+i)n_k|$. Also, $a+i$ is even. Were $\|n_{k-1}\theta\|$ the shortest distance between discontinuities, we would obviously have

$$|p_j - \varepsilon_j(a+i)n_k|\ \|n_{k-1}\theta\| < \tfrac{3}{2}\delta$$

or

$$|p_j - \varepsilon_j(a+i)n_k| < 3\delta n_k$$

implying (45). However $\|n_{k-1}\theta\|$ is not to be expected as the shortest length. The point of the argument is that smaller multiples of $\|n_{k-1}\theta\|$, e.g. $\frac{1}{4}\|n_{k-1}\theta\|$ will bound the shortest length from below, and so (45) will follow.

To compute the shortest lengths in (46), (47) we first note that anything shorter than $\|n_{k-1}\theta\|$ must come from crossed terms $\varphi_a\varphi_{b-n}$. The possibilities for $a-b+n$ are found in the intervals

$$M_i = \{m \mid n-(-1)^i(\varepsilon_j p_j-(a+i)n_k)+1 \leqq m \leqq n+(-1)^i(\varepsilon_j p_j-(a+i)n_k)-1\} \qquad (i = 0, 1).$$

The length of M_i is less than $2n_k$, and so in M_i there are at most two numbers of the form un_k and two of the form $n_{k-1}+vn_k$.

Our only requirements on n have been that $\delta n_{k+1}<n<\frac{1}{2}n_{k+1}$, and $\|n\theta\|<\varepsilon$, but by enlarging k if necessary, we can assume $\frac{1}{4}n_{k+1}<n<\frac{1}{3}n_{k+1}$. In fact, since $|p_j|\ \|n_k\theta\| < \delta$, we have $n_k\|n_k\theta\|<\delta$, or $n_k<2\delta n_{k+1}$. Letting δ be smaller if necessary, we can assume each $m \in M_i$ satisfies $\frac{1}{4}n_{k+1}<m<\frac{1}{3}n_{k+1}$. From $un_k>\frac{1}{4}n_{k+1}$ we obtain

$$\|un_k\theta\| > \tfrac{1}{4}\frac{n_{k+1}}{n_k}\|n_k\theta\| > \frac{1}{8n_k}.$$

From $vn_k<\frac{1}{3}n_{k+1}$ we obtain

$$\|(n_{k-1}+vn_k)\theta\| = \|n_{k-1}\theta\|-v\|n_k\theta\| > \frac{1}{2n_k}-\frac{1}{3n_k} = \frac{1}{6n_k}.$$

Collecting results we have

$$|p_j-\varepsilon_j(a+i)n_k| < 8n_k\cdot\tfrac{3}{2}\delta$$

which is (45). The lemma is proved since $d=\varepsilon_j(a+i)$ is even, $i=0, 1$.

As remarked earlier the right hand inclusion in (6) is now proved. We will next prove the left, but first we observe that $K(\theta)$ is a group.

LEMMA 13. *$K(\theta)$ is a subgroup of X.*

Proof. Suppose $t \in K(\theta)$, and let $f=\varphi_t$. If $n<0$ $f^{(n)}$ is defined by $f^{(n)}(x) =f(x+n\theta)f(x+(n+1)\theta)\cdots f(x-\theta)$, and the statement (4) remains true when negative values are used ($\alpha_n=\alpha_{-n}$!). Therefore, if $g=\varphi_{-t}$, then $g(x)=-f(-x)$, and if $n>0$, then $(-1)^n g^{(n)}(x)=f(-x)f(-x-n\theta)f^{(-n)}(-x)$. As $n\theta\to 0$, $f(-x-n\theta) \to f(-x)$, and therefore (4) holds for g. Thus $-t \in K(\theta)$. If $t_1, t_2 \in K(\theta)$, then we have $\varphi_{t_1+t_2}(x)=\varphi_{t_1}(x-t_2)\varphi_{t_2}(x)\varphi_{t_1}(-t_2)$, at least when $t_1, t_2, t_1+t_2\neq 0, 1$. From this (4) follows for t_1+t_2, and the lemma is proved.

It is convenient to introduce an ambiguous notation. We will generally be dealing with $t=\langle m; \varepsilon_1 b_1, \varepsilon_2 b_2, \ldots\rangle_\theta$, where an infinite number of the b_j's are not 0. When this is so, we eliminate those b_j with $b_j=0$, renumber, and write t as before. Possibly $m=0$, but now $b_l\neq 0$ for all l. In fact by incorporating $\varepsilon_1 b_1$ into m we can assume $m\neq 0$, where convenient. In our new notation $n_1, n_2, \ldots$ is not the entire sequence of denominators. This will be of no consequence.

