TOPOLOGICAL CONJUGACY OF AFFINE TRANSFORMATIONS OF COMPACT ABELIAN GROUPS

BY

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0. Introduction. We consider the following problem. If \( X \) and \( Y \) are compact connected metric abelian groups, \( T = a + A \) an affine transformation of \( X \) and \( S = b + B \) an affine transformation of \( Y \), what are necessary and sufficient conditions for every continuous mapping \( g \) of \( X \) onto \( Y \) satisfying \( gT = Sg \) to be affine? Sufficient conditions are obtained in Theorem 3 in the case when the character group \( \hat{F} \) of \( F \) is polynomially annihilated by \( B \) (see Definition 1). In Theorem 6 we show that these conditions are also necessary when \( F \) is a finite-dimensional group and in Theorem 8 we state (without proof) that this is also true in a more general setting. An example is given to show the assumption that \( \hat{Y} \) be polynomially annihilated by \( B \) cannot be dropped from Theorem 3. We also give an example which shows Theorem 6 to be false if \( Y \) is not finite-dimensional but \( \hat{Y} \) is polynomially annihilated by \( B \).

Theorem 7 deals with the case when \( T \) and \( S \) are endomorphisms of an \( n \)-dimensional group. A conjugacy property of affine transformations with quasi-discrete spectrum is given in Theorem 4, and Theorem 5 gives information on continuous roots of affine transformations.

The results of this paper were proved in [10] for the cases when \( X \) and \( Y \) are finite-dimensional tori. The idea of using Theorem 1 was obtained from the paper [2].

1. Definitions and notations. Let \( Y \) be a compact connected metric abelian (c.c.m.a.) group. We shall use additive notation in such groups. \( \hat{Y} \) will denote the discrete torsion-free countable abelian character group of \( Y \), and multiplicative notation will be used in \( \hat{Y} \). \( Y \) can be written as an inverse limit \( \text{inv lim} \ (Y_m, \sigma_m) \), where each \( Y_m \ (m \geq 1) \) is a finite-dimensional torus and \( \sigma_m \) is a homomorphism of \( Y_{m+1} \) onto \( Y_m \). If \( Y \) is \( n \)-dimensional then each \( Y_m \) can be chosen to be an \( n \)-dimensional torus.

An affine transformation \( S \) of a c.c.m.a. group \( Y \) is a transformation of the form \( S(y) = b + B(y), \ y \in Y \), where \( b \in Y \) and \( B \) is an endomorphism of \( Y \) onto \( Y \). We write \( S = b + B \). Every affine transformation of \( Y \) is continuous and preserves Haar measure. An endomorphism \( B \) of \( Y \) onto \( Y \) induces a one-to-one dual endomorphism, which we also denote by \( B \), of \( \hat{Y} \) into \( \hat{Y} \) defined by \( (By)(\gamma) = \gamma(B\gamma), \ y \in Y, \ \gamma \in \hat{Y} \).

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The affine transformation $S = b + B$ of $Y$ is ergodic if and only if $B^n\gamma = \gamma$, $\gamma \in \hat{Y}$, $n > 0$, implies $B^ny = y$, and $[b, (B - I)Y] = Y$ where $[b, (B - I)Y]$ denotes the smallest closed subgroup of $Y$ containing $b$ and $(B - I)Y$. ($I$ is the identity mapping of $Y$ [6].) Also, $S = b + B$ is ergodic if and only if there exists $y_0 \in Y$ such that $\{S^n(y_0) | n \geq 0\}$ is dense in $Y$ [10]. From the first condition it follows that an endomorphism $B$ of $Y$ onto $Y$ is ergodic if and only if $B^n\gamma = \gamma$, $\gamma \in \hat{Y}$, $n > 0$, implies $\gamma = 1$ (see also [5]). Also from the first condition we have that $S = b + B$ is strong mixing if and only if $B$ is ergodic (see also [3]).

$\mathbb{R}^n$ will denote real Euclidean $n$-space, $\mathbb{Z}^n$ the subgroup of $\mathbb{R}^n$ of points with integer coordinates and $\mathbb{K}^n = \mathbb{R}^n/\mathbb{Z}^n$ the $n$-dimensional torus. $\Re(p)$ and $\Im(p)$ will denote the real and imaginary parts of the complex number $p$, and if $P(x) = (P_1(x), P_2(x), \ldots, P_n(x))$ is a transformation from a set $X$ to complex $n$-space $\mathbb{C}^n$, then $\Re P$ and $\Im P$ will denote the transformations of $X$ to $\mathbb{R}^n$ defined by $(\Re P)(x) = (\Re P_1(x), \ldots, \Re P_n(x))$ and $(\Im P)(x) = (\Im P_1(x), \ldots, \Im P_n(x))$ respectively.

$\mathbb{Q}$ will denote the field of rational numbers and $\mathbb{Q}[\theta]$ the algebra of all polynomials in $\theta$ with coefficients from $\mathbb{Q}$.

2. Preliminary results.

**Theorem 1** (Van Kampen). Let $Y$ be a c.c.m.a. group and $f$ be a continuous function from $Y$ to the set of complex numbers of unit modulus. Then $f$ can be expressed in the form $f(y) = a(y)e^{\phi(y)}$, $y \in Y$, where $a \in \hat{Y}$ is uniquely determined by $f$, $\phi: Y \to \mathbb{R}$ is continuous and is uniquely determined up to an additive constant.

Proofs of this theorem can be found in [9] and [2]. The following is immediate from Theorem 1.

**Corollary 1.1.** Let $X$ and $Y$ be c.c.m.a. groups and $g: X \to Y$ a continuous mapping. For each $y \in Y$ there exists a unique $a_y \in \hat{X}$ and a continuous mapping $\phi_y: X \to \mathbb{R}$ unique up to an additive constant, such that $(\gamma \circ g)(x) = a_y(x)e^{\phi_y(x)}$, $x \in X$. Furthermore $\exp [i\phi_{y_1}(x)] \cdot \exp [i\phi_{y_2}(x)] = \exp [i\phi_{y_1}(x)]$, $x \in X$, $\gamma, \gamma^1 \in \hat{Y}$.

