ASYMPTOTIC BEHAVIOR OF MEROMORPHIC
FUNCTIONS WITH EXTREMAL DEFICIENCIES

BY

ALLEN WEITSMAN(1)

Introduction. This paper continues and completes the preceding one of A. Edrei. I shall adopt the terminology, the bibliographical references and all the notations and conventions of Edrei’s paper. Whenever necessary, I shall refer to it as [L]. In view of my frequent use of specific formulae of this paper, as well as of [2], I shall write, for instance, [L, (2.9)] or [2, (2.9)] to denote, respectively, formula (2.9) of [L] or of [2]. Other references will be denoted in the same way as is done in [L].

One of the aims of my investigation is the completion of the proof of Theorem A of [L]. Since the relation [L, (7)] is already proved I have only to examine [L, (8)]. Using Theorem 2 of [L], Edrei had previously proved [L, (8)] for values of $p$ belonging to the sequence

\[ \{1/2+1/2q\} \quad (q = 1, 2, \ldots). \]


The methods which I develop here enable me to prove [L, (8)] for all $\mu$ in the interval \((1/2, 1)\). They may be summarized as follows:

I. Consider the sets $E_0(r_m)$ and $E_\alpha(r_m)$ which appear in Theorem 1 of [L]. The limits of their measures have been determined but it is still possible that these sets be the union of many disjoint intervals. I first show that in some sense each of the sets $E_0(r_m)$ and $E_\alpha(r_m)$ is “essentially” an interval.

II. This enables me to return to the distribution of the zeros and poles lying in the annuli

\[ R'_m < r = |z| \leq R_m \quad (R'_m < r_m < R_m) \]

where $R'_m$, $r_m$, and $R'_m$ are quantities satisfying [L, (2.7)]. I prove that almost all the poles in (1) have arguments close to some quantity $\omega_m$ and almost all the zeros have arguments close to $\omega_m + \pi$.

III. This knowledge about the zeros and poles of $f$ in (1) is sufficient to determine the asymptotic behavior of $f(z)$ on some circles in the annuli (1).

IV. Theorem 2 of [L] shows that these arguments may be applied to $f'(z)$. The asymptotic evaluation mentioned above, applied to $f'(z)$, indicates that there exists a circle in the annulus (1) such that $f'(z)$ is very small on a single arc $C_m$ of

Received by the editors February 5, 1968.

(1) The research of the author was supported in part by a grant from the National Science Foundation GP-7407. It was done, under the guidance of Professor Edrei, in partial fulfillment of the Ph.D. requirements at Syracuse University.

333

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
the circumference. By an obvious integration we then verify that $f(z)$ is practically constant on $C_n$. On the complementary arc $f(z)$ is very large so that $f(z)$ can have only one finite deficient value.

1. **Statement of the main results.** In addition to the notations of [L] I require the following ones, which will enable me to conveniently refer to some sets which appear in my proofs.

Throughout this paper, I denote by $C$ the set of all the arguments $\theta$ such that $-\pi < \theta \leq \pi$.

Since we are only interested in the circular arrangement of the elements of $C$, the points $\theta = -\pi$ and $\theta = \pi$ will be "identified" and, more generally, all the values $\theta + 2k\pi$ ($k = 0, \pm 1, \pm 2, \pm 3, \ldots$) will be considered as different numerical representations of a single element of $C$.

Beside $C$, I introduce

I. The sector

$$S(\omega, \gamma; R', R^*) = \{z: \omega - \gamma < \arg z \leq \omega + \gamma; R' < |z| \leq R^*\}. $$

II. Put $\theta = \arg z$. The "interval" $\omega - \gamma < \theta \leq \omega + \gamma$, considered as a subset of $C$, will be denoted by $S(\omega, \gamma)$.

III. I extend Nevanlinna's notation and denote by $n(\mathscr{D}, f)$ the number of poles of $f(z)$ which fall in the bounded set $\mathscr{D}$. (Multiple poles are counted as often as indicated by their multiplicity.)

With these conventions, we obtain a natural complement to Theorem 1 of [L].

**Theorem 1.** Let $f(z)$ be a meromorphic function of lower order $\mu$ ($0 < \mu < 1$) and let

$$\limsup_{r \to \infty} \frac{N(r, 1/f)}{T(r, f)} = u, \quad \limsup_{r \to \infty} \frac{N(r, f)}{T(r, f)} = v, $$

where $\mathscr{D}$ is any fixed set of density zero.

Assume that $u$ and $v$ satisfy

$$u < 1, \quad v < 1$$

and

$$\sin^2 \pi \mu = u^2 + v^2 - 2uv \cos \pi \mu. $$

Then, with every sequence $\{r_m\}$ of Pólya peaks of order $\mu$ of $T(r, f)$, it is possible to associate four sequences $\{\omega_m\}$, $\{\eta_m\}$, $\{\rho'_m\}$, and $\{\rho^*_m\}$ having the following properties:

$$0 < \eta_m < \pi \quad (m = 1, 2, \ldots), \quad \eta_m \to \pi \quad as \ m \to \infty, $$

$$\rho'_m \to +\infty, \quad r_m/\rho'_m \to +\infty, \quad \rho^*_m/\rho'_m \to +\infty \quad as \ m \to \infty, $$

and

$$n(\mathscr{D}(\omega_m, \eta_m; \rho'_m, \rho^*_m), 1/f) = o(T(r_m, f)), $$

$$n(\mathscr{D}(\omega_m + \pi, \eta_m; \rho'_m, \rho^*_m), f) = o(T(r_m, f)), $$

as $m \to \infty$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Theorem 1 enables us to obtain an asymptotic evaluation of \( f(z) \) which leads to

**Theorem 2.** Let the assumptions and notations of Theorem 1 be unchanged, and let \( s(0) \) and \( s(\infty) \) be the quantities defined by \([L, (2.4)]\) and \([L, (2.5)]\), and let \( \varepsilon \) \((0 < \varepsilon < \frac{1}{2} \min \{\sigma(\infty), \sigma(0)\}) \) be given.

Then there exists a sequence \( \{\omega_m\} \), a positive sequence \( \{\sigma_m\} \) \((\sigma_m \to +\infty)\) and a constant \( K > 0 \), such that

\[
\log |f(re^{i\theta})| > KT(r, f) \quad (\theta \in \Gamma(\omega_m, s(\infty)/2 - \varepsilon)),
\]

\[
\log |f(re^{i\theta})| < -KT(r, f) \quad (\theta \in \Gamma(\omega_m + \pi, s(0)/2 - \varepsilon)),
\]

provided

(i) \( r \to +\infty \) in the intervals \( \sigma_m^{-1} r_m < r \leq \sigma_m r_m \);

(ii) \( r \) avoids in each of these intervals an exceptional set \( \mathcal{E}_m \) of measure not greater than \( \sigma_m^{-2} r_m \).

From this theorem we deduce at once that:

The values of \( r \) for which the inequalities (1.7) are valid have upper density one.

Theorem 2 and the well-known relations between a function and its derivative lead to

**Theorem 3.** Let \( f(z) \) be a meromorphic function satisfying the conditions of \( \text{Theorem 1}. \)

Then, if \( F'(z) = f(z) \), and if \( F(z) \) is meromorphic, it has at most two deficient values.