LEMMA 14. *Suppose $t=\langle m; \varepsilon_1 b_1, \varepsilon_2 b_2, \ldots\rangle_\theta$, where $b_j>0$ all j and*

$$\lim_{j\to\infty} b_j n_j\|n_j\theta\| = 0.$$

(We do not assume b_j even.) Let $q_k=m+\sum_{j=1}^k \varepsilon_j b_j n_j$. If k is sufficiently large, and if

$|q| \leqq |q_{k+1}|$ satisfies $\|q\theta - t\| \leqq \|q_k\theta - t\|$, then $q = q_k + \varepsilon_{k+1} c n_{k+1}$, where $0 \leqq c \leqq b_{k+1}$. Also, if $1 \leqq c \leqq b_{k+1}$, and if $q = q_k + \varepsilon_{k+1} c n_{k+1}$, then $\|q\theta - t\| < \|(q - \varepsilon_{k+1} n_{k+1})\theta - t\|$. Finally, $t \neq 0$.

Proof. If $0 < \delta < 1/2$, we have for large j, say $j \geqq j_0$, the inequalities

$$b_j \|n_j\theta\| < \delta/n_j, \qquad b_j/n_{j+1} < 2\delta/n_j, \quad \text{and} \quad 1/n_{j+1} < 2\delta/n_j.$$

Let $r = m + \sum_{j=1}^{j_0} b_j n_j$. We have for $k \geqq j_0$,

$$\begin{aligned} |q_k| &\leqq r + \sum_{j_0+1}^{k} b_j n_j \\ &< r + 2\delta \sum_{j_0+1}^{k} n_{j+1} \\ &< 2\delta n_{k+1} + r + 2\delta \sum_{l=0}^{k-j_0-1} (2\delta)^l n_k \\ &< \left\{ 2\delta + \frac{(2\delta)^2}{1-2\delta} \right\} n_{k+1} + r \end{aligned}$$

meaning $n_{k+1} \gg |q_k|$, large k.

Using the triangle inequality,

$$\begin{aligned} \|q_k\theta - t\| &\leqq \sum_{j=k+1}^{\infty} b_j \|n_j\theta\| \\ &< \delta \sum_{j=k+1}^{\infty} 1/n_j \\ &< \delta/n_{k+1} \sum_{j=0}^{\infty} (2\delta)^j \\ &= \frac{\delta}{1-2\delta} 1/n_{k+1}. \end{aligned} \tag{48}$$

If $|q| \leqq |q_k|$, then by the preceding paragraph $|q - q_k| < n_{k+1}$ for large k. Using (48) for $\delta/(1-2\delta) < 1/4$, we conclude for large k that if $\|q\theta - t\| \leqq \|q_k\theta - t\|$, and if $|q| \leqq |q_k|$, then $q = q_k$.

Suppose $|q| \leqq |q_k|$. The numbers $(q_k + \varepsilon_{k+1} c n_{k+1})\theta$, $0 \leqq c \leqq b_{k+1}$, are $\|n_{k+1}\theta\|$ dense in the short interval joining $q_k\theta$ and $q_{k+1}\theta$. Using (48) for q_{k+1}, we find for large k that $\|q_{k+1}\theta - t\| < \frac{1}{2}\|n_{k+1}\theta\|$, and from this inequality the second to last statement of the lemma is obvious. We now prove the main statement.

If $q\theta$ is closer to t than is $q_k\theta$, then either $q\theta$ lies in the short interval between $q_k\theta$ and $q_{k+1}\theta$, or else it lies in the longer interval, but nearer $q_{k+1}\theta$. In the former case $\|(q - (q_k + \varepsilon_{k+1} c n_{k+1}))\theta\| < \|n_{k+1}\theta\|$ for some c, $0 \leqq c \leqq b_{k+1}$, and if $|q| \leqq |q_{k+1}|$, then $q = q_k + \varepsilon_{k+1} c n_{k+1}$. In the latter case $(q - q_k)\theta$ has the same sense as $(q_{k+1} - q_k)\theta = \varepsilon_{k+1} b_{k+1} n_{k+1}\theta$, and therefore arguing as we have before $q = q_k - \varepsilon_{k+1}(n_k + c n_{k+1})$.

Since $n_{k+1}>|q_k|$, large k, $0\leqq c\leqq b_{k+1}$. Now $\|(n_k+cn_{k+1})\theta\|\geqq\frac{3}{4}\|n_k\theta\|$, large k, but also $\|(q-q_k)\theta\|<2\|q_k\theta-t\|<\frac{1}{4}\|n_k\theta\|$, large k, by (48). This is a contradiction, and q has the form asserted.