**Theorem 2.** Let $X$ and $Y$ be c.c.m.a. groups and suppose that for every $\gamma \in \hat{Y}$ there exists a continuous mapping $\phi_{\gamma}: X \to \mathbb{R}$ such that $\phi_{\gamma y} = \phi_{\gamma} + \phi_{y}$, $\gamma, \gamma^1 \in \hat{Y}$. Then there exists a continuous mapping $u: X \to Y$ such that $\gamma \circ u(x) = \exp [i\phi_{\gamma}(x)]$, $x \in X$, $\gamma \in \hat{Y}$, and $u$ is homotopic to a constant.

**Proof.** For each $x \in X$ the mapping $\gamma \to \exp [i\phi_{\gamma}(x)]$ is a character of $\hat{Y}$ and therefore there exists $y_{x} \in Y$ such that $\gamma(y_{x}) = \exp [i\phi_{\gamma}(x)]$. Define $u: X \to Y$ by $u(x) = y_{x}$. $u$ is clearly continuous.

For each $x \in X$ and each $t \in [0, 1]$ the mapping $\gamma \to \exp [it\phi_{\gamma}(x)]$ is a character of $\hat{Y}$ and, as above in the case $t = 1$, there exists a continuous mapping $u_t: X \to Y$ such that $\gamma(r(x)) = \exp [it\phi_{\gamma}(x)]$, $x \in X$, $\gamma \in \hat{Y}$. $u_t$ is a homotopy between $u$ and a constant.
3. Topological conjugacy and groups with polynomially annihilated character groups. Let \( Y \) be a c.c.m.a. group and \( B \) an endomorphism of \( Y \) onto \( Y \). Let \( p(\theta) = n_0 + n_1 \theta + \cdots + n_k \theta^k \) be a polynomial over \( \mathbb{Z} \). We shall say that \( p \) is an annihilating polynomial of \( \gamma \in \hat{Y} \) with respect to \( B \) if \( \gamma^p \cdot B^{\gamma^p} \cdots B^{\gamma^p} = 1 \).

Suppose \( \gamma \in \hat{Y} \) has a nontrivial annihilating polynomial with respect to \( B \). Let \( M_\gamma \) denote the set of all polynomials over \( \mathbb{Q} \) some integral multiple of which is an annihilating polynomial of \( \gamma \) with respect to \( B \). \( M_\gamma \) is an ideal in \( \mathbb{Q}[\theta] \) and therefore there exists a unique monic polynomial \( q_\gamma \in \mathbb{Q}[\theta] \) such that \( M_\gamma \) is the principal ideal generated by \( q_\gamma \) [8, p. 121]. If \( q_\gamma(\theta) = s_0 + s_1 \theta + \cdots + s_{l-1} \theta^{l-1} + \theta^l \) then \( s_0 \neq 0 \) for otherwise \( q_\gamma(\theta) = s_1 + s_2 \theta + \cdots + s_{l-1} \theta^{l-2} + \theta^{l-1} \) would be a monic polynomial generating \( M_\gamma \). If \( n_\gamma \) is the lowest common denominator of the nonzero members of \( s_0, s_1, \ldots, s_{l-1} \) then \( p_\gamma(\theta) = n_\gamma s_0 + n_\gamma s_1 \theta + \cdots + n_\gamma \theta^l \) is a polynomial over \( \mathbb{Z} \) which will be called the minimal annihilating polynomial of \( \gamma \) with respect to \( B \).

Definition 1. Let \( Y \) be a c.c.m.a. group and \( B \) an endomorphism of \( Y \) onto \( Y \). We say that \( \hat{Y} \) is polynomially annihilated by \( B \) if every element of \( \hat{Y} \) has a nontrivial annihilating polynomial with respect to \( B \).

If \( Y \) is an \( n \)-dimensional c.c.m.a. group then \( \hat{Y} \) is polynomially annihilated by any endomorphism \( B \) of \( Y \) onto \( Y \). This follows because \( \hat{Y} \) is isomorphic to a subgroup of the additive group \( \mathbb{Q}^n \) (the direct sum of \( n \) copies of \( \mathbb{Q} \)) and therefore the one-to-one endomorphism \( B \) of \( \hat{Y} \) corresponds to an \( n \times n \) matrix with rational entries and nonzero determinant. The Cayley-Hamilton theorem shows that some integral multiple of the characteristic polynomial of this matrix is an annihilating polynomial, with respect to \( B \), of every element of \( \hat{Y} \). If \( q(\theta) = s_0 + s_1 \theta + \cdots + s_n \theta^n \) is the characteristic polynomial of some matrix representation of \( B \) and if \( n_\gamma \) is the lowest common denominator of the nonzero members of \( s_0, s_1, \ldots, s_{n-1} \), then the polynomial \( p(\theta) = n_\gamma s_0 + n_\gamma s_1 \theta + \cdots + n_\gamma \theta^n \) is a polynomial over \( \mathbb{Z} \), which will be called the annihilating polynomial of \( \hat{Y} \) with respect to \( B \). This polynomial is independent of the matrix representation of \( B \).

The following lemma will be used in the proof of Theorem 3.

Lemma 1. Let \( X \) be a c.c.m.a. group and \( T = a + A \) an affine transformation of \( X \). Suppose \( \Phi: X \to \mathbb{R}^n \) is a nonconstant continuous function and \( M \) is a linear transformation of \( \mathbb{R}^n \) such that \( \Phi(Tx) = M\Phi(x) + d \), \( x \in X \), where \( d \in \mathbb{R}^n \). Then there exists \( \delta \in \hat{X}, \delta \neq 1 \), and a root \( \lambda \), with \( |\lambda| = 1 \), of the characteristic equation of \( M \) such that \( A^p \delta = \delta \) for some \( p \geq 1 \) and \( \delta(a + A(a) + \cdots + A^{p-1}(a)) = \lambda^p \) for all such \( p \).

