TOPOLOGICAL CONJUGACY OF AFFINE TRANSFORMATIONS OF COMPACT ABELIAN GROUPS

BY

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0. Introduction. We consider the following problem. If \( X \) and \( Y \) are compact connected metric abelian groups, \( T = a + A \) an affine transformation of \( X \) and \( S = b + B \) an affine transformation of \( Y \), what are necessary and sufficient conditions for every continuous mapping \( g \) of \( X \) onto \( Y \) satisfying \( gT = Sg \) to be affine? Sufficient conditions are obtained in Theorem 3 in the case when the character group \( \hat{Y} \) of \( Y \) is polynomially annihilated by \( B \) (see Definition 1). In Theorem 6 we show that these conditions are also necessary when \( Y \) is a finite-dimensional group and in Theorem 8 we state (without proof) that this is also true in a more general setting. An example is given to show the assumption that \( \hat{Y} \) be polynomially annihilated by \( B \) cannot be dropped from Theorem 3. We also give an example which shows Theorem 6 to be false if \( Y \) is not finite-dimensional but \( \hat{Y} \) is polynomially annihilated by \( B \).

Theorem 7 deals with the case when \( T \) and \( S \) are endomorphisms of an \( n \)-dimensional group. A conjugacy property of affine transformations with quasi-discrete spectrum is given in Theorem 4, and Theorem 5 gives information on continuous roots of affine transformations.

The results of this paper were proved in [10] for the cases when \( X \) and \( Y \) are finite-dimensional tori. The idea of using Theorem 1 was obtained from the paper [2].

1. Definitions and notations. Let \( Y \) be a compact connected metric abelian (c.c.m.a.) group. We shall use additive notation in such groups. \( \hat{Y} \) will denote the discrete torsion-free countable abelian character group of \( Y \), and multiplicative notation will be used in \( \hat{Y} \). \( Y \) can be written as an inverse limit \( \prod_{m \geq 1} (Y_m, \sigma_m) \), where each \( Y_m \) is a finite-dimensional torus and \( \sigma_m \) is a homomorphism of \( Y_{m+1} \) onto \( Y_m \). If \( Y \) is \( n \)-dimensional then each \( Y_m \) can be chosen to be an \( n \)-dimensional torus.

An affine transformation \( S \) of a c.c.m.a. group \( Y \) is a transformation of the form \( S(y) = b + B(y) \), \( y \in Y \), where \( b \in Y \) and \( B \) is an endomorphism of \( Y \) onto \( Y \). We write \( S = b + B \). Every affine transformation of \( Y \) is continuous and preserves Haar measure. An endomorphism \( B \) of \( Y \) onto \( Y \) induces a one-to-one dual endomorphism, which we also denote by \( B \), of \( \hat{Y} \) into \( \hat{Y} \) defined by \( (By)(y) = y(By) \), \( y \in Y \), \( y \in \hat{Y} \).

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95
The affine transformation $S=b+B$ of $Y$ is ergodic if and only if $B^n\gamma=\gamma$, $\gamma \in \hat{Y}$, $n>0$, implies $B^ny=\gamma$, and $[b, (B-I)Y] = Y$ where $[b, (B-I)Y]$ denotes the smallest closed subgroup of $Y$ containing $b$ and $(B-I)Y$. ($I$ is the identity mapping of $Y$ [6].) Also, $S=b+B$ is ergodic if and only if there exists $y_0 \in Y$ such that \( \{S^n(y_0) \mid n \geq 0 \} \) is dense in $Y$ [10]. From the first condition it follows that an endomorphism $B$ of $Y$ onto $Y$ is ergodic if and only if $B^n\gamma=\gamma$, $\gamma \in \hat{Y}$, $n>0$, implies $\gamma=1$ (see also [5]). Also from the first condition we have that $S=b+B$ is strong mixing if and only if $B$ is ergodic (see also [3]).

$R^n$ will denote real Euclidean $n$-space, $Z^n$ the subgroup of $R^n$ of points with integer coordinates and $K^n=R^n/Z^n$ the $n$-dimensional torus. $(p)$ and $J(p)$ will denote the real and imaginary parts of the complex number $p$, and if $F(x)=(P_1(x), P_2(x), \ldots, P_n(x))$ is a transformation from a set $X$ to complex $n$-space $C^n$, then $P$ and $J$ will denote the transformations of $X$ to $R^n$ defined by $(P)(x) = (P_1(x), \ldots, P_n(x))$ and $(J)(x) = (J_1(x), \ldots, J_n(x))$ respectively.

$Q$ will denote the field of rational numbers and $Q[\theta]$ the algebra of all polynomials in $\theta$ with coefficients from $Q$.

2. Preliminary results.

**Theorem 1 (Van Kampen).** Let $Y$ be a c.c.m.a. group and $f$ be a continuous function from $Y$ to the set of complex numbers of unit modulus. Then $f$ can be expressed in the form $f(y) = a(y)e^{\phi(y)}$, $y \in Y$, where $a \in \hat{Y}$ is uniquely determined by $f$, $\phi: Y \to R$ is continuous and is uniquely determined up to an additive constant.

Proofs of this theorem can be found in [9] and [2]. The following is immediate from Theorem 1.

**Corollary 1.1.** Let $X$ and $Y$ be c.c.m.a. groups and $g: X \to Y$ a continuous mapping. For each $y \in Y$ there exists a unique $a_y \in X$ and a continuous mapping $\phi_y: X \to R$ unique up to an additive constant, such that $(g \circ \phi)(x) = a(x)\exp [i\phi(x)]$, $x \in X$. Furthermore \( \exp [i\phi(y)(x)] = \exp [i\phi(x)] \cdot \exp [i\phi(y)(x)] \), $x \in X$, $y, y_1 \in \hat{Y}$.