If $t=0$, the inequality (48) for $\delta/(1-2\delta)<1/2$ implies $q_k=\pm c_k n_{k+1}$ for certain constants c_k [5, Theorem 19]. Since $|q_k|<n_{k+1}$, large k, it must be that $c_k=0$ and $q_k=0$, all k. This is not so, and the lemma is proved.

REMARK. Suppose $t=m_1\theta+\sum_{j=1}^{\infty}\varepsilon_j b_j n_j\theta$ and $t=m_2\theta+\sum_{j=1}^{\infty}\delta_j c_j n_j\theta$, where now $n_1, n_2, \ldots$ is the full sequence of denominators. Suppose also $\lim_{j\to\infty} b_j n_j\|n_j\theta\| =0=\lim_{j\to\infty} c_j n_j\|n_j\theta\|$. By our lemma we have $\delta_j=\varepsilon_j$ and $c_j=b_j$ for large j.

Our lemma will be applied later. Recall from (17) that

$$f^{(m+n)}(x) = f^{(m)}(x+n\theta)f^{(n)}(x). \tag{49}$$

Clearly, if two among α_{m+n}^2, α_m^2, α_n^2 are near 1, then so will the third be. We will establish (4) for $t\in K_1(\theta)$ by proving first

$$\lim_{\substack{\|n\theta\|\to 0\\ |q_k|<n<n_{k+1}-|q_k|}} \alpha_n^2 = 1. \tag{50}$$

A general value of n will not fall in an interval as in (50). However for all large l, $n+(b_l+1)n_l$ does, because $|q_l|<(b_l+1)n_l$ for large l. Moreover $m=(b_l+1)n_l$ has $\|m\theta\|$ small for large l, so that if $\|n\theta\|$ is small, (50) and (49) will prove $\alpha_n^2\to 1$.

Suppose $|q_k|<n<n_{k+1}-|q_k|$, in which case $\varepsilon_n=\|q_k\theta-t\|$. If k is large and $\|n\theta\|$ small, the error term in (35) is small. It is therefore sufficient to prove (50) for γ_n^2 instead of α_n^2.

We introduce notation, the significance of which will presently become clear. Define ξ_j, η_j, $j=2, 3, \ldots$, by $\xi_j=(1+\varepsilon_j\varepsilon_{j-1})\varepsilon_j/2$ and $\eta_j=\varepsilon_j-\xi_j$. Then, if $k\geqq l>2$, we define $\xi_{kl}=(1+\varepsilon_k\varepsilon_{l-1})\varepsilon_{l-1}/2$, $\eta_{kl}=\varepsilon_k-\xi_{kl}$, and $\mu_k, \ldots, \mu_{l-1}$ and $\sigma_k, \ldots, \sigma_{l-1}$ by

$$\begin{aligned}
\mu_k &= \varepsilon_k & \sigma_k &= \varepsilon_k b_k n_k\\
\mu_{k-1} &= \sigma_k+\xi_k & \sigma_{k-1} &= \sigma_k+\varepsilon_{k-1}b_{k-1}n_{k-1}+\eta_k\\
\mu_{k-2} &= \sigma_{k-1}+\xi_{k-1} & \sigma_{k-2} &= \sigma_{k-1}+\varepsilon_{k-2}b_{k-2}n_{k-2}+\eta_{k-1}\\
&\;\;\vdots\\
\mu_l &= \sigma_{l+1}+\xi_{l+1} & \sigma_l &= \sigma_{l+1}+\varepsilon_l b_l n_l+\eta_{l+1}\\
\mu_{l-1} &= \sigma_l+\xi_{kl} & \sigma_{l-1} &= q_k+\eta_l.
\end{aligned}$$

Next, define

$$\begin{aligned}
\Psi_k &= \varphi_{\mu_k}\cdots\varphi_{\sigma_k}\\
\Psi_{k-1} &= \varphi_{\mu_{k-1}}\cdots\varphi_{\sigma_{k-1}}\\
&\;\;\vdots\\
\Psi_l &= \varphi_{\mu_l}\cdots\varphi_{\sigma_l}\\
\Psi_0 &= \varphi_{\mu_{l-1}}\cdots\varphi_{\sigma_{l-1}}.
\end{aligned} \tag{51}$$

We notice that if $\varepsilon_j = \operatorname{sgn} q_j$, $j \geqq l-1$, then $\varphi_{\varepsilon_k} \cdots \varphi_{q_k} = \Psi_k \Psi_{k-1} \cdots \Psi_l \Psi_0$, and also Ψ_j contains $|\sigma_j - \mu_j| = b_j n_j$ terms. Finally, $\sigma_l = q_k - q_{l-1} + \eta_{l+1} + \eta_{l+2} + \cdots + \eta_k$. If $\eta_j \neq 0$ for some j, and if $\eta_i = 0$, $j_0 < i < j$, $\eta_{j_0} \neq 0$, then $\eta_{j_0} = -\eta_j$. Therefore, Ψ_0 has $|q_k - \sigma_l| \leqq |q_{l-1}| + 1$ terms.