Proof. We consider \( \mathbb{R}^n \) as a subset of \( \mathbb{C}^n \) (complex \( n \)-space) in the usual way and complexify \( M \). There exists an invertible linear transformation \( U \) of \( \mathbb{C}^n \) such that \( U^{-1}MU = D_M \), the Jordan normal form of the linear transformation \( M \). Therefore \( U^{-1}\Phi(Tx) = D_M U^{-1}\Phi(x) + U^{-1}d \), \( x \in X \). If \( w_1, w_2, \ldots, w_n \) denotes the fixed basis of \( \mathbb{R}^n \) then \( w_1, w_2, \ldots, w_n \) is also a basis, using complex coefficients for \( \mathbb{C}^n \). Suppose \( U^{-1}\Phi(x) = \sum_{i=1}^n f_i(x)w_i \). Each \( f_i: X \to \mathbb{C} \) is continuous, and if \( i_0 \) is the least positive integer for which \( f_{i_0} \) is nonconstant then \( f_{i_0}(Tx) = Mf_{i_0}(x) + e \), \( x \in X \), where \( e \in \mathbb{C} \).
and $\lambda$ is an eigenvalue of $M$. If $I: X \to C$ is defined by $I(x) = \int_X f(x) dm$, where $m$ denotes Haar measure on $X$, then $I(Tx) = \lambda I(x)$ and $I$ is nonconstant and continuous. Since $T$ maps $X$ onto $X$, $\sup_x |I(Tx)| = |\lambda| \sup_x |I(x)|$ implies $|\lambda| = 1$. But $I \in L^2(X)$ and therefore $I(x) = \sum b_i \delta_i(x)$ ($L^2$ convergence) where $\delta_i \in \hat{X}$ and $\sum |b_i|^2 < \infty$. From the equation $I(T^p x) = \lambda^p I(x)$, $p \geq 1$, we have

$$\sum b_i \delta_i(a + Aa + \cdots + A^{p-1}a)\delta_i(A^p x) = \lambda^p \sum b_i \delta_i(x) \quad (L^2 \text{ convergence}).$$

If $\delta_i, A\delta_i, A^2\delta_i, \ldots$ are all distinct then $b_i = 0$ for otherwise the condition $\sum |b_i|^2 < \infty$ is violated. Therefore $b_i \neq 0$ implies $A^p \delta_i = \delta_i$ for some $p \geq 1$ and when this occurs

$$\delta_i(a + Aa + \cdots + A^{p-1}a) = \lambda^p.$$

Since $I(x)$ is nonconstant there must be some $\delta_i \in \hat{X}$, $\delta_i \neq 1$, with this property.

**Theorem 3.** Let $X$ and $Y$ be c.c.m.a. groups. Let $T = a + A$ be an affine transformation of $X$ and $S = b + B$ an affine transformation of $Y$. Suppose further that $\hat{Y}$ is polynomially annihilated by $B$. If there exists a nonaffine continuous mapping $g : X \to Y$ such that $gT = Sg$ then there exists $\delta \in \hat{X}$, $\delta \neq 1$, and a root $\lambda$, with $|\lambda| = 1$, of the minimal annihilating polynomial with respect to $B$ of some element of $\hat{Y}$, such that $A^p \delta(a + Aa + \cdots + A^{p-1}a) = \lambda^p$ for all such $p$.

**Proof.** Using the notation of Corollary 1.1, for $\gamma \in \hat{Y}$ let

$$(\gamma \circ g)(x) = \alpha_\gamma(x) \exp [i\phi_\gamma(x)],$$

where $\alpha_\gamma \in \hat{X}$ and $\phi_\gamma : X \to R$ is continuous. Since $g$ is nonaffine there exists $\gamma_0 \in \hat{Y}$ such that $\phi_{\gamma_0}$ is nonconstant. Applying $\gamma \in \hat{Y}$ to the equation $gT = Sg$ and using the uniqueness asserted in Corollary 1.1 we have $\alpha_\gamma(a) \exp [i\phi_\gamma(Tx)] = \gamma(b) \exp [i\phi_{B\gamma}(x)]$. Since $X$ is connected this implies

$$\phi_\gamma(Tx) = \phi_{B\gamma}(x) + c_\gamma, \quad x \in X,$$

where $c_\gamma \in R$. Suppose that $p_{\gamma_0}$, the minimal annihilating polynomial of $\gamma_0$ with respect to $B$, is of degree $n$. Define $\Phi : X \to R^n$ by

$$\Phi(x) = \begin{bmatrix} \phi_{\gamma_0}(x) \\ \phi_{B\gamma_0}(x) \\ \vdots \\ \phi_{B^{n-1}\gamma_0}(x) \end{bmatrix}, \quad x \in X.$$

$\Phi$ is nonconstant and continuous. If $p_{\gamma_0}(\theta) = m_0 + m_1 \theta + \cdots + m_n \theta^n$, $m_i \in Z$ ($1 \leq i \leq n$), $m_n \neq 0$, then using the connectedness of $X$ we have that $m_n \phi_{B^n\gamma_0}(x) + m_{n-1} \phi_{B^{n-1}\gamma_0} + \cdots + m_0 \phi_{\gamma_0}(x)$ is a constant mapping. Let $M$ denote the linear transformation of $R^n$ given by the matrix
Then $\Phi(Tx) = M\Phi(x) + d$, $x \in X$, where $d \in \mathbb{R}^n$, and the result follows from Lemma 1 since $p_{\gamma_0}$ is the characteristic polynomial of $M$.

**COROLLARY 3.1.** Let $X$, $Y$, $T$, $S$ be as in Theorem 3 with the additional assumption that $T$ is ergodic. If there is a nonaffine continuous mapping $g : X \to Y$ such that $gT = Sg$ then there exists $\delta \in \hat{X}$, $\delta \neq 1$, and a root $\lambda$, with $|\lambda| = 1$, of the minimal annihilating polynomial with respect to $B$ of some element of $\hat{Y}$, such that $\lambda$ is not a root of unity, $A\delta = \delta$ and $\delta(a) = \lambda$.

Hence if $T$ is strong mixing, all continuous mappings $g : X \to Y$ such that $gT = Sg$ are affine.

**Proof.** Let $\delta$ be the element of $\hat{X}$ and $\lambda$ the complex number which are determined by Theorem 3. Since $A^p\delta = \delta$ for some $p \geq 1$, the ergodicity of $T$ implies $A\delta = \delta$ and hence $\delta(a) = \lambda$. If $\lambda$ were a root of unity then since $[a, (A-I)X] = X$, $\delta$ would only assume a finite number of values on $X$ and would have to be the identity character.