**Theorem 2.** Let $X$ and $Y$ be c.c.m.a. groups and suppose that for every $\gamma \in \hat{Y}$ there exists a continuous mapping $\phi_\gamma: X \to R$ such that $\phi_{\gamma^1} = \phi_\gamma + \phi_{\gamma^1}$, $\gamma, \gamma^1 \in \hat{Y}$. Then there exists a continuous mapping $u: X \to Y$ such that $\gamma \circ u(x) = \exp [i\phi(x)]$, $x \in X$, $\gamma \in \hat{Y}$, and $u$ is homotopic to a constant.

**Proof.** For each $x \in X$ the mapping $\gamma \to \exp [i\phi(x)]$ is a character of $\hat{Y}$ and therefore there exists $y_x \in Y$ such that $\gamma(y_x) = \exp [i\phi(x)]$. Define $u: X \to Y$ by $u(x) = y_x$. $u$ is clearly continuous.

For each $x \in X$ and each $t \in [0, 1]$ the mapping $\gamma \to \exp [it\phi(x)]$ is a character of $\hat{Y}$ and, as above in the case $t=1$, there exists a continuous mapping $u_t: X \to Y$ such that $\gamma(u_t(x)) = \exp [it\phi(x)]$, $x \in X$, $\gamma \in \hat{Y}$. $u_t$ is a homotopy between $u$ and a constant.
3. Topological conjugacy and groups with polynomially annihilated character groups. Let $Y$ be a c.c.m.a. group and $B$ an endomorphism of $Y$ onto $Y$. Let $p(\theta) = n_0 + n_1 \theta + \ldots + n_k \theta^k$ be a polynomial over $\mathbb{Z}$. We shall say that $p$ is an annihilating polynomial of $\gamma \in \hat{Y}$ with respect to $B$ if $\gamma^{n_0} \cdot B^{n_1} \gamma^{n_2} \cdots B^n \gamma^{n_k} = 1$.

Suppose $\gamma \in \hat{Y}$ has a nontrivial annihilating polynomial with respect to $B$. Let $M_\gamma$ denote the set of all polynomials over $\mathbb{Q}$ some integral multiple of which is an annihilating polynomial of $\gamma$ with respect to $B$. $M_\gamma$ is an ideal in $\mathbb{Q}[\theta]$ and therefore there exists a unique monic polynomial $q_\gamma \in \mathbb{Q}[\theta]$ such that $M_\gamma$ is the principal ideal generated by $q_\gamma$ [8, p. 121]. If $q_\gamma(\theta) = s_0 + s_1 \theta + \ldots + s_{l-1} \theta^{l-1} + \theta^l$ then $s_0 \neq 0$ for otherwise $q_\gamma(\theta) = s_1 + s_2 \theta + \ldots + s_{l-1} \theta^{l-2} + \theta^{l-1}$ would be a monic polynomial generating $M_\gamma$. If $n_\gamma$ is the lowest common denominator of the nonzero members of $s_0, s_1, \ldots, s_{l-1}$ then $p_\gamma(\theta) = n_\gamma s_0 + n_\gamma s_1 \theta + \ldots + n_\gamma \theta^l$ is a polynomial over $\mathbb{Z}$ which will be called the minimal annihilating polynomial of $\gamma$ with respect to $B$.

Definition 1. Let $Y$ be a c.c.m.a. group and $B$ an endomorphism of $Y$ onto $Y$. We say that $\hat{Y}$ is polynomially annihilated by $B$ if every element of $\hat{Y}$ has a nontrivial annihilating polynomial with respect to $B$.

If $Y$ is an $n$-dimensional c.c.m.a. group then $\hat{Y}$ is polynomially annihilated by any endomorphism $B$ of $Y$ onto $Y$. This follows because $\hat{Y}$ is isomorphic to a subgroup of the additive group $\mathbb{Q}^n$ (the direct sum of $n$ copies of $\mathbb{Q}$) and therefore the one-to-one endomorphism $B$ of $\hat{Y}$ corresponds to an $n \times n$ matrix with rational entries and nonzero determinant. The Cayley-Hamilton theorem shows that some integral multiple of the characteristic polynomial of this matrix is an annihilating polynomial, with respect to $B$, of every element of $\hat{Y}$. If $q(\theta) = s_0 + s_1 \theta + \ldots + s_n \theta^n$ is the characteristic polynomial of some matrix representation of $B$ and if $n_\gamma$ is the lowest common denominator of the nonzero members of $s_0, s_1, \ldots, s_{n-1}$, then the polynomial $p(\theta) = n_\gamma s_0 + n_\gamma s_1 \theta + \ldots + n_\gamma \theta^n$ is a polynomial over $\mathbb{Z}$, which will be called the annihilating polynomial of $\hat{Y}$ with respect to $B$. This polynomial is independent of the matrix representation of $B$.

The following lemma will be used in the proof of Theorem 3.

Lemma 1. Let $X$ be a c.c.m.a. group and $T = a + A$ an affine transformation of $X$. Suppose $\Phi: X \rightarrow \mathbb{R}^n$ is a nonconstant continuous function and $M$ is a linear transformation of $\mathbb{R}^n$ such that $\Phi(Tx) = M \Phi(x) + d$, $x \in X$, where $d \in \mathbb{R}^n$. Then there exists $\delta \in \hat{X}$, $\delta \neq 1$, and a root $\lambda$, with $|\lambda| = 1$, of the characteristic equation of $M$ such that $A^p \delta = \delta$ for some $p \geq 1$ and $\delta(a + A(a) + \ldots + A^{p-1}(a)) = \lambda^p$ for all such $p$.