If $|q_k| < n < n_{k+1} - |q_k|$, we let $\Psi_{j;n}$, $j=0$ or $l \leqq j \leqq n$ have a similar definition with φ_{a-n} replacing φ_a for each a. l is defined by $\|n_l\theta\| < \|n\theta\| < \|n_{l-1}\theta\|$, and $l > 2$ if $\|n\theta\|$ is small. Define

$$\rho_n = \left| \int_X \Psi_0(x) \Psi_{0;n}(x) \nu(dx) \right|.$$

We pair terms $\varphi_a \varphi_{a-n}$ with integrals $\pm(1-2\|n\theta\|)$ and conclude that

$$\rho_n \geqq (1-2(|q_{l-1}|+1)\|n\theta\|) \geqq (1-2(|q_{l-1}|+1)\|n_{l-1}\theta\|).$$

Therefore, as $\|n\theta\| \to 0$, with l defined as above, $\rho_n \to 1$.

Relation (50) is reduced to proving

$$\lim_{\substack{\|n\theta\| \to 0 \\ |q_k| < n < n_{k+1} - |q_k|}} \|\psi^{(n)} \pm \Psi_0 \Psi_{0;n}\|_1 = 0 \tag{52}$$

where the $\pm$ sign will vary with n, and l is as above. Assume $t \in K_1(\theta)$. Using the fact that Ψ_j, $l \leqq j \leqq k$, has $b_j n_j$ factors and b_j is even, we pair terms as in the proof of Lemma 13 to get

$$\int_X \Psi_j(x) \nu(dx) \geqq (1 - b_j n_j \|n_j \theta\|),$$

$$\left| \int_X \Psi_{j;n}(x) \nu(dx) \right| \geqq (1 - b_j n_j \|n_j \theta\|).$$

In (52), therefore, the error is less than $\sum_{j=l}^{k} 2 b_j n_j \|n_j \theta\|$. Since $t \in K_1(\theta)$, this tends to 0 as $l \to \infty$ or, what is the same, as $\|n\theta\| \to 0$. Thus (4) obtains for $t \in K_1(\theta)$, and Theorem 2 is proved.

We will prove Theorem 3 twice. For the first proof, $K_1(2\theta) \subseteq K(2\theta)$, and therefore by the remark following Proposition 6, one of the equations

$$\begin{aligned} g(x)g(x+\theta) &= \varphi_t(x) \\ &= -\varphi_t(x) \end{aligned} \tag{53}$$

has a measurable ± 1-valued solution. The proof is complete, however it does not indicate which of the equations (53) has a solution. Our second proof will be given presently.

If one of the equations (53) does have a solution, then only one does. We use $w(t) = \pm 1$ to denote which and g_t to denote the measurable ± 1-valued solution. Thus

$$g_t(x) g_t(x+\theta) = w(t) \varphi_t(x) \qquad (t \in K_1(2\theta)).$$

If $t \neq m\theta$, then $\mathscr{S}(w(t)\varphi_t, \theta, \{0, t\})$ is minimal by Corollary 1. If $\int_X g_t(x)\nu(dx) \neq 0$, then by Proposition 3 (with $f = w(t)\varphi_t$) (14) fails to exist for an uncountable number of x. Letting $I=\{s \mid \varphi_t(s)=-w(t)\}$ then $\mu_\theta(I-x)$ fails to exist when (14) fails to exist. To complete the necessity statement of Theorem 1 it is enough to find, for θ with unbounded partial quotients, a $t\in K_1(2\theta)$, $t\neq m\theta$, such that $\int_X g_t(x)\nu(dx)\neq 0$.

If $t_1, t_2\in K_1(2\theta)$, $t_1\neq t_2$, $t_1, t_2\neq 0$ (or 1), then

$$g_{t_1}g_{t_2}R_\theta(g_{t_1}g_{t_2}) = w(t_1)w(t_2)\varphi_{t_1}\varphi_{t_2}$$

and also

$$\begin{aligned}R_{t_2}(g_{t_1}g_{t_2})R_{t_2+\theta}g_{t_1}g_{t_2} &= w(t_1)w(t_2)R_{t_2}(\varphi_{t_1}\varphi_{t_2})\\ &= \varphi_{t_1-t_2}(-t_2)w(t_1)w(t_2)\varphi_{t_1-t_2}.\end{aligned} \tag{54}$$

In the last equation we have used

$$\varphi_{a+b} = \varphi_b(-a)\varphi_a R_{-a}\varphi_b \qquad (a, b, a+b \neq 0, 1) \tag{55}$$

which is easily checked.