Theorem 3 is not vacuous because the meromorphic function

\[
F(z) = \frac{\prod_{j=1}^{\infty} (1 + zn^{-1/\mu})}{\prod_{j=1}^{\infty} (1 - zn^{-1\mu})} \quad (\frac{1}{\mu} < \mu < 1)
\]

satisfies the conditions \( \delta(0, F) = 1 - \sin \pi \mu, \delta(\infty, F) = 1 \) [see for example R. Nevanlinna, *Eindeutige analytische Funktionen*, p. 232], and hence, in view of Theorem 2 of \([L]\), \( u(F') = \sin \pi \mu, \nu(F') = 0 \).

This shows that \( f = F' \) satisfies the conditions of \( \text{Theorem 1}. \)

It might be of interest to investigate whether there exist functions \( f(z) \), satisfying the conditions of \( \text{Theorem 1} \) with \( 0 < \delta(\infty, f) < 1 \), and having a meromorphic integral. I am at present unable to answer this question.

As an immediate consequence of \( \text{Theorem 3} \) and \([L, \text{Theorem 2}]\) we now obtain \([L, (8)]\) which I restate for completeness.

If \( f(z) \) is a meromorphic function of lower order \( \mu \) \((\frac{1}{\mu} < \mu < 1)\), if \( \delta(\infty, f) = 1 \), and if \( \Delta(f) = 2 - \sin \pi \mu \), then \( \nu(f) = 2 \).

Hence any function \( f(z) \) satisfying the above conditions has precisely one finite deficient value \( \tau \), such that \( \delta(\tau, f) = 1 - \sin \pi \mu \), and \( f(z) - \tau \) has the asymptotic behavior described in \( \text{Theorem 2} \).
2. Structure of the sets $E_0(r_m)$ and $E_\omega(r_m)$. Let $E$ denote a measurable subset of $C$ and let

$$\gamma = \frac{1}{2} \text{meas } E.$$  

Consider the function $M(\omega) = \text{meas } \{ E - \Gamma(\omega, \gamma) \}$ which is clearly a nonnegative function of $\omega$, defined and continuous on $C$. Let $\omega$ be any one of the values of $\omega$ such that

$$M(\omega) = \inf_{\omega \in C} M(\omega) = \chi.$$  

We shall say that $\omega$ is a center of $E$.

The inequalities $0 \leq M(\omega) \leq 2\gamma$, $M(\omega) \leq 2(\pi - \gamma)$, are obvious.

If $\gamma = 0$ or $\gamma = \pi$, we have $M(\omega) = 0$ and $\chi = 0$ (trivially); in both cases, every $\omega \in C$ is a center of $E$. If $0 < \gamma < \pi$, the inequality $\chi > 0$ is possible; the quantity $\chi$ then represents, in some sense, the total measure of the "gaps" in $E$.

If $\gamma > 0$ and $\chi = 0$, we may think of $E$ as being, apart from a set of zero measure, an interval on $C$. The following lemma shows that, for functions satisfying (1.3), the sets $E_\omega(r_m)$ and $E_0(r_m)$, tend, as $m \to +\infty$, toward this "single interval" structure.

**Lemma 1.** Let $f(z)$ satisfy the hypotheses of Theorem 1, and let $\{r_m\}$ be a sequence of Pólya peaks of order $\mu$ of $T(r, f)$. Let

$$\gamma_m = \frac{1}{2} \text{meas } E_\omega(r_m).$$  

Then, there exists a sequence $\{\omega_m\}$ such that

$$\lim_{m \to \infty} \text{meas } \{ E_\omega(r_m) - \Gamma(\omega_m, \gamma_m) \} = 0,$$

and

$$\lim_{m \to \infty} \text{meas } \{ E_0(r_m) - \Gamma(\omega_m + \pi, \pi - \gamma_m) \} = 0.$$  

Before proving Lemma 1 we prove two elementary lemmas.

**Lemma 2.** Let $E$ be a measurable subset of $C$. Then, if $w$ is any value, real or complex,

$$\frac{1}{2\pi} \int_E |\log |1 - we^{i\theta}| | d\theta \leq \left\{ \log (1 + |w|) + \left( 1 + \log^+ \frac{1}{\text{meas } E} \right) \right\} \text{meas } E.$$  

**Proof.** Put $\arg w + \theta = \phi$.

Then

$$|1 - we^{i\theta}| = |1 - |w|e^{i\phi}| = |e^{-i\theta} - |w| | \geq |\sin \phi|,$$

which may be sharpened to

$$|1 - we^{i\theta}| \geq 1,$$

if $\pi/2 \leq |\phi| \leq \pi$.  

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Now

\[(2.6) \quad |\sin \phi| \geq (2/\pi)|\phi| \quad (|\phi| \leq \pi/2)\]

and hence, if \(\text{meas } E = \mathcal{X}\) and

\[(2.7) \quad \alpha = \min \{\pi, \mathcal{X}\},\]

we observe, with Edrei and Fuchs [7, p. 338], that (2.4), (2.5), (2.6), and (2.7) imply

\[
I(E) = \frac{1}{2\pi} \int_E \log^+ \left| \frac{1}{1-we^{i\theta}} \right| d\theta \leq \frac{1}{\pi} \int_0^{\alpha/2} \log \left( \frac{\alpha/2}{\sin \phi} \right) d\phi
\]

\begin{align*}
\leq \frac{1}{\pi} \int_0^{\alpha/2} \log \left( \frac{\pi}{2\phi} \right) d\phi & \leq \int_0^{\alpha/2\pi} \log t \, dt.
\end{align*}

By definition \(\alpha/2\pi \leq \mathcal{X}/2\pi \leq 1\), so that (2.8) yields

\[(2.9) \quad I(E) \leq -\int_0^{\mathcal{X}/2\pi} \log t \, dt = \frac{\mathcal{X}}{2\pi} + \frac{\mathcal{X}}{2\pi} \log \left( \frac{2\pi}{\mathcal{X}} \right) \leq \mathcal{X} \left( 1 + \log^+ \frac{1}{\mathcal{X}} \right).\]

We now obtain Lemma 2 by integrating over \(E\) the obvious relation

\[\log |1-we^{i\theta}| \leq \log (1+|w|) + \log^+ |1/(1-we^{i\theta})|,\]

and using the estimate (2.9).

**Lemma 3.** Let \(E\) be a measurable subset of \(C\), and let

\[(2.10) \quad \text{meas } E = 2\gamma.\]

Assume

\[(2.11) \quad \text{meas } \{E - \Gamma(0, \gamma)\} \geq 2\xi.\]

Then, if \(t\) is restricted to the range

\[(2.12) \quad \sigma^{-1} \leq t \leq \sigma \quad (1 < \sigma < +\infty),\]

we have

\[
\frac{1}{2\pi} \int_{-\gamma}^\gamma \log |1+te^{i\theta}| \, d\theta - \frac{1}{2\pi} \int_E \log |1+te^{i\theta}| \, d\theta \geq K = K(\sigma, \xi).
\]

The constant \(K\) which appears in (2.13) may be chosen equal to

\[(2.14) \quad K(\sigma, \xi) = 2\xi \sin^2 (\xi/2)/\sigma(4+\sigma)(1+\sigma)^2,\]

which is clearly positive for \(0 < \xi \leq \pi/2\).