Proof. We consider $\mathbb{R}^n$ as a subset of $\mathbb{C}^n$ (complex $n$-space) in the usual way and complexify $M$. There exists an invertible linear transformation $U$ of $\mathbb{C}^n$ such that $U^{-1}MU = D_M$, the Jordan normal form of the linear transformation $M$. Therefore $U^{-1} \Phi(Tx) = D_M U^{-1} \Phi(x) + U^{-1}d$, $x \in X$. If $w_1, w_2, \ldots, w_n$ denotes the fixed basis of $\mathbb{R}^n$ then $w_1, w_2, \ldots, w_n$ is also a basis, using complex coefficients for $\mathbb{C}^n$. Suppose $U^{-1} \Phi(x) = \sum_{i=1}^n f_i(x) w_i$. Each $f_i: X \rightarrow \mathbb{C}$ is continuous, and if $i_0$ is the least positive integer for which $f_{i_0}$ is nonconstant then $f_{i_0}(Tx) = M f_{i_0}(x) + e$, $x \in X$, where $e \in \mathbb{C}$.
and $\lambda$ is an eigenvalue of $M$. If $I: X \rightarrow C$ is defined by $I(x) = \int_X f(x) \, dm$, where $m$ denotes Haar measure on $X$, then $I(Tx) = \lambda(I(x))$ and $I$ is nonconstant and continuous. Since $T$ maps $X$ onto $X$, $\|I(Tx)\| = |\lambda| \sup_X |I(x)|$ implies $|\lambda| = 1$. But $I \in L^2(X)$ and therefore $I(x) = \sum_i b_i \delta_i(x)$ ($L^2$ convergence) where $\delta_i \in \hat{X}$ and $\sum_i |b_i|^2 < \infty$. From the equation $I(T^px) = \lambda^p I(x)$, $p \geq 1$, we have

$$\sum_i b_i \delta_i(a + Aa + \cdots + A^{p-1}a) \delta_i(A^p x) = \lambda^p \sum_i b_i \delta_i(x) \quad (L^2 \text{ convergence}).$$

If $\delta_i, A\delta_i, A^2\delta_i, \ldots$ are all distinct then $b_i = 0$ for otherwise the condition $\sum_i |b_i|^2 < \infty$ is violated. Therefore $b_i \neq 0$ implies $A^p \delta_i = \delta_i$ for some $p \geq 1$ and when this occurs $\delta_i(a + Aa + \cdots + A^{p-1}a) = \lambda^p$. Since $I(x)$ is nonconstant there must be some $\delta_i \in \hat{X}$, $\delta_i \neq 1$, with this property.

**Theorem 3.** Let $X$ and $Y$ be c.c.m.a. groups. Let $T = a + A$ be an affine transformation of $X$ and $S = b + B$ an affine transformation of $Y$. Suppose further that $\hat{Y}$ is polynomially annihilated by $B$. If there exists a nonaffine continuous mapping $g : X \rightarrow Y$ such that $gT = Sg$ then there exists $\delta \in \hat{X}$, $\delta \neq 1$, and a root $\lambda$, with $|\lambda| = 1$, of the minimal annihilating polynomial with respect to $B$ of some element of $\hat{Y}$, such that $A^p \delta = \delta$ for some $p \geq 1$ and $\delta(a + Aa + \cdots + A^{p-1}a) = \lambda^p$ for all such $p$.

**Proof.** Using the notation of Corollary 1.1, for $\gamma \in \hat{Y}$ let

$$(\gamma \circ g)(x) = \alpha_{\gamma}(x) \exp [i\phi_{\gamma}(x)],$$

where $\alpha_{\gamma} \in \hat{X}$ and $\phi_{\gamma} : X \rightarrow \mathbb{R}$ is continuous. Since $g$ is nonaffine there exists $\gamma_0 \in \hat{Y}$ such that $\phi_{\gamma_0}$ is nonconstant. Applying $\gamma \in \hat{Y}$ to the equation $gT = Sg$ and using the uniqueness asserted in Corollary 1.1 we have $\alpha_{\gamma}(a) \exp [i\phi_{\gamma}(Tx)] = \gamma(b) \exp [i\phi_{\gamma}(x)]$. Since $X$ is connected this implies

$$\phi_{\gamma}(Tx) = \phi_{B\gamma}(x) + c_{\gamma}, \quad x \in X,$$

where $c_{\gamma} \in \mathbb{R}$. Suppose that $p_{\gamma_0}$, the minimal annihilating polynomial of $\gamma_0$ with respect to $B$, is of degree $n$. Define $\Phi : X \rightarrow \mathbb{R}^n$ by

$$\Phi(x) = \begin{bmatrix} \phi_{\gamma_0}(x) \\ \phi_{B\gamma_0}(x) \\ \vdots \\ \phi_{B^{n-1}\gamma_0}(x) \end{bmatrix}, \quad x \in X.$$
Then $\Phi(Tx) = M\Phi(x) + d, x \in X$, where $d \in \mathbb{R}^n$, and the result follows from Lemma 1 since $p_{r_0}$ is the characteristic polynomial of $M$.

**Corollary 3.1.** Let $X, Y, T, S$ be as in Theorem 3 with the additional assumption that $T$ is ergodic. If there is a nonaffine continuous mapping $g: X \to Y$ such that $gT = Sg$ then there exists $\delta \in \hat{X}, \delta \neq 1$, and a root $\lambda$, with $|\lambda| = 1$, of the minimal annihilating polynomial with respect to $B$ of some element of $\hat{Y}$, such that $\lambda$ is not a root of unity, $A\delta = \delta$ and $\delta(a) = \lambda$.

Hence if $T$ is strong mixing, all continuous mappings $g: X \to Y$ such that $gT = Sg$ are affine.

**Proof.** Let $\delta$ be the element of $\hat{X}$ and $\lambda$ the complex number which are determined by Theorem 3. Since $A^p \delta = \delta$ for some $p \geq 1$, the ergodicity of $T$ implies $A\delta = \delta$ and hence $\delta(a) = \lambda$. If $\lambda$ were a root of unity then since $[a, (A - I)X] = X$, $\delta$ would only assume a finite number of values on $X$ and would have to be the identity character.

Lastly, if $T$ is strong mixing then $A$ is ergodic and there is no $\delta \in \hat{X}, \delta \neq 1$, with $A\delta = \delta$.


The notion of a measure-preserving transformation with quasi-discrete spectrum has been defined by Abramov [1], and the notion of a homeomorphism with quasi-discrete spectrum has been defined by Hahn and Parry [4]. An ergodic affine transformation $S = b + B$ of a c.c.m.a. group $Y$ has quasi-discrete spectrum as a (Haar) measure-preserving transformation if and only if it has quasi-discrete spectrum as a homeomorphism. In fact $S = b + B$, assumed to be ergodic, has quasi-discrete spectrum in either sense if and only if $\bigcap_{n=0}^{\infty} (B-I)^n Y = \{0\}$, where $I$ denotes the identity mapping of $Y$ [7]. The following result extends Theorem 6 of the paper [4].