Lastly, if $T$ is strong mixing then $A$ is ergodic and there is no $\delta \in \hat{X}$, $\delta \neq 1$, with $A\delta = \delta$.


The notion of a measure-preserving transformation with quasi-discrete spectrum has been defined by Abramov [1], and the notion of a homeomorphism with quasi-discrete spectrum has been defined by Hahn and Parry [4]. An ergodic affine transformation $S = b + B$ of a c.c.m.a. group $Y$ has quasi-discrete spectrum as a (Haar) measure-preserving transformation if and only if it has quasi-discrete spectrum as a homeomorphism. In fact $S = b + B$, assumed to be ergodic, has quasi-discrete spectrum in either sense if and only if $\bigcap_{n=0}^{\infty} (B-I)^n Y = \{0\}$, where $I$ denotes the identity mapping of $Y$ [7]. The following result extends Theorem 6 of the paper [4].

**THEOREM 4.** Let $X$ and $Y$ be c.c.m.a. groups and let $T = a + A$ be an ergodic affine transformation of $X$ and $S = b + B$ an ergodic affine transformation of $Y$. If $S$ has quasi-discrete spectrum then all continuous mappings $g : X \to Y$ satisfying $gT = Sg$ are affine.

**Proof.** Let $\gamma \in \hat{Y}$. There exists $n \geq 1$ such that $(\theta - I)^n$ is an annihilating polynomial of $\gamma$ with respect to $B$. It follows that the roots of the minimal annihilating
polynomial of \( y \) with respect to \( B \) are equal to 1. The result follows from Corollary 3.1.

**Theorem 5.** Let \( Y \) be a c.c.m.a. group and \( S = b + B \) a strong mixing affine transformation of \( Y \) such that \( \hat{Y} \) is polynomially annihilated by \( B \). Then every continuous \( p \)th root \((p \geq 1)\) of \( S \) is an affine transformation and \( S \) has a continuous \( p \)th root if and only if there is an endomorphism \( C \) of \( Y \) onto \( Y \) with \( C^p = B \).

**Proof.** Suppose \( g \) is a continuous \( p \)th root of \( S \). Then \( gS = Sg \) and \( g \) is affine by Corollary 3.1. Since \( S \) is strong mixing \( B \) is ergodic and therefore \((B - I)Y = Y\). Choose \( y_0 \in Y \) so that \((B - I)y_0 = b \) and the homeomorphism \( h: Y \to Y \), defined by \( h(y) = y_0 + y \), satisfies \( hS = Bh \). Therefore \( S \) has a continuous \( p \)th root if and only if \( B \) has a continuous \( p \)th root. Any continuous \( p \)th root of \( B \) is affine and the \( p \)th power of its endomorphism part will be \( B \). Conversely if \( C \) is an endomorphism of \( Y \) onto \( Y \) with \( C^p = B \) then \( C \) is a continuous \( p \)th root of \( B \).

As a special case of Corollary 3.1 we have the following result. If \( Y \) is a c.c.m.a. group and \( B \) is an ergodic endomorphism of \( Y \) onto \( Y \) which polynomially annihilates \( \hat{Y} \), then every continuous mapping commuting with \( B \) is affine. The example below shows that this result is false (and therefore Theorem 3 is false) if the assumption that \( Y \) be polynomially annihilated by \( B \) is dropped.

Let \( K^n \) denote the two-sided infinite-dimensional torus (i.e. the two-sided infinite direct sum of copies of \( K \)) and let \( B \) denote the shift automorphism of \( K^n \) defined by \((Bz)_n = z_{n+1} \) if \( z = (z_n) \). No nontrivial element of \( \hat{K}^n \) is polynomially annihilated by \( B \). Let \( f: K \to K \) be any homeomorphism and define \( F: K^n \to K^n \) by \((F(z))_n = f(z_n)\), \(-\infty < n < \infty\). \( F \) is a homeomorphism and \( FB = BF \). Moreover \( F \) can be chosen to be nonaffine by choosing \( f \) nonaffine.

It would be interesting to know if the condition that \( \hat{Y} \) be polynomially annihilated by \( B \) follows from the fact that every continuous mapping commuting with \( B \) (\( B \) ergodic) is affine.

4. **Converses of Theorem 3.**

**Lemma 2.** Let \( X \) and \( Y \) be c.c.m.a. groups and let them be represented as \( X = \operatorname{inv lim} (X_q, \tau_q) \) and \( Y = \operatorname{inv lim} (Y_m, \sigma_m) \) where \( X_q \) \((q \geq 1)\) and \( Y_m \) \((m \geq 1)\) are finite-dimensional tori. Let \( C \) be a homomorphism of \( X \) onto \( Y \) and let \( u: X \to Y \) be a continuous mapping which depends only on \( X_{q_0} \) and which is homotopic to a constant by a homotopy which depends only on \( X_{k_0} \). Then \( C + u \) maps \( X \) onto \( Y \).

**Proof.** Let \( C_m \) and \( u_m \) \((m \geq 1)\) denote the mappings of \( X \) to \( Y_m \) obtained by projecting \( C \) and \( u \) onto \( Y_m \). \( C + u \) will map \( X \) onto \( Y \) if and only if \( C_m + u_m \) maps \( X \) onto \( Y_m \) for each \( m \geq 1 \). For each \( m \geq 1 \) there exists \( q_m \geq 1 \) such that \( C_m \) only depends on \( X_{q_m} \). Let \( k_m = \max (q_m, k_0) \). Then \( C_m \) can be considered as a homomorphism of \( X_{k_m} \) onto \( Y_m \) and \( u_m \) can be considered as a continuous mapping of \( X_{k_m} \) into \( Y_m \) which is homotopic (on \( X_{k_m} \)) to a constant. The result will follow if we can show that whenever \( C \) is a homomorphism of \( K^n \) onto \( K^m \) and \( u: K^n \to K^m \) is a continuous
mapping homotopic to a constant then \( C + u \) maps \( K^n \) onto \( K^m \). However this result follows from Lemma 1 of [10].