**Proof.** Put

\[(2.15) \quad \{E - \Gamma(0, \gamma)\} = G_1, \quad \{E \cap \Gamma(0, \gamma)\} = G_2,\]

so that \(E = \{G_1 \cup G_2\}, \{G_1 \cap G_2\} = 0\). Hence, in view of (2.10) and (2.11), we have

\[(2.16) \quad 0 \leq 2\xi \leq 2\gamma = \text{meas } G_1 \leq 2(\pi - \gamma), \quad \text{meas } G_2 = 2(\gamma - \eta).\]
We now use the familiar remark that, for any fixed $t > 0$, $\log |1 + te^{i\theta}|$ is an even function of $\theta$, strictly decreasing as $\theta$ varies from 0 to $\pi$. By (2.15) and (2.16), this leads to the obvious inequalities:

$$\int_{\alpha_2} \log |1 + te^{i\theta}| \, d\theta \leq 2 \int_{0}^{\pi - \eta} \log |1 + te^{i\theta}| \, d\theta,$$

and

$$\int_{\alpha_1} \log |1 + te^{i\theta}| \, d\theta \leq 2 \int_{\pi}^{\pi + \eta} \log |1 + te^{i\theta}| \, d\theta = 2 \int_{\pi - \eta}^{\pi} \log |1 + te^{i(\phi + \eta)}| \, d\phi,$$

which, when added, yield

$$\int_{\eta} \log |1 + te^{i\theta}| \, d\theta \leq 2 \int_{0}^{\pi - \eta} \log |1 + te^{i\theta}| \, d\theta - 2 \int_{\pi - \eta}^{\pi} \log |1 + te^{i(\theta + \eta)}| \, d\theta.$$ (2.17)

Consider now the positive function

$$H(t, \theta, \eta) = \left| \frac{1 + te^{i\theta}}{1 + te^{i(\theta + \eta)}} \right|^2 = 1 + \frac{2t(\cos \theta - \cos (\theta + \eta))}{1 + t^2 + 2t \cos (\theta + \eta)},$$

which appears in the last integral of (2.17). From (2.16) we deduce $\gamma + \eta/2 \leq \pi - \eta/2$, $\eta/2 \leq \gamma - \eta/2$, and hence

$$\cos \theta - \cos (\theta + \eta) = 2 \sin (\theta + \eta/2) \sin \eta/2 \geq 2 \sin^2 (\eta/2) \quad (\gamma - \eta \leq \theta \leq \gamma).$$ (2.19)

Combining (2.18), (2.12) and (2.19), we find

$$\log H(t, \theta, \eta) \geq \log (1 + \xi) > \frac{\xi}{1 + \xi} \geq \frac{4 \sin^2 (\eta/2)}{4 + \sigma(1 + \sigma)^2},$$

$$\int_{\pi - \eta}^{\pi} \log H(t, \theta, \eta) \, d\theta \geq \frac{4 \eta \sin^2 (\eta/2)}{4 + \sigma(1 + \sigma)^2},$$

and, since $0 < \xi \leq \eta$, it is obvious that (2.17), (2.18) and (2.19) imply (2.13) and (2.14). This completes the proof of Lemma 3.

**Proof of Lemma 1.** The assumptions of Lemma 1 coincide with those of Theorem 1 so that (1.1), (1.2) and (1.3) hold. Then, in view of assertion II of [L, Theorem 1], we have

$$0 < \beta = \lim_{m \to \infty} \gamma_m = \frac{1}{\mu} \cos^{-1} v = \frac{s(\infty)}{2} < \pi \quad \left(0 \leq \cos^{-1} v \leq \frac{\pi}{2}\right),$$ (2.20)

where $2\gamma_m = \text{meas} \ E_{\infty}(r_m)$.

Let $\omega_m$ be a center of $E_{\infty}(r_m)$; we first examine the implications of

$$\limsup_{m \to \infty} \{E_{\infty}(r_m) - \Gamma(\omega_m, \gamma_m)\} \neq 0.$$ (2.21)
From (2.21) we deduce the existence of a constant $\xi > 0$ and of an unbounded sequence $\mathcal{M}$, of positive integers, such that
\begin{equation}
\text{meas} \{ E_\omega(r_m) - \Gamma(\omega_m, \gamma_m) \} \geq 2\xi \quad (m \in \mathcal{M}).
\end{equation}

Let $\sigma > 1$ be a given, fixed quantity and let $a = |a|e^{i\theta}$ be any one of the zeros of $f(z)$ such that
\begin{equation}
\sigma^{-1}r_m < |a| \leq \sigma r_m.
\end{equation}

In view of the extremal character of the centers $\omega_m$, the inequalities (2.22) remain true if $\omega_m$ is replaced by any other point of $C$; in particular
\begin{equation}
\text{meas} \{ E_\omega(r_m) - \Gamma(\psi + \pi, \gamma_m) \} \geq 2\xi \quad (m \in \mathcal{M}).
\end{equation}

The transformation of the set $C$ defined by
\begin{equation}
\phi = \theta - \psi - \pi \quad (\theta \in C),
\end{equation}
is a "translation" which leaves $C$ invariant and transforms the subsets of $C$ without affecting their measures. In particular, the sets $E_\omega(r_m)$, $\Gamma(\psi + \pi, \gamma_m)$ are transformed, respectively, into sets $\tilde{E}_m$ and $\Gamma(0, \gamma_m)$ and the inequalities (2.24) become
\begin{equation}
\text{meas} \{ \tilde{E}_m - \Gamma(0, \gamma_m) \} \geq 2\xi \quad (m \in \mathcal{M}).
\end{equation}

Hence, in view of (2.23) and (2.25), Lemma 3 yields
\begin{equation}
\frac{1}{2\pi} \int_{E_\omega(r_m)} \log \left| 1 - \frac{r_m e^{i\theta}}{a} \right| d\theta = \frac{1}{2\pi} \int_{\tilde{E}_m} \log \left| 1 + \frac{r_m e^{i\theta}}{|a|} \right| d\phi
\leq -K + \frac{1}{\pi} \int_0^{r_m} \log \left| 1 + \frac{r_m e^{i\theta}}{|a|} \right| d\theta
(m \in \mathcal{M}; K = K(\sigma, \xi)),
\end{equation}
where the positive constant $K$ depends on no parameters other than $\sigma$ and $\xi$.

There are $\tilde{r}_m = n(\sigma r_m, 1/f) - n(\sigma^{-1}r_m, 1/f)$ zeros of $f(z)$ characterized by the inequalities (2.23). Since our assumptions imply the validity of assertion II of [L, Theorem 1], we deduce from [L, (2.9)] and [L, (2.11)]
\begin{equation}
\lim_{m \to +\infty ; m \in \mathcal{M}} \frac{\tilde{r}_m}{T(r_m, f)} = \mu(\sigma^a - \sigma^{-a}).
\end{equation}

We denote by $a_j$ the zeros of $f(z)$ and by $b_j$ its poles and, in the following inequality, confine our attention, and our summations, to the zeros satisfying (2.23). Then (2.26) and (2.27) yield
\begin{equation}
\sum \frac{1}{2\pi} \int_{E_\omega(r_m)} \log \left| 1 - \frac{r_m e^{i\theta}}{a_j} \right| d\theta
\leq \frac{1}{\pi} \int_0^{r_m} \left\{ \sum \log \left| 1 + \frac{r_m e^{i\theta}}{|a_j|} \right| \right\} d\theta - K_1 T(r_m, f)\mu(1 + o(1))
(m \to +\infty, m \in \mathcal{M}),
\end{equation}
where $K_1 = K(\sigma, \xi)\mu(\sigma^a - \sigma^{-a})$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
We now examine a proof of Edrei [2, pp. 87–94] and consider, in particular, the fundamental inequality [2, (2.18)]. With our notations this inequality implies

\[ m(r_m, f) \leq \sum_{0 < \|a_j\| \leq R_m} \frac{1}{2\pi} \int_{\partial a(r_m)} \log \left| 1 - \frac{r_m e^{i\theta}}{a_j} \right| d\theta \]

\[ - \sum_{0 < \|b_j\| \leq R_m} \frac{1}{2\pi} \int_{\partial a(r_m)} \log \left| 1 - \frac{r_m e^{i\theta}}{b_j} \right| d\theta \]

\[ + 15 \frac{r_m}{R_m} T(2R_m) + o(T(r_m)) \]

\[ (m \to \infty, r_m \leq \frac{1}{2} R_m, T(r) = T(r, f)), \]

where it is understood that, subject to the restriction \( R_m \geq 2r_m \), the size of the error term is not affected by the choice of \( R_m \).