**Theorem 4.** Let $X$ and $Y$ be c.c.m.a. groups and let $T = a + A$ be an ergodic affine transformation of $X$ and $S = b + B$ an ergodic affine transformation of $Y$. If $S$ has quasi-discrete spectrum then all continuous mappings $g: X \to Y$ satisfying $gT = Sg$ are affine.

**Proof.** Let $\gamma \in \hat{Y}$. There exists $n \geq 1$ such that $(\theta - I)^n$ is an annihilating polynomial of $\gamma$ with respect to $B$. It follows that the roots of the minimal annihilating
polynomial of $y$ with respect to $B$ are equal to 1. The result follows from Corollary 3.1.

**Theorem 5.** Let $Y$ be a c.c.m.a. group and $S = b + B$ a strong mixing affine transformation of $Y$ such that $\hat{Y}$ is polynomially annihilated by $B$. Then every continuous pth root ($p \geq 1$) of $S$ is an affine transformation and $S$ has a continuous pth root if and only if there is an endomorphism $C$ of $Y$ onto $Y$ with $C^p = B$.

**Proof.** Suppose $g$ is a continuous pth root of $S$. Then $gS = Sg$ and $g$ is affine by Corollary 3.1. Since $S$ is strong mixing $B$ is ergodic and therefore $(B - I)Y = Y$. Choose $y_0 \in Y$ so that $(B - I)y_0 = b$ and the homeomorphism $h: Y \to Y$, defined by $h(y) = y_0 + y$, satisfies $hS = Bh$. Therefore $S$ has a continuous pth root if and only if $B$ has a continuous pth root. Any continuous pth root of $B$ is affine and the pth power of its endomorphism part will be $B$. Conversely if $C$ is an endomorphism of $Y$ onto $Y$ with $C^p = B$ then $C$ is a continuous pth root of $B$.

As a special case of Corollary 3.1 we have the following result. If $Y$ is a c.c.m.a. group and $B$ is an ergodic endomorphism of $Y$ onto $Y$ which polynomially annihilates $\hat{Y}$, then every continuous mapping commuting with $B$ is affine. The example below shows that this result is false (and therefore Theorem 3 is false) if the assumption that $Y$ be polynomially annihilated by $B$ is dropped.

Let $K^\infty$ denote the two-sided infinite-dimensional torus (i.e. the two-sided infinite direct sum of copies of $K$) and let $B$ denote the shift automorphism of $K^\infty$ defined by $(Bz)_n = z_{n+1}$ if $z = (z_n)$. No nontrivial element of $\hat{K}^\infty$ is polynomially annihilated by $B$. Let $f: K \to K$ be any homeomorphism and define $F: K^\infty \to K^\infty$ by $(F(z))_n = f(z_n)$, $-\infty < n < \infty$. $F$ is a homeomorphism and $FB = BF$. Moreover $F$ can be chosen to be nonaffine by choosing $f$ nonaffine.

It would be interesting to know if the condition that $Y$ be polynomially annihilated by $B$ follows from the fact that every continuous mapping commuting with $B$ (if ergodic) is affine.

4. **Converses of Theorem 3.**

**Lemma 2.** Let $X$ and $Y$ be c.c.m.a. groups and let them be represented as $X = \text{inv lim } (X_q, \tau_q)$ and $Y = \text{inv lim } (Y_m, \sigma_m)$ where $X_q$ ($q \geq 1$) and $Y_m$ ($m \geq 1$) are finite-dimensional tori. Let $C$ be a homomorphism of $X$ onto $Y$ and let $u: X \to Y$ be a continuous mapping which depends only on $X_{q_0}$ and which is homotopic to a constant by a homotopy which depends only on $X_{q_0}$. Then $C + u$ maps $X$ onto $Y$.

**Proof.** Let $C_m$ and $u_m$ ($m \geq 1$) denote the mappings of $X$ to $Y_m$ obtained by projecting $C$ and $u$ onto $Y_m$. $C + u$ will map $X$ onto $Y$ if and only if $C_m + u_m$ maps $X$ onto $Y_m$ for each $m \geq 1$. For each $m \geq 1$ there exists $q_m \geq 1$ such that $C_m$ only depends on $X_{q_m}$. Let $k_m = \max(q_m, k_0)$. Then $C_m$ can be considered as a homomorphism of $X_{k_m}$ onto $Y_m$ and $u_m$ can be considered as a continuous mapping of $X_{k_m}$ into $Y_m$ which is homotopic (on $X_{k_m}$) to a constant. The result will follow if we can show that whenever $C$ is a homomorphism of $K^n$ onto $K^m$ and $u: K^n \to K^m$ is a continuous
mapping homotopic to a constant then \( C + u \) maps \( K^n \) onto \( K^m \). However this result follows from Lemma 1 of [10].

**Lemma 3.** Let \( P: \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a continuous mapping such that \( P(v + \tau) = P(v) \), \( v \in \mathbb{R}^n \), \( \tau \in \mathbb{Z}^n \) and \( \|P(v) - P(v')\| < \|v - v'\| \), \( v, v' \in \mathbb{R}^n \) where \( \| \cdot \| \) denotes the usual norm in \( \mathbb{R}^n \). Let \( \psi: \mathbb{K}^n \rightarrow \mathbb{K}^n \) be the continuous mapping defined by \( \psi \pi = \pi P \), where \( \pi: \mathbb{R}^n \rightarrow \mathbb{K}^n \) is the natural projection. Then \( I + \psi \) is a one-to-one mapping of \( \mathbb{K}^n \). (I denotes the identity mapping of \( \mathbb{K}^n \).