**Lemma 3.** Let \( P : \mathbb{R}^n \to \mathbb{R}^n \) be a continuous mapping such that \( P(v + \tau) = P(v), \) \( v \in \mathbb{R}^n, \) \( \tau \in \mathbb{Z}^n \) and \( \|P(v) - P(v')\| < \|v - v'\|, \) \( v, v' \in \mathbb{R}^n \) where \( \| \cdot \| \) denotes the usual norm in \( \mathbb{R}^n \). Let \( \psi : K^n \to K^n \) be the continuous mapping defined by \( \psi \pi = \pi P, \) where \( \pi : \mathbb{R}^n \to \mathbb{R}^n \) is the natural projection. Then \( I + \psi \) is a one-to-one mapping of \( K^n \).

(I denotes the identity mapping of \( K^n \).)

**Proof.** Let \( I' \) denote the identity mapping of \( \mathbb{R}^n \). \( I' + P \) is a one-to-one mapping because \( v + P(v) = v' + P(v') \) implies \( v - v' = P(v') - P(v) \) and hence \( v = v' \). Suppose \((I + \psi)\pi(v) = (I + \psi)\pi(v')\). Then \( \pi(I' + P)(v) = \pi(I' + P)(v') \) and
\[
(I' + P)(v) = (I' + P)(v') + \tau, \quad \tau \in \mathbb{Z}^n
\]
Therefore \( v = v' + \tau \) and \( \pi(v) = \pi(v') \). This proves that \( I + \psi \) is one-to-one.

The following theorem gives a converse to Theorem 3 in the cases when \( F \) is a finite-dimensional group.

**Theorem 6.** Let \( X \) and \( Y \) be c.c.m.a. groups and suppose that \( Y \) is \( n \)-dimensional. Let \( T = a + A \) be an affine transformation of \( X \), \( S = b + B \) an affine transformation of \( Y \) and suppose there exists a continuous mapping \( h \) of \( X \) onto \( Y \) such that \( h T = S h \). Suppose further there exists \( \delta \in \hat{X}, \delta \neq 1, \) and a root \( \lambda, \) with \( |\lambda| = 1, \) of the annihilating polynomial of \( \hat{Y} \) with respect to \( B \) such that \( A^p \delta = \delta \) for some \( p \geq 1 \) and \( \delta (a + A(a) + \cdots + A^{p-1}(a)) = \lambda^p \) for all such \( p \). Then there exists a nonaffine continuous mapping \( g \) of \( X \) onto \( Y \) such that \( g T = S g \). Moreover, if \( h \) is given to be a homeomorphism then \( g \) can be chosen to be a homeomorphism.

**Proof.** We may as well assume that the given mapping \( h \) is affine or there is nothing to prove. Suppose \( h = c + C, \) where \( c \in Y \) and \( C \) is a homomorphism of \( X \) onto \( Y \). We shall use Theorem 2 and to do this we have to construct continuous mappings \( \phi_y : X \to \mathbb{R} \) for each \( y \in \hat{Y} \).

Since \( Y \) is \( n \)-dimensional \( \hat{Y} \) is isomorphic to a subgroup \( \mathbb{Q}^n \) of the additive group \( \mathbb{Q}^n \) and we can choose \( \mathbb{Q}^n \) so that if \( d_i = (d_{i1}, d_{i2}, \ldots, d_{in}) \) where
\[
d_{ij} = 1 \text{ if } i = j,
\]
\[
= 0 \text{ if } i \neq j,
\]
then \( d_i \in \mathbb{Q}^n \) \((1 \leq i \leq n)\). Let \([M] \) be the matrix (with rational entries) representing the action of \( B \) on \( \mathbb{Q}^n \), and let \( \gamma_i \in \hat{Y} \) correspond under the above isomorphism to \( d_i \in \mathbb{Q}^n \) \((1 \leq i \leq n)\). Let \( M \) denote the linear transformation of \( \mathbb{R}^n \) induced by the matrix \([M] \).

Suppose that \( p \) is the smallest positive integer such that \( A^p \delta = \delta \). Define \( f : X \to \mathbb{C} \) by
\[
f(x) = \sum_{j=0}^{p-1} \frac{\delta (a + A(a) + \cdots + A^{j-1}(a))}{\lambda^j} \delta (A^jx), \quad x \in X,
\]
$f$ is a nonconstant continuous function satisfying $f(Tx) = \lambda f(x)$, $x \in X$. If $w_1, w_2, \ldots, w_n$ denotes the fixed basis of $R^n$ it is also a basis for $C^n$. Let $U$ be the invertible linear transformation of $C^n$ such that $U^{-1}MU = D_M$, the Jordan normal form of the complexified linear transformation $M$. Let $j_0$ be the largest integer for which $w_{j_0}$ corresponds to the eigenvalue $\lambda$ of $D_M$. Then $U(f(x)w_{j_0})$ is nonconstant and so either $RU(f(x)w_{j_0})$ or $SU(f(x)w_{j_0})$ is nonconstant. Suppose, without loss of generality, that $RU(f(x)w_{j_0})$ is nonconstant and define the mappings $\phi_{\gamma}: X \to R$ by

$$\sum_{i=1}^{n} \phi_{\gamma}(x)w_i = RU(f(x)w_{j_0}), \quad x \in X.$$ 

Let $\gamma \in \bar{Y}$. If $\gamma^{m_0} = \gamma^{n_0} \cdot \gamma^{n_1} \cdot \gamma^{n_2} \cdots \lambda^{n_m}$, $m_0, m_1, \ldots, m_n \in Z$, $m_0 \neq 0$, define $\phi_{\gamma}: X \to R$ by

$$\phi_{\gamma}(x) = \frac{m_1}{m_0} \phi_{\gamma_1}(x) + \frac{m_2}{m_0} \phi_{\gamma_2}(x) + \cdots + \frac{m_n}{m_0} \phi_{\gamma_n}(x), \quad x \in X.$$ 

Then $\phi_{\gamma^{m_1}} = \phi_{\gamma} + \phi_{\gamma^{m_1}}$, $\gamma, \gamma^{m_1} \in \bar{Y}$. Also

$$\sum_{i=1}^{n} \phi_{\gamma_i}(Tx)w_i = RU(f(Tx)w_{j_0}) = RU D_M(f(x)w_{j_0}) = M \sum_{i=1}^{n} \phi_{\gamma_i}(x)w_i.$$ 

Therefore $\phi_{B_{\gamma}}(Tx) = \phi_{B_{\gamma}}(x), x \in X, 1 \leq i \leq n$, and hence $\phi_{\gamma}(Tx) = \phi_{B_{\gamma}}(x), x \in X, \gamma \in \bar{Y}$. By Theorem 2 there exists a continuous mapping $u: X \to Y$ such that $\gamma(u(x)) = \exp [i\phi_{\gamma}(x)], x \in X, \gamma \in \bar{Y}, u(Tx) = Bu(x), x \in X$, and $u$ is homotopic to a constant. Let $g: X \to Y$ be defined by $g(x) = c + C(x) + u(x), x \in X$. $g(Tx) = c + C(Tx) + u(Tx) = S(c + C(x)) + Bu(x) = Sg(x), x \in X$.