The arguments in [2, p. 90] may be repeated with the following minor modification: instead of using [2, (2.20)] to estimate all the terms of the first sum in the right-hand side of (2.29), we use (2.28) to evaluate the contribution of all the \( a_j \) such that \( \gamma_1 = \gamma_\infty \leq r_m \leq r_m \).

We thus obtain

\[ T(r_m) \leq \frac{1}{\pi} \int_0^{r_m} \left\{ \sum_{0 < \|a_j\| \leq R_m} \log \left| 1 + \frac{r_m e^{i\theta}}{|a_j|} \right| \right\} d\theta + \int_0^{\pi - \gamma_m} \left\{ \sum_{0 < \|b_j\| \leq R_m} \log \left| 1 + \frac{r_m e^{i\theta}}{|b_j|} \right| \right\} d\theta \]

\[ + 15 \frac{r_m}{R_m} T(2R_m) + o(T(r_m)) - K_i u T(r_m) \]

\[ (m \to +\infty, m \in \mathcal{M}, r_m \leq \frac{1}{2} R_m), \]

instead of [2, (2.22)].

In view of (2.20), \( 0 < \gamma_m < \pi (m > m_0) \) and we obtain, as in [2],

\[ T(r_m) \leq \int_0^{r_m} N_0(t)P(t, r_m, \gamma_m) dt + \int_0^{r_m} N_\infty(t)P(t, r_m, \pi - \gamma_m) dt - K_i u T(r_m) \]

\[ + A \frac{r_m}{R_m} T(2R_m) + o(T(r_m)) \]

\[ (m \to +\infty, m \in \mathcal{M}), \]

where \( A (>0) \) is an absolute constant and the symbols \( N_0, N_\infty, P \) have the same meaning as in [2] or in [L, (4.8)].

The main difference between (2.30) and [L, (4.8)] is the presence, in (2.30), of the negative quantity \( -K_i u T(r_m) \). The arguments which, in [L], lead to [L, (4.12)] now yield

\[ \sin \pi \mu \leq u \sin \beta \mu + v \sin (\pi - \beta) \mu - K_i u \sin \pi \mu. \]

Using the Cauchy-Schwarz inequality, as in [L, (5.1)], we deduce from (2.31)

\[ (K_i u + 1) \sin^2 \pi \mu \leq u^2 + v^2 - 2uv \cos \pi \mu, \]

and hence, by (1.3), \( u=0 \).
We have thus shown that, if $u \neq 0$, the relation (2.21) is impossible and therefore

$$\lim_{m \to \infty} \text{meas} \{ E_\infty(r_m) - \Gamma(\omega_m, \gamma_m) \} = 0.$$  

This is the first of the relations (2.3).

From (2.32) and (2.20) we deduce

$$\lim_{m \to \infty} \text{meas} \{ E_\infty(r_m) \cap \Gamma(\omega_m, \gamma_m) \} = \lim_{m \to \infty} (2\gamma_m) = s(\infty).$$

The sets $E_0(r_m)$ and $E_\infty(r_m)$ are disjoint by definition so that $C$ may be represented as the union of three pairwise disjoint sets:

$$C = \{ E_0(r_m) \cup E_\infty(r_m) \cup E_1(r_m) \}.$$  

Then, by [L, (2.6)]

$$\lim_{m \to \infty} \text{meas} E_1(r_m) = 0.$$  

From (2.34) we deduce

$$\text{meas} \{ E_0(r_m) \cap \Gamma(\omega_m, \gamma_m) \} + \text{meas} \{ E_\infty(r_m) \cap \Gamma(\omega_m, \gamma_m) \} + \text{meas} \{ E_1(r_m) \cap \Gamma(\omega_m, \gamma_m) \} = 2\gamma_m,$$

and hence, by (2.33) and (2.35)

$$\lim_{m \to \infty} \text{meas} E_0(r_m) \cap \Gamma(\omega_m, \gamma_m) = 0.$$  

Finally, $\Gamma(\omega_m, \gamma_m)$ and $\{(\pi + \omega_m, \pi - \gamma_m) \}$ are disjoint and their union is $C$. Therefore

$$\text{meas} \{ E_0(r_m) \cap \Gamma(\pi + \omega_m, \pi - \gamma_m) \} + \text{meas} \{ E_0(r_m) \cap \Gamma(\omega_m, \gamma_m) \} = \text{meas} E_0(r_m),$$

which, in view of (2.36), yields

$$\lim_{m \to \infty} \text{meas} \{ E_0(r_m) \cap \Gamma(\pi + \omega_m, \pi - \gamma_m) \} = \lim_{m \to \infty} \text{meas} E_0(r_m),$$

and proves the second relation in (2.3).

We have thus completed the proof of Lemma 1 in the case $u \neq 0$. If $u = 0$, we certainly have $v \neq 0$ (by (1.3)) and hence Lemma 1 follows from the consideration of the function $1/f$, instead of $f$.

3. Arguments of the zeros and poles of $f(z)$. Lemma 1 gives a precise meaning to step I of the general argument outlined in the Introduction. The following Lemma 4 clarifies, in a similar manner, step II.

**Lemma 4.** Let the assumptions and notations of Lemma 1 be unchanged. Let $\sigma$ and $\eta$ be given, fixed quantities such that

$$1 < \sigma, \quad 0 < \pi - \eta < \min(s(0), s(\infty)).$$  

$s(0)$ and $s(\infty)$ are defined as in [L, Theorem 1].
I. Then, if $K(\sigma, \xi)$ is the constant in (2.14), and if

\[ 2K_2 = K(\sigma, (\pi - \eta)/2), \]

we have

\[ \frac{1}{2\pi} \int_{K(\sigma, \xi)} \log \left| 1 - \frac{r_m e^{i\theta}}{a} \right| d\theta \leq \frac{1}{\pi} \int_0^{\pi} \log \left| 1 + \frac{r_m e^{i\theta}}{|a|} \right| d\theta - K_2, \]

provided

\[ a \in \mathcal{S}(\omega_m, \eta; \sigma^{-1}r_m, \sigma r_m) \]

and $m > m_2$.

The bound $m_2$, which depends on $f, \{r_m\}, \sigma$ and $\eta$, holds uniformly for all $a$ satisfying (3.4).

II. The counting functions of all zeros and poles of $f(z)$ satisfy the relations

\[ n(\mathcal{S}(\omega_m, \eta; \sigma^{-1}r_m, \sigma r_m), 1/f) = o(T(r_m, f)), \]

\[ n(\mathcal{S}(\omega_m + \pi, \eta; \sigma^{-1}r_m, \sigma r_m), f) = o(T(r_m, f)), \]

as $m \to \infty$.