**Proof.** Let \( I' \) denote the identity mapping of \( \mathbb{R}^n \). \( I' + P \) is a one-to-one mapping because \( v + P(v) = v' + P(v') \) implies \( v - v' = P(v') - P(v) \) and hence \( v = v' \). Suppose \( (I' + \psi)\pi(v) = (I' + \psi)\pi(v') \). Then \( \pi(I' + P)(v) = \pi(I' + P)(v') \) and

\[
(I' + P)(v) = (I' + P)(v') + \tau, \quad \tau \in \mathbb{Z}^n
\]

Therefore \( v = v' + \tau \) and \( \pi(v) = \pi(v') \). This proves that \( I + \psi \) is one-to-one.

The following theorem gives a converse to Theorem 3 in the cases when \( F \) is a finite-dimensional group.

**Theorem 6.** Let \( X \) and \( Y \) be c.c.m.a. groups and suppose that \( Y \) is \( n \)-dimensional. Let \( T = a + A \) be an affine transformation of \( X \), \( S = b + B \) an affine transformation of \( Y \) and suppose there exists a continuous mapping \( h \) of \( X \) onto \( Y \) such that \( hT = Sh \). Suppose further there exists \( \delta \in \hat{X} \), \( \delta \neq 1 \), and a root \( \lambda \), with \( |\lambda| = 1 \), of the annihilating polynomial of \( \hat{Y} \) with respect to \( B \) such that \( A^p \delta = \delta \) for some \( p \geq 1 \) and \( \delta(a + A(a) + \cdots + A^{p-1}(a)) = \lambda^p \) for all such \( p \). Then there exists a nonaffine continuous mapping \( g \) of \( X \) onto \( Y \) such that \( gT = Sg \). Moreover, if \( h \) is given to be a homeomorphism then \( g \) can be chosen to be a homeomorphism.

**Proof.** We may as well assume that the given mapping \( h \) is affine or there is nothing to prove. Suppose \( h = c + C \), where \( c \in Y \) and \( C \) is a homomorphism of \( X \) onto \( Y \). We shall use Theorem 2 and to do this we have to construct continuous mappings \( \psi_i: X \rightarrow \mathbb{R} \) for each \( \gamma_i \in \hat{Y} \).

Since \( Y \) is \( n \)-dimensional \( \hat{Y} \) is isomorphic to a subgroup \( Q^n \) of the additive group \( Q^n \) and we can choose \( Q^n \) so that if

\[
d_i = (d_{i1}, d_{i2}, \ldots, d_{in}) \text{ where } d_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}
\]

then \( d_i \in Q^n \) (\( 1 \leq i \leq n \)). Let \([M]\) be the matrix (with rational entries) representing the action of \( B \) on \( Q^n \), and let \( \gamma_i \in \hat{Y} \) correspond under the above isomorphism to \( d_i \in Q^n \) (\( 1 \leq i \leq n \)). Let \( M \) denote the linear transformation of \( \mathbb{R}^n \) induced by the matrix \([M]\).

Suppose that \( p \) is the smallest positive integer such that \( A^p \delta = \delta \). Define \( f: X \rightarrow C \) by

\[
f(x) = \sum_{j=0}^{p-1} \frac{\delta(a + A(a) + \cdots + A^{j-1}(a))}{\lambda^j} A^j(x), \quad x \in X,
\]
$f$ is a nonconstant continuous function satisfying $f(Tx) = \lambda f(x)$, $x \in X$. If $w_1, w_2, \ldots, w_n$ denotes the fixed basis of $\mathbb{R}^n$ it is also a basis for $\mathbb{C}^n$. Let $U$ be the invertible linear transformation of $\mathbb{C}^n$ such that $U^{-1}MU = D_\lambda$, the Jordan normal form of the complexified linear transformation $M$. Let $j_0$ be the largest integer for which $w_{j_0}$ corresponds to the eigenvalue $\lambda$ of $D_\lambda$. Then $U(f(x)w_{j_0})$ is nonconstant and so either $\mathcal{R}U(f(x)w_{j_0})$ or $\mathcal{F}U(f(x)w_{j_0})$ is nonconstant. Suppose, without loss of generality, that $\mathcal{R}U(f(x)w_{j_0})$ is nonconstant and define the mappings $\phi_r : X \to \mathbb{R}$ by

$$\sum_{i=1}^{n} \phi_r(x)w_i = \mathcal{R}U(f(x)w_{j_0}), \quad x \in X.$$ 

Let $\gamma \in \tilde{Y}$. If $\gamma^m = \gamma_{2}^m \cdot \gamma_{3}^m \cdot \gamma_{n}^m$, $m_0, m_1, \ldots, m_n \in \mathbb{Z}$, $m_0 \neq 0$, define $\phi_\gamma : X \to \mathbb{R}$ by

$$\phi_\gamma(x) = \frac{m_1}{m_0} \phi_{r_1}(x) + \frac{m_2}{m_0} \phi_{r_2}(x) + \cdots + \frac{m_n}{m_0} \phi_{r_n}(x), \quad x \in X.$$ 

Then $\phi_\gamma = \phi_{r_1} + \phi_{r_2} + \cdots + \phi_{r_n}$, $\gamma \in \tilde{Y}$. Also

$$\sum_{i=1}^{n} \phi_{r_i}(Tx)w_i = \mathcal{R}U(f(Tx)w_{j_0}) = \mathcal{R}UD_M(f(x)w_{j_0})$$

$$= M \sum_{i=1}^{n} \phi_{r_i}(x)w_i = \sum_{i=1}^{n} \phi_{Br_i}(x)w_i.$$ 

Therefore $\phi_{Br_i}(Tx) = \phi_{Br_i}(x)$, $x \in X$, $1 \leq i \leq n$, and hence $\phi_r(Tx) = \phi_{Br_i}(x)$, $x \in X$, $\gamma \in \tilde{Y}$.

By Theorem 2 there exists a continuous mapping $u : X \to Y$ such that $\gamma(u(x)) = \exp [it\phi_r(x)]$, $x \in X$, $\gamma \in \tilde{Y}$, $u(Tx) = Bu(x)$, $x \in X$, and $u$ is homotopic to a constant.