If $t_1, t_2\in K_1(2\theta)$, $t_1, t_2, t_1-t_2\neq 0$, then $t_1-t_2\in K_1(2\theta)$, and

$$\begin{aligned}g_{t_1-t_2} &= R_{t_2}(g_{t_1}g_{t_2}),\\ w(t_1-t_2) &= \varphi_{t_1-t_2}(-t_2)w(t_1)w(t_2).\end{aligned}$$

Clearly, $\int_X g_{t_1-t_2}(x)\nu(dx)=0$ if and only if $g_{t_1}\perp g_{t_2}$ in $\mathscr{L}^2(X,\nu)$.

When θ has unbounded partial quotients, $K_1(2\theta)$ is uncountable, and we can make choices $t\in K_1(2\theta)$ from each of an uncountable number of distinct cosets of $\{m\theta\}$. For each pair of choices $t_1-t_2\neq m\theta$, and because $\mathscr{L}^2(X,\nu)$ is separable, it cannot be that $g_{t_1}\perp g_{t_2}$ for all such pairs. As remarked above, the necessity statement of Theorem 1 is established. We have left to establish (14) for $t=m\theta$ (which does not require bounded partial quotients) and this will appear below.

To (55) we add the consequence

$$\varphi_{y_1+\cdots+y_m} = \varphi_{y_1}(x)\varphi_{y_2}(x-y_1)\varphi_{y_2}(-y_1)\cdots\varphi_{y_m}\left(x-\sum_1^{m-1}y_j\right)\varphi_{y_m}\left(-\sum_1^{m-1}y_j\right) \tag{56}$$

valid so long as $y_j\neq 0$ for all j and $\sum_{j=1}^i y_j\neq 0$ all i.

If $\varepsilon=1$ or -1, we use (55) with $a=b=\varepsilon$ to get

$$\varphi_{2\varepsilon}(x) = \varphi_\varepsilon(x)\varphi_\varepsilon(x-\varepsilon\theta)\varphi_\varepsilon(-\varepsilon\theta). \tag{57}$$

More generally, for $n>0$ we have from (56)

$$\varphi_{2n\varepsilon}(x) = \varphi_{2\varepsilon}(x)\varphi_{2\varepsilon}(x-2\varepsilon\theta)\varphi_{2\varepsilon}(-2\varepsilon\theta)\cdots\varphi_{2\varepsilon}(x-2(n-1)\varepsilon\theta)\varphi_{2\varepsilon}(-2(n-1)\varepsilon\theta)$$

and from (57)

$$\begin{aligned}\varphi_{2n\varepsilon}(x) &= \prod_{j=0}^{n-1}\varphi_\varepsilon(x-2\varepsilon j\theta)\varphi_\varepsilon(x-(2j+1)\varepsilon\theta)\varphi_\varepsilon(-\varepsilon\theta)\\ &\quad\cdot\prod_{j=1}^{n-1}\varphi_\varepsilon(-2\varepsilon j\theta)\varphi_\varepsilon(-(2j+1)\varepsilon\theta)\varphi_\varepsilon(-\varepsilon\theta).\end{aligned} \tag{58}$$

Define

$$w(2n\varepsilon) = \prod_{j=1}^{2n-1} \varphi_\varepsilon(-\varepsilon j\theta)$$

and

$$H_{2n\varepsilon}(x) = \prod_{j=0}^{n-1} \varphi_\varepsilon(x-2\varepsilon j\theta).$$

Using (58)

$$H_{2n\varepsilon}(x)H_{2n\varepsilon}(x-\varepsilon\theta) = w(2n\varepsilon)\varphi_{2n\varepsilon}(x). \tag{59}$$

Let $g_{2n\varepsilon}(x) = H_{2n\varepsilon}(x-(1+\varepsilon)\theta/2)$. Then by (59)

$$g_{2n\varepsilon}(x)g_{2n\varepsilon}(x+\theta) = w(2n\varepsilon\theta)\varphi_{2n\varepsilon}(x)$$

which is (53).