Proof. The assumptions of Lemma 4 coincide with those of Lemma 1 as well as with those of [L, Theorem 1, assertion II]. Hence if $\{r_m\}$ is a sequence of Pólya peaks, of order $\mu$, of $T(r) = T(r, f)$, we see that (2.2) and (2.3) hold and that

\[ 0 < \lim_{m \to \infty} 2y_m = s(\infty) < 2\pi, \quad 0 < \lim_{m \to \infty} 2(\pi - y_m) = s(0) < 2\pi. \]

Consider the sets

\[ C_{m1} = \{E_{\omega}(r_m) \cap \Gamma(\omega_m, y_m)\}, \]

\[ C_{m2} = \{E_{\omega}(r_m) - \Gamma(\omega_m, y_m)\} = \{E_{\omega}(r_m) - C_{m1}\}, \]

\[ C_{m3} = \{\Gamma(\omega_m, y_m) - E_{\omega}(r_m)\} = \{\Gamma(\omega_m, y_m) - C_{m1}\}, \]

and notice that, in view of the first of the relations (2.3), we have

\[ \lim_{m \to \infty} \text{meas } C_{m1} = \lim_{m \to \infty} \text{meas } E_{\omega}(r_m) = s(\infty), \]

and hence, as $m \to \infty$,

\[ \text{meas } C_{m2} + \text{meas } C_{m3} = c_m \to 0. \]

Let $\int$ indicate integration of some measurable function defined on $C$; then, by (3.8),

\[ \int_{E_{\omega}(r_m)} - \int_{\Gamma(\omega_m, y_m)} = \int_{C_{m2}} - \int_{C_{m3}} = \zeta_m. \]
In particular, if we apply (3.11) to the function

(3.12) \[ \log |1 - r_m e^{i\theta}/a| \quad (|a| \leq \sigma_m), \]

we obtain, in view of (3.10) and Lemma 2,

(3.13) \[ |\zeta_m| \leq 4\pi c_m (\log (1 + \sigma) + 1 + \log (1/|c_m|)) \quad (m > m_0(\sigma)). \]

Now let

(3.14) \[ a = |a|e^{i\varphi} \]

satisfy the condition (3.4) so that

(3.15) \[ \sigma^{-1} \leq r_m/|a| < \sigma, \]

(3.16) \[ \psi = \omega_m + \kappa \eta \quad (-1 < \kappa \leq 1). \]

With a suitable choice of \( m_1(\sigma, \eta) \) we may, in view of (3.1), assume

(3.17) \[ 0 < \pi - \eta < \min (2\gamma_m, 2\pi - 2\gamma_m) \quad (m > m_1(\sigma, \eta)). \]

The change of variable \( \phi = \theta - \pi - \psi \) leads to

(3.18) \[ \int_{\Gamma(\omega_m, \gamma_m)} \log \left| 1 - \frac{r_m e^{i\theta}}{a} \right| \, d\theta = \int_{\Gamma(\tilde{\omega}_m, \gamma_m)} \log \left| 1 + \frac{r_m e^{i\theta}}{|a|} \right| \, d\phi, \]

where

(3.19) \[ \tilde{\omega}_m = \omega_m - \psi - \pi. \]

Using (3.16) in (3.19), we find \( \tilde{\omega}_m = -\pi - \kappa \eta \), and therefore

\[ -\pi < \tilde{\omega}_m < -\pi + \eta \quad \text{if} \quad -1 < \kappa < 0, \]

\[ \pi - \eta \leq 2\pi + \tilde{\omega}_m \leq \pi \quad \text{if} \quad 0 \leq \kappa \leq 1. \]

Hence if \( \omega'_m \) is defined by the relations

\[ \omega'_m = \tilde{\omega}_m \quad \text{if} \quad -1 < \kappa < 0, \]

\[ \omega'_m = 2\pi + \tilde{\omega}_m \quad \text{if} \quad 0 \leq \kappa \leq 1, \]

we always have

(3.20) \[ \Gamma(\tilde{\omega}_m, \gamma_m) = \Gamma(\omega'_m, \gamma_m), \]

with

(3.21) \[ 0 < \pi - \eta \leq |\omega'_m| \leq \pi. \]

Before applying Lemma 3 to the last integral in (3.18), we require the following elementary remark:

(3.22) \[ \text{meas} \{\Gamma(\omega'_m, \gamma_m) - \Gamma(0, \gamma_m)\} = \min \{|\omega'_m|, 2\gamma_m, 2(\pi - \gamma_m)| \}

\[ (|\omega'_m| \leq \pi, 0 < \gamma_m < \pi). \]
In order to verify this relation consider the set

\[ \mathcal{H}(\omega) = \{ \Gamma(\omega, \gamma) - \Gamma(0, \gamma) \}, \]

for \( \gamma \) fixed and \( \omega \) variable.

A. First assume \( 0 < 2\gamma \leq \pi \) and, as \( \omega \) increases from 0 to \( \pi \), follows the positions of the points \( \omega - \gamma, \omega + \gamma \) in the interval \( (-\gamma, 2\pi - \gamma] \). Obviously

\[ \mathcal{H}(\omega) = (\gamma, \omega + \gamma) \quad \text{if} \quad 0 < \omega < 2\gamma, \]

\[ \mathcal{H}(\omega) = (\omega - \gamma, \omega + \gamma) \quad \text{if} \quad 2\gamma < \omega \leq \pi. \]

B. Similarly, if \( -\pi < 2\gamma < 2\pi \), then

\[ \mathcal{H}(\omega) = (\gamma, \omega + \gamma) \quad \text{if} \quad 0 < \omega \leq 2(\pi - \gamma), \]

\[ \mathcal{H}(\omega) = (\gamma, 2\pi - \gamma) \quad \text{if} \quad 2(\pi - \gamma) < \omega \leq \pi. \]

The relations (3.23), (3.24) and (3.25) yield

\[ \text{meas } \mathcal{H}(\omega) = \text{meas } \mathcal{H}(-\omega) = \min (|\omega|, 2\gamma, 2(\pi - \gamma)) \quad (|\omega| \leq \pi, 0 < \gamma \leq \pi), \]

and (3.22) follows.

In view of (3.17) and (3.21) we have

\[ 0 < \pi - \eta \leq \min (|\omega_m|, 2\gamma_m, 2\pi - 2\gamma_m) \quad (m > m_1(\sigma, \eta)), \]

which used in (3.22) yields

\[ \text{meas } \{ \Gamma(\omega', \gamma_m) - \Gamma(0, \gamma_m) \} \geq \pi - \eta > 0. \]

Consider the constant \( K(\sigma, \xi) \) defined by (2.14) and set

\[ 2K_2 = 2K_2(\sigma, \eta) = K(\sigma, \frac{1}{2}(\pi - \eta)) > 0. \]

By (3.15), (3.26) and Lemma 3, we find

\[ \frac{1}{2\pi} \int_{\Gamma(\omega_m, \gamma_m)} \log \left| 1 + \frac{r_m e^{i\theta}}{|a|} \right| d\theta \leq -2K_2 - \frac{1}{\pi} \int_0^{\gamma_m} \log \left| 1 + \frac{r_m e^{i\theta}}{|a|} \right| d\theta \]

\[ (m > m_1(\sigma, \eta)). \]