Let $g : X \to Y$ be defined by $g(x) = c + C(x) + u(x)$, $x \in X$. $g(Tx) = c + C(Tx) + u(Tx)$.

It remains to show that $g$ maps $X$ onto $Y$. Suppose $X = \text{inv lim } (X_q, \tau_q)$ and $Y = \text{inv lim } (Y_m, \sigma_m)$ where the $X_q$ ($q \geq 1$) are finite-dimensional tori and the $Y_m$ ($m \geq 1$) are $n$-dimensional tori. Suppose the given character $\delta \in \tilde{X}_{k_0}$. Then each mapping $\phi_\gamma : X \to \mathbb{R}$ only depends on $X_{k_0}$ and therefore $u$ only depends on $X_{k_0}$ and is homotopic to a constant by a homotopy depending only on $X_{k_0}$ (Theorem 2). The fact that $g$ maps $X$ onto $Y$ now follows from Lemma 2.

We now show that if $h$ is given to be a homeomorphism then $g$ can be chosen to be a homeomorphism. Since we are assuming $h = c + C$, $C$ will be an isomorphism of $X$ onto $Y$. Let $g_t : X \to Y$, $t \in [0, 1]$, be defined by $g_t(x) = c + C(x) + ut(x)$, $x \in X$, where $u_t : X \to Y$ satisfies $\gamma(u_t(x)) = \exp [it\phi_r(x)]$, $x \in X$, $\gamma \in \tilde{Y}$ (Theorem 2). By Lemma 2 $g_t$ is a continuous mapping of $X$ onto $Y$ and $g_T = Sg_0$, $t \in [0, 1]$. We shall show that $g_t$ is one-to-one for sufficiently small $t$.

It suffices to show that $g_t \circ C^{-1} : Y \to Y$ is one-to-one for sufficiently small $t$. We have $g_t \circ C^{-1}(y) = c + y + ut \circ C^{-1}y$, $y \in Y$. Let $k$ be the smallest integer for which $\delta \circ C^{-1} \in \tilde{X}_k$, where $\delta$ is the given element of $\tilde{X}$. By the definition of $\phi_r$,
\( \gamma \in \hat{Y} \), each \( \phi_{\gamma} \circ C^{-1} \) can be considered as a real-valued function of \( Y_k \), and therefore induces a mapping \( P_{\gamma} : \mathbb{R}^n \rightarrow \mathbb{R} \) defined by \( P_{\gamma}(v) = \phi_{\gamma} \circ C^{-1}(y_v), \ v \in \mathbb{R}^n \), where \( y_v \) is any point of \( Y \) which has component \( v + Z^n \) in \( Y_k \). Since each \( P_{\gamma} \) is a linear combination of sines and cosines of the coordinates of \( \mathbb{R}^n \), there exists a constant \( N \) such that if \( \beta \) is a generator of any \( \hat{Y}_m (m \geq 1) \) then
\[
|P_{\beta}(v) - P_{\beta}(v')| \leq N \|v - v'\|, \quad v, v' \in \mathbb{R}^n,
\]
where \( \|\cdot\| \) denotes the usual norm in \( \mathbb{R}^n \).

Choose \( t_0 \in [0, 1] \) so that \( nt_0N < 1 \). Let \( y, y' \in Y, y \neq y' \). Suppose \( y = (y_1, y_2, \ldots), \ y' = (y'_1, y'_2, \ldots) \) where \( y_i, y'_i \in Y_i, i \geq 1 \). We shall show that \( g_{t_0} \circ C^{-1}(y) \neq g_{t_0} \circ C^{-1}(y') \). If \( y_k = y'_k \) then \( \phi_{\gamma} \circ C^{-1}(y) = \phi_{\gamma} \circ C^{-1}(y'), \ y \in \hat{Y}, \) and therefore \( u_{t_0} \circ C^{-1}(y) = u_{t_0} \circ C^{-1}(y') \). Hence \( g_{t_0} \circ C^{-1}(y) - g_{t_0} \circ C^{-1}(y') = y - y' \neq 0 \). Now suppose \( y_k \neq y'_k \). Considering \( Y_k \) as an \( n \)-torus let \( \beta_1, \beta_2, \ldots, \beta_n \in \hat{Y}_k \) be defined by \( \beta_j(z_1, \ldots, z_n) = \exp(2\pi iz_j) \). Define \( G : Y_k \rightarrow Y_k \) by
\[
G(z_1, \ldots, z_n) = (z_1 + t_0\phi_{\beta_1} \circ C^{-1}(y_2), \ldots, z_n + t_0\phi_{\beta_n} \circ C^{-1}(y_2)) + Z^n
\]
where \( y_z \) is any point of \( Y \) having \( z = (z_1, \ldots, z_n) \) as its component in \( Y_k \). By Lemma 3, since \( t_0 \) is chosen so that \( nt_0N < 1 \), we have that \( G \) is one-to-one. Since \( y_k \neq y'_k \), \( G(y_k) \neq G(y'_k) \), i.e. \( \beta_j(y_k + u_{t_0} \circ C^{-1}(y)) \neq \beta_j(y'_k + u_{t_0} \circ C^{-1}(y')) \) for some \( j \). Therefore \( y + u_{t_0} \circ C^{-1}(y) \neq y' + u_{t_0} \circ C^{-1}(y') \), i.e. \( \phi_{\gamma} \circ C^{-1}(y) \neq \phi_{\gamma} \circ C^{-1}(y') \).

The following is a direct consequence of Theorems 3 and 6.

**Corollary 6.1.** Let \( X \) be a c.c.m.a. group and let \( Y \) be a c.c.m.a. \( n \)-dimensional group. Let \( T = a + A \) be an affine transformation of \( X \) and \( S = b + B \) an affine transformation of \( Y \) for which there exists a continuous mapping \( h \) of \( X \) onto \( Y \) satisfying \( hT = Sh \). There exists a nonaffine continuous mapping \( g \) of \( X \) onto \( Y \) such that \( gT = Sg \) if and only if there exists two \( |A| = 1 \), and a root \( \alpha \) of the annihilating polynomial of \( X \) with respect to \( B \) such that \( \alpha^p = \delta \) for some \( p \geq 1 \) and \( \delta(a + A(a) + \cdots + A^{p-1}(a)) = \lambda^p \) for all such \( p \). If \( h \) is a homeomorphism, the above conditions are necessary and sufficient for the existence of a nonaffine homeomorphism \( g \) of \( Y \) such that \( gT = Sg \).