Suppose $t = m\theta$ and that (53) has a solution for $w(t) = 1$ or -1. If $m = 2n\varepsilon$, we have exhibited this solution, and it is Riemann-integrable, indeed a step function. Since $f^{(n)}(x) = g_t(x)g_t(x+n\theta)$, where $f = w(t)\varphi_t$, (14) exists by the Kronecker-Weyl theorem. If $m = 2n\varepsilon+1$, we will show (53) has no solution. For if it does, then by (59) and (54) so does

$$g(x)g(x+\theta) = w(\theta)\varphi_1(x).$$

Let $h(y, x)$ be as in the proof of Proposition 4. We would have

$$\alpha(y) = \int_X h(y, x)\nu(dx)$$

continuous on X with $\alpha(n\theta) = \alpha_n$. It has been observed that $|\alpha_n| = |1-2\|n\theta\||$, and therefore by continuity $|\alpha(y)| = (1-2\|y\|)$. $\alpha(y) \neq 0$ for $y \neq 1/2$. It follows that $\alpha(y) \geqq 0$ or $\alpha(y) \leqq 0$ for all y. Since $\alpha(0) = 1$, $\alpha(y) \geqq 0$.

Now $w(\theta)^n\varphi_1^{(n)}(x) = w(\theta)^n\varphi_1^{(n)}(0)\varphi_1(x)\varphi_{1-n}(x)$, and

$$\alpha_n = w(\theta)^n\varphi_1^{(n)}(0)(\pm 1)(1-2\|n\theta\|)$$

where ± 1 is determined by whether the Euclidean distance from θ to $(1-n)\theta$ in X is less than or greater than $1/2$. For concreteness assume $0 < \theta < 1/2$. Then

$$\alpha_n = w(\theta)^n\varphi_1^{(n)}(0)\varphi_{1/2+\theta}((1-n)\theta)(1-2\|n\theta\|).$$

Since $\alpha_n > 0$ for all n,

$$w(\theta)^n\varphi_1^{(n)}(0) = \varphi_{1/2+\theta}((1-n)\theta).$$

Therefore, if $g(x)g(x+\theta) = w(\theta)\varphi_1(x)$, then

$$g(x)g(x+n\theta) = \varphi_{1/2+\theta}((1-n)\theta)\varphi_1(x)\varphi_{1-n}(x)$$

and it follows for $y \neq \theta$, $1/2$ that for almost all x

$$g(x)g(x+y) = \varphi_{1/2+\theta}((\theta-y))\varphi_1(x)\varphi_{\theta-y}(x).$$

Letting $x \neq 0$ be such that equality holds for almost all y, we see that $g(x+y)$ is equivalent to a function with discontinuities only at $y=1/2$ and $y=\theta-x$. (There is an apparent discontinuity at $y=\theta$, but it appears in the first and third factors, thus being cancelled out.) Thus g has only two discontinuities, but the position of one varies with x, a contradiction. We conclude that $\mathscr{S}(\pm\varphi_t, \theta\{0, t\})$ is strictly ergodic for $t=2n\varepsilon+1$. Theorem 1 is proved.

REMARK. Let $0<\theta<1$, and consider the numbers $k\theta$, $k \geqq 1$, not reduced modulo 1. Considering φ_1 to be periodic with period 1, we note that $\varphi_1(k\theta)=1$ precisely when $[(k-1)\theta] \neq [k\theta]$. Thus, $\varphi_1^{(n)}(0)=\varphi_1(0)\varphi_1(\theta)\cdots\varphi_1((n-1)\theta)=(-1)^{n-1}(-1)^{[(n-1)\theta]}$, and $(-\varphi_1)^{(n)}(0)=-(-1)^{[(n-1)]\theta}$. We conclude from (21) that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (-1)^{[k\theta]} = 0$$

which is a special case of a result of Niven [8].

We proceed with the second proof of Theorem 3. Suppose $t \in K_1(2\theta)$, from which it follows, using our first proof of Theorem 3 and also Theorem 2, that $t \in K_0(\theta)$. In terms of θ, $t=\langle m; \varepsilon_1 b_1, \ldots\rangle_\theta$, and for large j, $\|n_j\theta\| < 1/2n_j$. Let m_j be such that $(n_j, m_j)=1$ and

$$|n_j\theta - m_j| = \|n_j\theta\|.$$

CASE 1. $n_j = 2l_j$ even. Here

$$|l_j(2\theta)-m_j| < \frac{1}{2n_j} < \frac{1}{2l_j}$$

for large j, and $(l_j, m_j)=1$. Therefore [5, Theorem 19] l_j is a denominator for 2θ.

CASE 2. n_j odd. Here $(n_j, 2m_j)=1$, and

$$|n_j(2\theta)-2m_j| < \frac{1}{2n_j},$$

large j (since $\|n_j\theta\| < 1/4n_j$, large j). Here n_j is a denominator for 2θ.