In view of (3.18) and (3.20) the left-hand side of (3.27) may be replaced by

\[ \frac{1}{2\pi} \int_{\Gamma(\omega_m, \gamma_m)} \log \left| 1 - \frac{r_m e^{i\theta}}{a} \right| d\theta. \]

If we combine the resulting inequality with (3.11), we find

\[ \frac{1}{2\pi} \int_{E(\omega_m)} \log \left| 1 - \frac{r_m e^{i\theta}}{a} \right| d\theta \leq -2K_2 + \frac{\gamma_m}{2\pi} + \frac{1}{\pi} \int_0^{\gamma_m} \log \left| 1 + \frac{r_m e^{i\theta}}{|a|} \right| d\theta \]

\[ (m > m_1(\sigma, \eta)). \]

Now by (3.10) and (3.13)

\[ \lim_{m \to \infty} \xi_m = 0, \]
uniformly for all \( a \) satisfying (3.4). Hence, if \( m_2 \) is chosen large enough, the inequality \( m > m_2 \) implies \( m > m_1(\alpha, \eta) \), \( \xi_\eta / 2\pi \leq K_2 \), and (3.3) follows from (3.28).

We have thus proved assertion I of Lemma 4.

Proof of assertion II of Lemma 4. The parameters \( \sigma, \eta \), as well as the sequence \( \{\omega_n\} \), are fixed. Explicit reference to all these quantities is unnecessary and we simplify our notation by setting

\[
\tilde{n}_m = n(\mathcal{A}(\omega_m, \eta; \sigma^{-1}r_m, \sigma r_m), 1/f).
\]

Assume that (3.5) is false. Then there exists some constant \( \xi > 0 \) and some unbounded sequence \( \mathscr{N} \), of positive integers such that

\[
\tilde{n}_m > \xi T(r_m) \quad (m \in \mathscr{N}).
\]

This yields a contradiction as may be seen by a repetition, with minor modifications, of the proof of Lemma 1:

(i) start from (2.29). Consider its right-hand side and use (3.3) (instead of (2.28)) to estimate the contribution of the \( \tilde{n}_m \) terms involving the zeros of \( f(z) \) in

\[
\mathcal{A}(\omega_m, \eta; \sigma^{-1}r_m, \sigma r_m);
\]

(ii) we are thus led to an inequality such as (2.30) with \( -K_1 u T(r_m) \) replaced by \( -K_2 \xi T(r_m) \), and finally to

\[
\sin \pi \mu \leq u \sin \beta \mu + v \sin (\pi - \beta) \mu - K_2 \xi \sin \pi \mu
\]

(instead of (2.31)).

Hence \( \xi = 0 \), a contradiction which proves (3.5). The relation (3.6) is obtained by applying our arguments to \( 1/f \) instead of \( f \). This completes the proof of Lemma 4.

4. Proof of Theorem 1. Let \( l > 2 \) be a fixed integer. By Lemma 4 it is possible to determine \( m_1 \) so that \( m > m_1 \) implies

\[
n(\mathcal{A}(\omega_m, \pi - 1/l; r_m/l, l r_m), 1/f) + n(\mathcal{A}(\omega_m + \pi, \pi - 1/l; r_m/l, l r_m), f) < T(r_m)/l.
\]

We then set

\[
(4.1) \quad \eta_m = \pi - 1/l, \quad \rho'_m = r_m/l, \quad \rho''_m = l r_m
\]

for

\[
(4.2) \quad m_1 < m \leq m_{l+1} \quad (l = 3, 4, 5, \ldots).
\]

Theorem 1 is now obvious since the quantities defined by (4.1) and (4.2) clearly satisfy the relations (1.4), (1.5) and (1.6).

5. Preliminary steps leading to Theorem 2. Let the assumptions of Theorem 2 be satisfied. Since they include those of Theorem 1, the existence and the properties of the four sequences \( \{\omega_m\}, \{\eta_m\}, \{\rho'_m\}, \{\rho''_m\} \) may be taken for granted. In particular,
consider the left-hand sides of the two relations (1.6) and let \( n_m \) denote their sum; by Theorem 1

\[
(5.1) \quad \frac{n_m}{T_r(m)} = \delta_m \to 0 \quad (m \to +\infty, T(t) = T(t, f)).
\]

We define

\[
(5.2) \quad \sigma^2_m = \frac{1}{2} \min \{r_m/r_m', r_m/R_m', \rho_m^2/r_m, R_m^2/r_m, 1 / \sqrt{\delta_m}\},
\]

where \( R_m' \) and \( R_m'' \) are the quantities in [L, Theorem 1]; by (1.5), [L, (2.7)], and (5.1) this implies

\[
(5.3) \quad \lim_{m \to \infty} \sigma_m = +\infty,
\]
as well as

\[
(5.4) \quad \lim_{m \to +\infty} \frac{n_m \sigma^2_m}{T_r(m)} = 0.
\]

Given \( e (0 < e < \frac{1}{2} \min \{s(0), s(\infty)\}) \), we define \( \eta \) by the relation

\[
(5.5) \quad \pi - \eta = e/2,
\]

and from now on write

\[
(5.6) \quad \mathcal{S}_{0m} = \mathcal{S}(\omega_m, \eta; \sigma^{-2}_{m} r_m, \sigma^2_{m} r_m),
\]

\[
\mathcal{S}_{\omega m} = \mathcal{S}(\pi + \omega_m, \eta; \sigma^{-2}_{m} r_m, \sigma^2_{m} r_m),
\]

\[
\mathcal{A}_m = \{z : \sigma^{-2}_{m} r_m < |z| \leq \sigma^2_{m} r_m\},
\]

\[
\mathcal{H}_{0m} = \mathcal{A}_m - \mathcal{S}_{0m}, \quad \mathcal{H}_{\omega m} = \mathcal{A}_m - \mathcal{S}_{\omega m}.
\]

By (5.2), (5.4), (5.5) and (5.6)

\[
(5.7) \quad \lim_{m \to +\infty} \frac{n(\mathcal{S}_{0m}, 1/f) + n(\mathcal{S}_{\omega m}, f) \sigma^2_m}{T_r(m)} = 0.
\]

We propose to study the asymptotic behavior of \( f(z) \) as \( z \to \infty \) by values such that

\[
(5.8) \quad \sigma^{-1}_{m} r_m < r \leq \sigma_m r_m \quad (z = re^{i\theta}),
\]

and

\[
(5.9) \quad \theta \in \Gamma \left( \omega_m, \frac{s(\infty)}{2} - e \right) = \Gamma_m.
\]

Consider the fundamental representation [2, (2.6)]; for our purposes this relation may be rewritten in the form

\[
(5.10) \quad \log |f(z)| = \log \left| \prod_{a \in \mathcal{A}_m} \left( 1 - \frac{z}{a_j} \right) \right| - \log \left| \prod_{b \in \mathcal{B}_m} \left( 1 - \frac{z}{b_j} \right) \right| + \log \left| \prod_{0 < |z| \leq \sigma^{-2}_{m} r_m} \left( 1 - \frac{z}{a_j} \right) \right| - \log \left| \prod_{0 < |z| \leq \sigma^{-2}_{m} r_m} \left( 1 - \frac{z}{b_j} \right) \right| + \log (|c| r^e) + S(z, \sigma^2_m r_m) \quad \left( 0 < |z| = r \leq \frac{\sigma^2_m r_m}{2} \right),
\]
where \( c \neq 0 \) and \( q \) (an integer) are constants and the "error term" \( S(z, \sigma_m^2 r_m) \) satisfies the inequality
\[
|S(z, \sigma_m^2 r_m)| \leq 15 \frac{r}{\sigma_m^2 r_m} T(2\sigma_m^2 r_m).
\]