It remains to show that $g$ maps $X$ onto $Y$. Suppose $X = \text{inv lim} (X_q, \tau_q)$ and $Y = \text{inv lim} (Y_m, \sigma_m)$ where the $X_q$ ($q \geq 1$) are finite-dimensional tori and the $Y_m$ ($m \geq 1$) are $n$-dimensional tori. Suppose the given character $\delta \in \bar{X}_{k_0}$. Then each mapping $\phi_{\gamma}: X \to R$ only depends on $X_{k_0}$ and therefore $u$ only depends on $X_{k_0}$ and is homotopic to a constant by a homotopy depending only on $X_{k_0}$ (Theorem 2). The fact that $g$ maps $X$ onto $Y$ now follows from Lemma 2.

We now show that if $h$ is given to be a homeomorphism then $g$ can be chosen to be a homeomorphism. Since we are assuming $h = c + C$, $C$ will be an isomorphism of $X$ onto $Y$. Let $g_t: X \to Y, t \in [0, 1]$, be defined by $g_t(x) = c + C(x) + u_t(x), x \in X$, where $u_t: X \to Y$ satisfies $\gamma(u_t(x)) = \exp [it\phi_{\gamma}(x)], x \in X, \gamma \in \bar{Y}$. By Lemma 2 $g_t$ is a continuous mapping of $X$ onto $Y$ and $g_T = Sg_t, t \in [0, 1]$. We shall show that $g_t$ is one-to-one for sufficiently small $t$.

It suffices to show that $g_t \circ C^{-1}: Y \to Y$ is one-to-one for sufficiently small $t$. We have $g_t \circ C^{-1}(y) = c + y + u_t \circ C^{-1}y, y \in Y$. Let $k$ be the smallest integer for which $\delta \circ C^{-1} \in \bar{Y}_k$, where $\delta$ is the given element of $\bar{X}$. By the definition of $\phi_{\gamma}$,
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\[ \gamma \in \hat{Y}, \] each \( \phi_{\gamma} \circ C^{-1} \) can be considered as a real-valued function of \( Y_{k} \), and therefore induces a mapping \( P_{\gamma} : \mathbb{R}^{n} \to \mathbb{R} \) defined by \( P_{\gamma}(v) = \phi_{\gamma} \circ C^{-1}(v_{y}), v \in \mathbb{R}^{n} \), where \( y_{y} \) is any point of \( Y \) which has component \( v + \mathbb{Z}^{n} \) in \( Y_{k} \). Since each \( P_{\gamma} \) is a linear combination of sines and cosines of the coordinates of \( \mathbb{R}^{n} \), there exists a constant \( N \) such that if \( \beta \) is a generator of any \( Y_{m} (m \geq 1) \) then

\[ |P_{\beta}(v) - P_{\beta}(v')| \leq N \|v - v'\|, \quad v, v' \in \mathbb{R}^{n}, \]

where \( \|\cdot\| \) denotes the usual norm in \( \mathbb{R}^{n} \).

Choose \( t_{0} \in [0, 1] \) so that \( nt_{0}N < 1 \). Let \( y, y' \in Y, y \neq y' \). Suppose \( y = (y_{1}, y_{2}, \ldots), y' = (y'_{1}, y'_{2}, \ldots) \) where \( y_{i}, y'_{i} \in Y_{i}, i \geq 1 \). We shall show that \( g_{t_{0}} \circ C^{-1}(y) \neq g_{t_{0}} \circ C^{-1}(y') \), \( y \in \hat{Y} \), and therefore \( u_{t_{0}} \circ C^{-1}(y) = u_{t_{0}} \circ C^{-1}(y') \). Hence \( g_{t_{0}} \circ C^{-1}(y) - g_{t_{0}} \circ C^{-1}(y') = y - y' \neq 0 \). Now suppose \( y_{k} = y'_{k} \). Considering \( Y_{k} \) as an \( n \)-torus let \( \beta_1, \beta_2, \ldots, \beta_n \in \hat{Y}_{k} \) be defined by \( \beta_j(z_1, \ldots, z_n) = \exp(2\pi iz_j) \). Define \( G : Y_{k} \to Y_{k} \) by

\[ G(z_1, \ldots, z_n) = (z_1 + t_{0}\phi_{\beta_1} \circ C^{-1}(y_2), \ldots, z_n + t_{0}\phi_{\beta_n} \circ C^{-1}(y_n)) + \mathbb{Z}^{n} \]

where \( y_{k} \) is any point of \( Y \) having \( z = (z_1, \ldots, z_n) \) as its component in \( Y_{k} \). By Lemma 3, since \( t_{0} \) is chosen so that \( nt_{0}N < 1 \), we have that \( G \) is one-to-one. Since \( y_{k} \neq y'_{k} \), \( G(y_{k}) \neq G(y'_{k}) \), i.e., \( \beta_j(y_{k} + u_{t_{0}} \circ C^{-1}(y)) \neq \beta_j(y'_{k} + u_{t_{0}} \circ C^{-1}(y')) \) for some \( j \). Therefore \( y + u_{t_{0}} \circ C^{-1}(y) \neq y' + u_{t_{0}} \circ C^{-1}(y') \), i.e., \( g_{t_{0}} \circ C^{-1}(y) \neq g_{t_{0}} \circ C^{-1}(y') \).

The following is a direct consequence of Theorems 3 and 6.