Since $t \in K_1(2\theta)$,

$$t = 2m_1\theta + \sum_{j=1}^{\infty} \delta_j c_j N_j(\dot{2}\theta) \qquad (c_j \text{ even}).$$

From Cases 1 and 2 we also have

$$t = m_2\theta + \sum_{\text{Case 1}} \varepsilon_j b_j l_j(2\theta) + \sum_{\text{Case 2}} \varepsilon_j \frac{b_j}{2} n_j(2\theta).$$

Using the remark following Lemma 14 these representations are eventually the same, and in particular b_j is divisible by 4 in Case 2, large j.

We now assume

$$t = 2m\theta + \sum_{j=1}^{\infty} \varepsilon b_j n_j \theta$$

where either n_j is a denominator for 2θ and $4|b_j$ or $n_j=2l_j$ and l_j is a denominator for 2θ.

As before, set $q_l=2m+\sum_{j=1}^{l}\varepsilon_j b_j n_j$, and also define $A_0=2m\theta$, $A_l=q_l\theta$, $l>0$. We may assume the representation of t is such that no partial sums are 0 (by incorporating more terms into the $2m\theta$ part). Therefore, by (56) for an infinite sum

$$\varphi_t(x) = \varphi_{2m}(x)\prod_{j=1}^{\infty}\varphi_{\varepsilon_j b_j n_j}(x-A_{j-1})\varphi_{\varepsilon_j b_j n_j}(-A_{j-1}). \tag{60}$$

Let $m=n\varepsilon$, $n>0$. Using (59) in (60) we get

$$\begin{aligned}\varphi_t(x) &= H_{2n\varepsilon}(x)H_{2n\varepsilon}(x-\varepsilon\theta)w(2n\varepsilon\theta)\\ &\quad\cdot\prod_{j=1}^{\infty}H_{\varepsilon_j b_j n_j}(x-A_{j-1})\cdot H_{\varepsilon_j b_j n_j}(x-A_{j-1}-\varepsilon_j\theta)\\ &\quad\cdot H_{\varepsilon_j b_j n_j}(-A_{j-1})H_{\varepsilon_j b_j n_j}(-A_{j-1}-\varepsilon_j\theta).\end{aligned} \tag{61}$$

We have also used $w(\varepsilon_j b_j n_j\theta)^2=1$. Formally, define

$$g_t(x) = g_{2n\varepsilon}(x)\prod_{j=1}^{\infty}g_{\varepsilon_j b_j n_j}(x-A_{j-1}) \tag{62}$$

and let

$$w(t) = w(2n\varepsilon)\prod_{j=1}^{\infty}H_{\varepsilon_j b_j n_j}(-A_{j-1})H_{\varepsilon_j b_j n_j}(-A_{j-1}-\varepsilon_j\theta). \tag{63}$$

Then by (61) we have

$$g_t(x)g_t(x+\theta) = w(t)\varphi_t(x). \tag{64}$$

To justify (64) we must prove (63) is a convergent product and that (62) is convergent in $L^1(X,\nu)$.

Let $b_j=2c_j$, $j=1,2,\ldots$. By the definition

$$H_{2\varepsilon_l c_l n_l}(x-A_{l-1}) = \prod_{j=0}^{c_l n_l-1}\varphi_{\varepsilon_l}(x-A_{l-1}-2\varepsilon_l j\theta) \tag{65}$$

and

$$H_{2\varepsilon_l c_l n_l}(-A_{l-1}) = \prod_{j=0}^{c_l n_l-1}\varphi_{\varepsilon_l}(-A_{l-1}-2\varepsilon_l j\theta). \tag{66}$$

LEMMA 15. *If $t\in K_1(2\theta)$, then for large l*

$$\begin{aligned}\int_X H_{2\varepsilon_l c_l n_l}(x-A_{l-1})\nu(dx) &= \int_X H_{2\varepsilon_l c_l n_l}(x)\nu(dx)\\ &\geqq (1-2c_l n_l\|n_l\theta\|).\end{aligned} \tag{67}$$

Proof. For large l either n_l is a denominator for 2θ and c_l is even, or else $n_l=2k_l$ and k_l is a denominator for 2θ. We treat the first case only. The second is the same if one replaces c_l by $2c_l$ and n_l by k_l.

Since c_l is even we may pair factors in (65) in the usual way. That is, if $0\leqq j \leqq n_l-1$, the terms are paired as $j, j+n_l$, and $j+2n_l, j+3n_l$, and ..., and $j+(c_l-2)n_l$, $j+(c_l-1)n_l$. The corresponding products of pairs will be -1 on intervals of length $2\|n_l\theta\|$, and there being $c_ln_l/2$ such pairs, the lemma follows.