By (5.2), (5.8) and [L, (2.9)], it follows that
\[
(5.11) \quad |S(z, \sigma_m^2 r_m)| \leq 30\sigma_m^{-\alpha - \delta} T(r) \quad (m > m_0).
\]

Since for any nonrational meromorphic function \( f \),
\[
\log r = o(T(r, f)) \quad (r \to +\infty),
\]
it is obvious that (5.3) and (5.11) yield
\[
(5.12) \quad |\log |c| r^s| + |S(z, \sigma_m^2 r_m)| = o(T(r)) \quad (r \to +\infty).
\]

Let \( L_m(z) \) denote the sum of the third and fourth terms in the right-hand side of (5.10). In order to estimate \( L_m(z) \) we observe that if \(|z|\) satisfies (5.8) and \(|a| \leq \sigma_m^2 r_m\), then \(|z|/|a| > \sigma_m\) and therefore
\[
0 < -\log 2 + \log (r/|a|) < \log |1 - z/a| < \log (r/|a|) + \log 2 \quad (m > m_0),
\]
which yields
\[
(5.13) \quad \sum_{0 < |a| \leq \sigma_m^2 r_m} \log \left| 1 - \frac{z}{a} \right| \leq (\log 2 + 3 \log \sigma_m) n\left(\sigma_m^{-2} r_m, \frac{1}{\sigma_m}\right)
+ N\left(\sigma_m^{-2} r_m, \frac{1}{\sigma_m}\right) + o(\log r) \quad (m \to \infty).
\]

There is a similar formula involving the poles of \( f(z) \).

By [L, (2.9)] and (5.8)
\[
(5.14) \quad T(\sigma_m^{-2} r_m) < 2\sigma_m^{-\alpha} T(r) \quad (m > m_0).
\]

We now use (5.14) in (5.13), and in the analogous inequality for poles, and take into account [L, (2.10)], [L, (2.11)] and (5.3). This yields
\[
(5.15) \quad L_m(z) = o(T(r)) \quad (m \to +\infty).
\]

Denote by \( \Lambda_m(z) \) the sum of the two first terms in the right-hand side of (5.10); in view of (5.12) and (5.15) we have
\[
(5.16) \quad \log |f(z)| = \Lambda_m(z) + o(T(r)),
\]
uniformly as \( r \to \infty \) in the intervals (5.8).

The next two sections are devoted to the study of \( \Lambda_m(z) \).

6. Bounds for the primary factors. Consider a zero \( a \) of \( f(z) \) such that
\[
(6.1) \quad a = |a|e^{i\varphi}, \quad a \in \mathcal{X}_m
\]
and let \( z \) satisfy the conditions (5.8) and (5.9).
By (6.1) and (5.5) there exists a determination of \( \psi \) such that \( \omega_+ - \psi < \psi \leq \omega_+ + e/2 \), and by (5.9) \( \omega_+ - s(\infty) + e < \theta \leq \omega_+ - s(\infty)/2 - e \).

Hence

\[
(-s(\infty) + e)/2 < \theta - \pi - \psi < (s(\infty) - e)/2,
\]

which, in view of the fact that \( \log |1 + re^{i\theta}| \) decreases as \( |\psi| \) increases from 0 to \( \pi \), yields

\[
\log \left| 1 - \frac{z}{a} \right| = \log \left| 1 + \frac{r \exp \left[i(\theta + \pi - \psi)\right]}{|a|} \right|
\]

\[
> \log \left| 1 + \frac{r e^{i\lambda}}{|a|} \right| \quad (a \in \mathcal{A}_m, \theta \in \Gamma_m),
\]

where

\[
\lambda = (s(\infty) - e)/2, \quad 0 < \lambda < \pi.
\]

Similarly, if \( b \) is a pole of \( f(z) \) lying in \( \mathcal{A}_m \), and if \( \theta \in \Gamma_m \), then

\[
\log |1 - re^{i\theta}/b| < \log |1 - re^{i\lambda}| |b|.
\]

The inequalities (6.2) and (6.4), and our definition of \( \Lambda_m(z) \), yield

\[
\Lambda_m(z) > \log \left| \prod_{a_j \in \mathcal{A}_m} \left( 1 + \frac{r e^{i\lambda}}{|a_j|} \right) \right| - \log \left| \prod_{b_j \in \mathcal{B}_m} \left( 1 - \frac{r e^{i\lambda}}{|b_j|} \right) \right|
\]

\[
- \log \left| \prod_{a_j \in \mathcal{A}_m} \left( 1 - \frac{z}{a_j} \right) \right| + \log \left| \prod_{b_j \in \mathcal{B}_m} \left( 1 - \frac{z}{b_j} \right) \right| \quad (\theta \in \Gamma_m).
\]

If \( a \in \mathcal{A}_m \) and \( r \) satisfies (5.8), we have \( \sin \lambda \leq |1 + re^{i\lambda}/|a|| < 2 \sigma_m^3 \), and hence, by (5.4) and [L, (2.9)],

\[
\sum_{a_j \in \mathcal{A}_m} \left| \log \left| 1 + \frac{r e^{i\lambda}}{|a_j|} \right| \right| + \sum_{b_j \in \mathcal{B}_m} \left| \log \left| 1 - \frac{r e^{i\lambda}}{|b_j|} \right| \right| \leq n_m (\log 2 + 3 \log \sigma_m) = o(T(r)) \quad (m > m_0, r \rightarrow +\infty).
\]

The two last terms of (6.5) are estimated by the following straight-forward application of the lemma of Boutroux-Cartan: if \( z \in \mathcal{A}_m \) and if \( z \) avoids finitely many disks with sum of diameters equal to \( \sigma_m^2 r_m/2 \), we have

\[
\prod_{a_j \in \mathcal{A}_m} |z - a_j| \geq \left( \frac{\sigma_m^2 r_m}{8e} \right)^n \quad (n = n(\mathcal{A}_m, 1/f)),
\]

\[
(1 + \sigma_m^4)^n \geq \prod_{a_j \in \mathcal{A}_m} \left| 1 - \frac{z}{a_j} \right| \geq (8e \sigma_m^3)^{-n}.
\]
The same bounds hold for the polynomial formed with the poles $b_j (\in \mathcal{M}_m)$. Hence, the arguments used in the proof of (6.6), yield

$$
(6.7) \quad \left| \log \prod \left(1 - \frac{z}{a_j} \right) \right| + \left| \log \prod \left(1 - \frac{z}{b_j} \right) \right| \leq n_m \log (8e) + 4 \log \sigma_m = o(T(r)) \quad (r \to \infty),
$$

provided $r$ (confined to $\mathcal{M}_m$) avoids a set $E_m$, of measure not greater than $\sigma_m^{-2}r_m$.