If \( B \) is an endomorphism of a c.c.m.a. \( n \)-dimensional group \( Y \) onto \( Y \) then it follows from the ergodicity conditions stated in §1 that \( B \) is ergodic if and only if no root of the annihilating polynomial of \( \hat{Y} \) with respect to \( B \) is a root of unity. Moreover there is an element \( \gamma \in \hat{Y}, \gamma \neq 1 \) such that \( B^p \gamma = \gamma \) if and only if the annihilating polynomial of \( \hat{Y} \) with respect to \( B \) has a \( p \)-th root of unity as a root.

**Theorem 7.** Let \( Y \) be a c.c.m.a. \( n \)-dimensional group and let \( A \) and \( B \) be endomorphisms of \( Y \) onto \( Y \). Suppose there exists a continuous mapping \( h \) of \( Y \) onto \( Y \) such that \( hA = Bh \). There exists a nonaffine continuous mapping \( g \) of \( Y \) onto \( Y \) such that \( gA = Bg \) if and only if \( A \) and \( B \) are not ergodic. If \( h \) is given to be a homeomorphism then there exists a nonaffine homeomorphism \( g \) of \( Y \) such that \( gA = Bg \) if and only if \( A \) and \( B \) are not ergodic.
Proof. If there exists a nonaffine continuous mapping $g$ of $Y$ into $Y$ satisfying $gA =Bg$ then Theorem 3 asserts the existence of $\delta \in \hat{Y}$, $\delta \neq 1$, and a root $\lambda$ of the annihilating polynomial of $\hat{Y}$ with respect to $B$ such that $A^p\delta = \delta$ for some $p \geq 1$ and $\lambda^p = 1$ for all such $p$. Therefore $A$ and $B$ are not ergodic.

Conversely suppose $A$ and $B$ are not ergodic. Suppose $h$ is affine or there is nothing to prove. Let $h = c + C$, where $c \in Y$ and $C$ is an endomorphism of $Y$ onto $Y$. If $\hat{Y}$ is considered as an (additive) subgroup of $Q^n$ the nonsingular matrix representing $C$ is a conjugacy between the matrix representing $A$ and the matrix representing $B$. Hence the annihilating polynomial of $\hat{Y}$ with respect to $A$ is the same as the annihilating polynomial of $\hat{Y}$ with respect to $B$. Let $\delta \in \hat{Y}$, $\delta \neq 1$, be such that $A^p\delta = \delta$ for some $p \geq 1$. Let $p$ be the least positive integer for which $A^p\delta = \delta$. Then the annihilating polynomial of $\hat{Y}$ with respect to $B$ has a root $\lambda$ which is a $p$th root of unity. The result now follows from Theorem 6.

We now give an example to show that Theorem 6 is false if the assumption that $\hat{Y}$ is finite-dimensional is replaced by the assumption that $\hat{Y}$ is polynomially annihilated by $B$, i.e. the converse of Theorem 3 is false.

Let $E$ denote the automorphism of the 4-torus $K^4$ determined by the matrix

$$
[E] = \begin{bmatrix}
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 8 \\
0 & 1 & 0 & -6 \\
0 & 0 & 1 & 8 
\end{bmatrix}.
$$

The matrix $[E]$ has two eigenvalues $\lambda_1$, $\bar{\lambda}_1$ of unit modulus which are not roots of unity and two distinct real eigenvalues $\lambda_2$, $\lambda_3$ [10]. Let $W$ denote the one-sided direct sum of an infinite number of copies of $K^4$. Let $Y = K + W$. $Y$ is an infinite-dimensional torus. Let $S = b + B : Y \to Y$ be defined by

$$
S(y_0, y_1, y_2, \ldots) = (b_0, 0, 0, \ldots) + (y_0, Ey_1, y_1 + Ey_2, y_2 + Ey_3, \ldots)
$$

$y_0 \in K$, $y_i \in K^4$ ($i \geq 1$), where $\exp [2\pi ib_0] = \lambda_1$. It is not difficult to show that $S$ is ergodic and $\hat{Y}$ is polynomially annihilated by $B$. The characteristic polynomial of $[E]$ is the minimal annihilating polynomial with respect to $B$ of some of the elements of $\hat{Y}$ and $\lambda_1$ is a root of this polynomial. If $\delta \in \hat{Y}$ is defined by

$$
\delta(y_0, y_1, y_2, \ldots) = \exp [2\pi iy_0],
$$

then $B\delta = \delta$ and $\delta(b) = \lambda_1$. Hence (with $X = Y$ and $T = S$) all the assumptions of Theorem 6 (except that $Y$ be finite-dimensional) are satisfied by this example. However, we shall show that every continuous mapping commuting with $S$ is affine.

Suppose $gS = Sg$ where $g$ is continuous. Let $g_n$ ($n \geq 0$) be the projection of $g$ onto the $n$th factor in the representation $Y = K + K^4 + K^4 + \cdots$. $g_0$ is a continuous mapping of $Y$ into $K$ and $g_n$ ($n \geq 1$) are continuous mappings of $Y$ into $K^4$. We shall show that each $g_n$ ($n \geq 0$) is affine and this implies $g$ is affine. By Theorem 1 there

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exists a homomorphism $\mu_0: Y \to K$ and a continuous mapping $\phi_0: Y \to R$ such that $g_0(y) = \mu_0(y) + \phi_0(y) + Z$. Since $g_0(Sy) = b_0 + g_0(y)$ we have

$$\phi_0(Sy) = \phi_0(y) + a, \quad y \in Y,$$

where $a \in R$.