Equation (67) implies for $t\in K_1(2\theta)$ that the product (62) is convergent in the $\mathscr{L}^1(X,\nu)$ norm. We next check that (63) is convergent. We continue to suppose c_l is even.

In (66) we use the same pairing as in our lemma. In each pair the product $\varphi_{\varepsilon_l}(-A_{l-1}-2\varepsilon_l(j+cn_l)\theta)\varphi_{\varepsilon_l}(-A_{l-1}-2\varepsilon_l(j+(c+1)n_l)\theta)$ is 1 unless either 0 or $\varepsilon_l\theta$ is between the two points. Should this occur, then for a choice of $d=c$ or $c+1$ and $s=0$ or ε_l, we would have for $q=q_{l-1}+2\varepsilon_l(j+dn_l)+s$ that

$$\|q\theta\| \leqq \|n_l\theta\|.$$

Now $|q|<n_{l+1}$ for large l, and therefore either $\|q\theta\|=\|n_l\theta\|$, $q=\pm n_l$, or $q=0$. In the first case $s\theta$ is midway between the two points, and therefore if $d'=c+1$ or c as $d=c$ or $c+1$, $q'=q_{l-1}+2\varepsilon_l(j+d'n_l)+s$, we have $\|q'\theta\|=\|n_l\theta\|$, $q'=\pm n_l$. This is impossible because at least one of d, d' is positive, $j\geqq 0$, and $|q_{l-1}|<n_l$, large l. We conclude that $q=0$. Since q_{l-1} is even, $s=0$. Also of course $d=0$. We conclude that for large l the product (66) can fail to have the value 1 only when $\varepsilon_l=-\varepsilon_{l-1}$, and when this is so, the product will have value $\varphi_{\varepsilon_l}(-2n_l\varepsilon_l\theta)$.

Consider next $H_{2\varepsilon_lc_ln_l}(-A_{l-1}-\varepsilon_l\theta)$. If $\varepsilon_{l-1}=\varepsilon_l$, then as above the value is 1. If $\varepsilon_{l-1}=-\varepsilon_l$, we let j be such that the number q defined above is 0. One sees easily that only the pair

$$\begin{aligned}\varphi_{\varepsilon_l}(-A_{l-1}-2\varepsilon_l(j-1)\theta-\varepsilon_l\theta)&\varphi_{\varepsilon_l}(-A_{l-1}-2\varepsilon_l(j-1+n_l)\theta-\varepsilon_l\theta)\\ &= \varphi_{\varepsilon_l}(\varepsilon_l\theta)\varphi_{\varepsilon_l}(\varepsilon_l(1-2n_l)\theta)\\ &= -\varphi_{\varepsilon_l}(\varepsilon_l(1-2n_l)\theta)\end{aligned}$$

can have product -1 in the usual pairing. Thus, we have for large l that

$$\begin{aligned}H_{2\varepsilon_lc_ln_l}(-A_{l-1})H_{2\varepsilon_lc_ln_l}(-A_{l-1}-\varepsilon_l\theta) &= 1, \quad \varepsilon_l=\varepsilon_{l-1},\\ &= -\varphi_{\varepsilon_l}((-2\varepsilon_ln_l)\theta)\varphi_{\varepsilon_l}((1-2n_l)\theta), \quad \varepsilon_l=-\varepsilon_{l-1}.\end{aligned}$$

Since $\varphi_{\varepsilon_l}(-2\varepsilon_ln_l\theta)=-\varphi_{\varepsilon_l}(\varepsilon_l(1-2n_l)\theta)$, large l, the second possibility is also 1. We conclude that (63) is convergent, and our second proof of Theorem 3 is complete.

We conclude by raising the following questions:

(1) Does (64) imply $t\in K(2\theta)$?

(2) What is the precise nature of $K(\theta)$? (E.g., does $K(\theta)=K_1(\theta)$?)

(3) Can our techniques be used for a Kronecker-Weyl theorem mod n? (We believe so.)

(4) Suppose (64) is true. Then 1 will be a cyclic vector for $U(f, \theta)$ ($f=\varphi_t$) precisely when g_t is cyclic for R_θ; that is, precisely when g_t has all Fourier coefficients nonzero. When does this happen? Are there conditions on θ, t guaranteeing that 1 be cyclic?

(5) If t is rational, one can show $t \notin K_0(\theta)$ for any θ(7). Thus (1) exists for all θ and all intervals with rational endpoints. Are there any other numbers t with this property?

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(7) Apply Lemma 14 to qt where $t \equiv p/q$ (mod 1).