**Corollary 6.1.** Let \( X \) be a c.c.m.a. group and let \( Y \) be a c.c.m.a. \( n \)-dimensional group. Let \( T = a + A \) be an affine transformation of \( X \) and \( S = b + B \) an affine transformation of \( Y \) for which there exists a continuous mapping \( h \) of \( X \) onto \( Y \) satisfying \( hT = Sh \). There exists a nonaffine continuous mapping \( g \) of \( X \) onto \( Y \) such that \( gT = Sg \) if and only if there exists a root \( \lambda \) of the annihilating polynomial of \( Y \) with respect to \( B \) such that \( \lambda^{n} = 1 \) and \( \delta(a + A(a) + \cdots + A^{n-1}(a)) \neq 0 \) for all such \( p \). If \( h \) is a homeomorphism, the above conditions are necessary and sufficient for the existence of a nonaffine homeomorphism \( g \) of \( Y \) such that \( gT = Sg \).

If \( B \) is an endomorphism of a c.c.m.a. \( n \)-dimensional group \( Y \) onto \( Y \) then it follows from the ergodicity conditions stated in §1 that \( B \) is ergodic if and only if no root of the annihilating polynomial of \( Y \) with respect to \( B \) is a root of unity. Moreover there is an element \( \gamma \in \hat{Y}, \gamma \neq 1 \) such that \( B^{n}\gamma = \gamma \) if and only if the annihilating polynomial of \( Y \) with respect to \( B \) has a \( p \)th root of unity as a root.

**Theorem 7.** Let \( Y \) be a c.c.m.a. \( n \)-dimensional group and let \( A \) and \( B \) be endomorphisms of \( Y \) onto \( Y \). Suppose there exists a continuous mapping \( h \) of \( Y \) onto \( Y \) such that \( hA = Bh \). There exists a nonaffine continuous mapping \( g \) of \( Y \) onto \( Y \) such that \( gA =Bg \) if and only if \( A \) and \( B \) are not ergodic. If \( h \) is given to be a homeomorphism then there exists a nonaffine homeomorphism \( g \) of \( Y \) such that \( gA =Bg \) if and only if \( A \) and \( B \) are not ergodic.
Proof. If there exists a nonaffine continuous mapping \( g \) of \( Y \) into \( Y \) satisfying \( gA =Bg \) then Theorem 3 asserts the existence of \( \delta \in \hat{Y}, \delta \neq 1 \), and a root \( \lambda \) of the annihilating polynomial of \( \hat{Y} \) with respect to \( B \) such that \( A^p\delta = \delta \) for some \( p \geq 1 \) and \( \lambda^p = 1 \) for all such \( p \). Therefore \( A \) and \( B \) are not ergodic.

Conversely suppose \( A \) and \( B \) are not ergodic. Suppose \( h \) is affine or there is nothing to prove. Let \( h = c+ C \), where \( c \in Y \) and \( C \) is an endomorphism of \( Y \) onto \( Y \). If \( \hat{Y} \) is considered as an (additive) subgroup of \( \mathbb{Q}^n \) the nonsingular matrix representing \( C \) is a conjugacy between the matrix representing \( A \) and the matrix representing \( B \). Hence the annihilating polynomial of \( \hat{Y} \) with respect to \( A \) is the same as the annihilating polynomial of \( \hat{Y} \) with respect to \( B \). Let \( \delta \in \hat{Y}, \delta \neq 1 \), be such that \( A^p\delta = \delta \) for some \( p \geq 1 \). Let \( p \) be the least positive integer for which \( A^p\delta = \delta \). Then the annihilating polynomial of \( \hat{Y} \) with respect to \( B \) has a root \( \lambda \) which is a \( p \)th root of unity. The result now follows from Theorem 6.

We now give an example to show that Theorem 6 is false if the assumption that \( \hat{Y} \) is finite-dimensional is replaced by the assumption that \( \hat{Y} \) is polynomially annihilated by \( B \), i.e. the converse of Theorem 3 is false.

Let \( E \) denote the automorphism of the 4-torus \( K^4 \) determined by the matrix

\[
\begin{bmatrix}
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 8 \\
0 & 1 & 0 & -6 \\
0 & 0 & 1 & 8
\end{bmatrix}
\]

The matrix \([E]\) has two eigenvalues \( \lambda_1, \lambda_2 \) of unit modulus which are not roots of unity and two distinct real eigenvalues \( \lambda_3, \lambda_4 \) [10]. Let \( W \) denote the one-sided direct sum of an infinite number of copies of \( K^4 \). Let \( Y = K + W \). \( Y \) is an infinite-dimensional torus. Let \( S = b + B : Y \to Y \) be defined by

\[
S(y_0, y_1, y_2, \ldots) = (b_0, 0, 0, \ldots) + (y_0, Ey_1, y_1 + Ey_2, y_2 + Ey_3, \ldots)
\]

\( y_0 \in K, \, y_i \in K^4 \) (\( i \geq 1 \)), where \( \exp [2\pi ib_0] = \lambda_1 \). It is not difficult to show that \( S \) is ergodic and \( Y \) is polynomially annihilated by \( B \). The characteristic polynomial of \([E]\) is the minimal annihilating polynomial with respect to \( B \) of some of the elements of \( \hat{Y} \) and \( \lambda_1 \) is a root of this polynomial. If \( \delta \in \hat{Y} \) is defined by

\[
\delta(y_0, y_1, y_2, \ldots) = \exp [2\pi iy_0],
\]

then \( B\delta = \delta \) and \( \delta(b) = \lambda_1 \). Hence (with \( X = Y \) and \( T = S \)) all the assumptions of Theorem 6 (except that \( Y \) be finite-dimensional) are satisfied by this example. However, we shall show that every continuous mapping commuting with \( S \) is affine.

Suppose \( gS = Sg \) where \( g \) is continuous. Let \( g_n \) (\( n \geq 0 \)) be the projection of \( g \) onto the \( n \)th factor in the representation \( Y = K + K^4 + K^4 + \cdots \). \( g_0 \) is a continuous mapping of \( Y \) into \( K \) and \( g_n \) (\( n \geq 1 \)) are continuous mappings of \( Y \) into \( K^4 \). We shall show that each \( g_n \) (\( n \geq 0 \)) is affine and this implies \( g \) is affine. By Theorem 1 there
exists a homomorphism $\mu_0: Y \to K$ and a continuous mapping $\phi_0: Y \to R$ such that $g_0(y) = \mu_0(y) + \phi_0(y) + Z$. Since $g_0(Sy) = b_0 + g_0(y)$ we have

$$\phi_0(Sy) = \phi_0(y) + a, \quad y \in Y,$$

where $a \in R$.