Therefore $\phi_0(y) - \int_Y \phi_0(y) \, dm$ (where $m$ denotes Haar measure on $Y$) is an invariant function under $S$ and therefore constant. Hence $\phi_0$ is constant and $g_0$ is affine. Suppose that some $g_n (n \geq 1)$ is nonaffine. Let $k$ be the least integer for which $g_k$ is nonaffine. By Theorem 1 there exist homomorphisms $\mu_i: Y \to K$ and continuous mappings $\phi_i: Y \to R$ ($1 \leq i \leq 4$) such that

$$g_k(y) = \begin{bmatrix}
\mu_1(y) + \phi_1(y) \\
\mu_2(y) + \phi_2(y) \\
\mu_3(y) + \phi_3(y) \\
\mu_4(y) + \phi_4(y)
\end{bmatrix} + Z.$$ 

Since $gS = Sg$ we have $g_kS = g_{k-1} + Eg_k$ and $g_{k+1}S = g_k + Eg_{k+1}$. Since $g_{k-1}$ is affine the uniqueness in Theorem 1 gives

$$g_k(y) = \begin{bmatrix}
\mu_1(y) + \phi_1(y) \\
\mu_2(y) + \phi_2(y) \\
\mu_3(y) + \phi_3(y) \\
\mu_4(y) + \phi_4(y)
\end{bmatrix} + Z.$$ 

Let $D$, with matrix

$$[D] = \begin{bmatrix}
\lambda_1 & 0 & 0 & 0 \\
0 & \lambda_1 & 0 & 0 \\
0 & 0 & \lambda_2 & 0 \\
0 & 0 & 0 & \lambda_3
\end{bmatrix}$$

be the Jordan normal form of $E$ and let $U: C^4 \to C^4$ be the linear transformation such that $U^{-1}EU = D$. By the type of argument used in the proof of Theorem 3 it follows that

$$\begin{bmatrix}
\phi_1(y) \\
\phi_2(y) \\
\phi_3(y) \\
\phi_4(y)
\end{bmatrix} = U \begin{bmatrix}
c\delta(y) \\
d\delta^{-1}(y) \\
0 \\
0
\end{bmatrix} + e', \quad y \in Y,$$

where $c, d \in C$ and $e' \in R^4$. By Theorem 1 again

$$g_{k+1}(y) = \begin{bmatrix}
\mu_6(y) + \phi_6(y) \\
\mu_8(y) + \phi_8(y) \\
\mu_7(y) + \phi_7(y) \\
\mu_0(y) + \phi_0(y)
\end{bmatrix}, \quad y \in Y,$$
where \( \mu_i : Y \to K \) are homomorphisms and \( \phi_i : Y \to R \) are continuous (5 \( \leq i \leq 8 \)).
Since \( g_{k+1}(Sy) = g_k(y) + Eg_{k+1}(y) \) we have
\[
\begin{bmatrix}
\phi_2(Sy) \\
\phi_0(Sy) \\
\phi_7(Sy) \\
\phi_9(Sy)
\end{bmatrix} = E
\begin{bmatrix}
\phi_2(y) \\
\phi_0(y) \\
\phi_7(y) \\
\phi_9(y)
\end{bmatrix} + U
\begin{bmatrix}
e \delta(y) \\
d \delta(y)^{-1}(y) \\
0 \\
0
\end{bmatrix} + e^u, \quad y \in Y,
\]
where \( e^u \in R^4 \). Apply \( U^{-1} \) to this equation and set
\[
\begin{bmatrix}
f_1(y) \\
f_2(y) \\
f_3(y) \\
f_4(y)
\end{bmatrix} = U^{-1}
\begin{bmatrix}
\phi_2(y) \\
\phi_0(y) \\
\phi_7(y) \\
\phi_9(y)
\end{bmatrix}.
\]
Then \( f_1(Sy) = \lambda_1 f_1(y) + c \delta(y) + c^1 \), where \( c^1 \in C \). Since \( f_2 \in L^2(Y) \) let \( f_2(y) = \sum a_i \gamma_i(y) \) (L^2 convergence) where \( \gamma_i \in \hat{Y} \) and \( \sum |a_i|^2 < \infty \). If \( \delta = \gamma_{i_0} \) then \( a_{i_0} \delta(b) = \lambda_1 a_{i_0} + c \), and since \( \delta(b) = \lambda_1 \) this gives \( c = 0 \). Consideration of the equation for \( f_2 \) implies \( d = 0 \). Therefore \( \phi_i \), \( 1 \leq i \leq 8 \), are constant and \( g_k \) is affine, a contradiction.
Therefore each \( g_n \) (\( n \geq 0 \)) is affine.

Thus we have shown that every continuous mapping commuting with \( S \) is affine.

We shall now state, without proof, a generalization of Theorem 6. If \( B \) is an endomorphism of a c.c.m.a. group \( Y \) we denote by \( \gamma(B, A) \) the subgroup of \( \gamma \) generated by those elements of \( \gamma \) whose minimal annihilating polynomials with respect to \( B \) have \( \lambda \) as a root.

**Theorem 8.** Suppose \( X \) and \( Y \) are c.c.m.a. groups, \( T = a + A \) an affine transformation of \( X \), and \( S = b + B \) an affine transformation of \( Y \) such that \( \hat{Y} \) is polynomially annihilated by \( B \). Suppose there exists a continuous mapping \( h \) of \( X \) onto \( Y \) such that \( hT = Sh \). Also assume there exists \( \delta \in \hat{X} \), \( \delta \neq 1 \), and a complex number \( \lambda \) with \( |\lambda| = 1 \), such that \( \hat{Y}(B, \lambda) \) is a subgroup of \( \hat{Y} \) of finite rank with the properties that \( A^p \delta = \delta \) for some \( p \geq 1 \) and \( \delta(a + A(a) + \cdots + A^{p-1}(a)) = \lambda^p \) for all such \( p \). Then there exists a nonaffine continuous mapping \( g \) of \( X \) onto \( Y \) such that \( gT = Sg \). If \( h \) is a homeomorphism then \( g \) can be chosen to be a homeomorphism.

**References**


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