Therefore $\phi_0(y) - \int_y \phi_0(y) \, dm$ (m denotes Haar measure on $Y$) is an invariant function under $S$ and therefore constant. Hence $\phi_0$ is constant and $g_0$ is affine. Suppose that some $g_n (n \geq 1)$ is nonaffine. Let $k$ be the least integer for which $g_k$ is nonaffine. By Theorem 1 there exist homomorphisms $\mu_i: Y \to K$ and continuous mappings $\phi_i: Y \to R$ ($1 \leq i \leq 4$) such that

$$g_k(y) = \begin{bmatrix} \mu_1(y) + \phi_1(y) \\ \mu_2(y) + \phi_2(y) \\ \mu_3(y) + \phi_3(y) \\ \mu_4(y) + \phi_4(y) \end{bmatrix} + Z^4.$$

Since $g_S = Sg$ we have $g_kS = g_{k-1} + Eg_k$ and $g_{k+1}S = g_k + Eg_{k+1}$. Since $g_{k-1}$ is affine the uniqueness in Theorem 1 gives

$$g_k(y) = \begin{bmatrix} \mu_1(y) + \phi_1(y) \\ \mu_2(y) + \phi_2(y) \\ \mu_3(y) + \phi_3(y) \\ \mu_4(y) + \phi_4(y) \end{bmatrix} + e,$$

where $e \in R^4$.

Let $D$, with matrix

$$[D] = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix},$$

be the Jordan normal form of $E$ and let $U: C^4 \to C^4$ be the linear transformation such that $U^{-1}EU = D$. By the type of argument used in the proof of Theorem 3 it follows that

$$\begin{bmatrix} \phi_1(y) \\ \phi_2(y) \\ \phi_3(y) \\ \phi_4(y) \end{bmatrix} = U \begin{bmatrix} c\delta(y) \\ d\delta^{-1}(y) \\ 0 \\ 0 \end{bmatrix} + e', \quad y \in Y,$$

where $c, d \in C$ and $e' \in R^4$. By Theorem 1 again

$$g_{k+1}(y) = \begin{bmatrix} \mu_5(y) + \phi_5(y) \\ \mu_6(y) + \phi_6(y) \\ \mu_7(y) + \phi_7(y) \\ \mu_8(y) + \phi_8(y) \end{bmatrix}, \quad y \in Y,$$
where $\mu_i: Y \to K$ are homomorphisms and $\phi_i: Y \to R$ are continuous ($5 \leq i \leq 8$).

Since $g_{k+1}(Sy) = g_k(y) + Eg_{k+1}(y)$ we have

$$
\begin{pmatrix}
\phi_0(Sy) \\
\phi_0(y) \\
\phi_\tau(Sy) \\
\phi_\tau(y)
\end{pmatrix}
= E
\begin{pmatrix}
\phi_0(y) \\
\phi_\tau(y) \\
\phi_\tau(y) \\
\phi_\tau(y)
\end{pmatrix}
+ U
\begin{pmatrix}
e \delta(y) \\
d \delta^{-1}(y) \\
0 \\
0
\end{pmatrix}
+ e^\tau,
\quad y \in Y,
$$

where $e^\tau \in R^4$. Apply $U^{-1}$ to this equation and set

$$
\begin{pmatrix}
f_1(y) \\
f_2(y) \\
f_3(y) \\
f_4(y)
\end{pmatrix}
= U^{-1}
\begin{pmatrix}
\phi_0(y) \\
\phi_\tau(y) \\
\phi_\tau(y) \\
\phi_\tau(y)
\end{pmatrix}.
$$

Then $f_1(Sy) = \lambda_1 f_1(y) + e \delta(y) + c^1$, where $c^1 \in C$. Since $f_2 \in L^2(Y)$ let $f_2(y) = \sum a_i \gamma_i(y)$ ($L^2$ convergence) where $\gamma_i \in \hat{Y}$ and $\sum |a_i|^2 < \infty$. If $\delta = \gamma_{i_0}$ then $a_{i_0} \delta(b) = \lambda_1 a_{i_0} + c$, and since $\delta(b) = \lambda_1$ this gives $c = 0$. Consideration of the equation for $f_2$ implies $d = 0$. Therefore $\phi_i, 1 \leq i \leq 4$, are constant and $g_k$ is affine, a contradiction. Therefore each $g_n (n \geq 0)$ is affine.

Thus we have shown that every continuous mapping commuting with $S$ is affine.

We shall now state, without proof, a generalization of Theorem 6. If $B$ is an endomorphism of a c.c.m.a. group $Y$ onto $Y$ we denote by $\hat{Y}(B, A)$ the subgroup of $\hat{Y}$ generated by those elements of $\hat{Y}$ whose minimal annihilating polynomials with respect to $B$ have $\lambda$ as a root.

**Theorem 8.** Suppose $X$ and $Y$ are c.c.m.a. groups, $T = a + A$ an affine transformation of $X$, and $S = b + B$ an affine transformation of $Y$ such that $\hat{Y}$ is polynomially annihilated by $B$. Suppose there exists a continuous mapping $h$ of $X$ onto $Y$ such that $hT = Sh$. Also assume there exists $\delta \in \hat{X}$, $\delta \neq 1$, and a complex number $\lambda$ with $|\lambda| = 1$, such that $\hat{Y}(B, \lambda)$ is a subgroup of $\hat{Y}$ of finite rank with the properties that $A^p \delta = \delta$ for some $p \geq 1$ and $\delta(a + A(a) + \cdots + A^{p-1}(a)) = \lambda^p$ for all such $p$. Then there exists a nonaffine continuous mapping $g$ of $X$ onto $Y$ such that $gT = Sg$. If $h$ is a homeomorphism then $g$ can be chosen to be a homeomorphism.

**References**


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