### 7. Proof of Theorem 2

Let $I_1(r)$ denote the first term in the right-hand side of (6.5). The elementary identity

$$
(7.1) \quad \log \left| 1 + \frac{z}{a} \right| = \Re \left( z \int_{|z|=1}^{+\infty} \frac{dt}{|t(z+t)|} \right) \quad (a \neq 0),
$$

valid if $z$ is not real and negative, shows that

$$
I_1(r) = \Re \left\{ z \int_{\sigma_m^{-2}r_m}^{\sigma_m^2r_m} \frac{n(t, 1/f) - n(\sigma_m^{-2}r_m, 1/f)}{t(z+t)} dt \right\} + \int_{\sigma_m^{-2}r_m}^{+\infty} \frac{n(\sigma_m^2r_m, 1/f) - n(\sigma_m^{-2}r_m, 1/f)}{t(z+t)} dt \quad (z = re^{i\alpha}),
$$

and using again (7.1)

$$
(7.2) \quad I_1(r) = \Re \left\{ z \int_{\sigma_m^{-2}r_m}^{\sigma_m^2r_m} \frac{n(t, 1/f) dt}{t(z+t)} + \frac{n(\sigma_m^2r_m, 1/f) dt}{t(z+t)} \log \left| 1 + \frac{z}{\sigma_m^2r_m} \right| \right\} \quad (z = re^{i\alpha}).
$$

Now $\log \left| 1 + z/\sigma_m^2r_m \right| = O(r/\sigma_m^2r_m)$, and

$$
(7.3) \quad -n(\sigma_m^{-2}r_m, 1/f) \log \left| 1 + \frac{z}{\sigma_m^{-2}r_m} \right| \quad (z = re^{i\alpha}).
$$

Rewriting (7.4) in the form

$$
(7.5) \quad \frac{I_1(r)}{T(r)} = \Re \left\{ z \mu u \int_{\sigma_m^{-2}r_m}^{\sigma_m^2r_m} \left( \frac{t}{r} \right)^{n} dt \right\} + o(1),
$$

we obtain, in view of [L, (2.9)] and [L, (2.11)],

$$
(7.6) \quad \frac{I_1(r)}{T(r)} = \Re \left\{ \mu u e^{i\alpha} \int_{0}^{1} \frac{d\theta}{x^{\alpha} + e^{i\alpha}} \right\} + o(1).
$$
as \( r \to +\infty \) in the intervals \( (\sigma_m^{-1} r_m, \sigma_m r_m] \). The value of the last integral in (7.5) is well known to be \( \pi e^{\delta(\mu-1)/\sin \pi \mu} \), and hence we are finally led to

\[
(7.6) \quad \frac{L(r)}{T(r)} = \frac{1}{T(r)} \log \left| \prod_{a_i \in \mathcal{A}_m} \left( 1 + \frac{r e^{i\lambda}}{|a_i|} \right) \right| = \frac{\pi \mu}{\sin \pi \mu} \cos \lambda \mu + o(1).
\]

The same method yields

\[
(7.7) \quad \frac{1}{T(r)} \log \left| \prod_{b_i \in \mathcal{B}_m} \left( 1 - \frac{r e^{i\lambda}}{|b_i|} \right) \right| = \frac{\pi \mu}{\sin \pi \mu} \cos (\pi - \lambda) \mu + o(1).
\]

Combining (5.16), (6.5), (6.6), (6.7), (6.8) and (7.7) we obtain, uniformly in \( z \),

\[
(7.8) \quad \log |f(z)| \geq \frac{\pi \mu T(r)}{\sin \pi \mu} \{ u \cos (\lambda \mu) - v \cos (\pi - \lambda) \mu \} + o(T(r)),
\]

provided

(i) \( |z| = r \to +\infty \) in the intervals \( (\sigma_m^{-1} r_m, \sigma_m r_m] \);
(ii) \( r \) avoids the exceptional sets \( \mathcal{E}_m \) (meas \( \mathcal{E}_m \leq \sigma_m^{-2} r_m \));
(iii) \( \arg z = \theta \in \Gamma \left( \omega_m, \frac{s(\infty)}{2} \right) \).

If we choose any \( K \) such that

\[
(7.9) \quad 0 < K < \frac{\pi \mu}{\sin \pi \mu} \left( u \cos \left( \frac{s(\infty) \mu}{2} - \frac{\epsilon \mu}{2} \right) - v \cos \left( \frac{\pi - s(\infty)}{2} + \frac{\epsilon \mu}{2} \right) \right) = \bar{K},
\]

we see that (7.8) and (6.3) imply the first of the inequalities (1.7). We must still verify that \( \bar{K} > 0 \) since otherwise it will be impossible to find a \( K \) satisfying (7.9). The relations \( [L, (2.4)] \), \( [L, (2.5)] \) and \( [L, (2.6)] \) yield an explicit value of \( \bar{K} \):

\[
\bar{K} = \frac{\pi \mu}{\sin \pi \mu} \left( u \sin \left( \frac{s(\infty) \mu}{2} \right) + v \sin \left( \frac{s(\infty)}{2} \right) \right) \sin \frac{\epsilon \mu}{2} > 0.
\]

The second inequality (1.7) is obtained by considering \( 1/f \) instead of \( f \). Our proof of Theorem 2 is now complete.

8. Proof of Theorem 3. Assume that the Theorem is false. Then there exists a meromorphic function \( F(z) \) of lower order \( \mu \) (\( 0 < \mu < 1 \)) having at least two finite, distinct, deficient values \( \tau_1, \tau_2 \) and such that \( f(z) = F'(z) \) satisfies the conditions of Theorem 1.

By the elements of Nevanlinna's theory

\[
(8.1) \quad m(r, F) + m(r, F(F - \tau_1)) + m(r, F(F - \tau_2)) = o(T(r, F)) \quad (r \notin \mathcal{E}, r \to +\infty),
\]

where \( \mathcal{E} \) is an exceptional set of finite measure. It is well known that this relation implies

\[
(8.2) \quad T(r, F) \leq 2T(r, F)(1 + o(1)) \quad (r \notin \mathcal{E}, r \to +\infty),
\]
and also (since the relation [L, (9.3)] is valid with \( g \) replaced by \( F \)),

\[
N(r, 1/f) + m(r, 1/(F - \tau_1)) + m(r, 1/(F - \tau_2)) \leq T(r, f) + o(T(r, F))
\]

\((r \notin \mathcal{E}, r \to +\infty).\)

From the definition of deficient value, we deduce

\[
m(r, 1/(F - \tau_k)) > \frac{1}{4} \delta(\tau_k, F) T(r, F) \quad (r > r_0; k = 1, 2),
\]

and hence, in view of (8.2) and (8.3), there exist two constants \( \kappa_1, \kappa_2 \) such that

\[
0 < \kappa_1 < T(r, f)/T(r, F) < \kappa_2 < +\infty \quad (r \notin \mathcal{E}, r > r_0).
\]

Let \( J \) be any measurable subset of \( C \) such that \( \text{meas} \{ J \} = 4\varepsilon > 0 \); then, by a lemma of Edrei and Fuchs [7, p. 322, Lemma III],

\[
\frac{1}{2\pi} \int_J \log^+ \left| \frac{1}{F - \tau_k} \right| d\theta = m\left( r, \frac{1}{F - \tau_k}; J \right) \leq A_0 T(2r, F) e \left( 1 + \log^+ \frac{1}{\varepsilon} \right)
\]

\((r > r_0; k = 1, 2),\)

where \( A_0 \) is an absolute constant.

From now on \( r \) will be restricted to the intervals \((r_m, 2r_m]\), and \( \{r_m\} \) is the sequence of Pólya peaks (of \( T(r, f) \)) which appears in Theorems 1 and 2. By [L, (2.9)], (8.5) and the fact that the characteristic functions are increasing,

\[
T(2r, F) < K_0 T(r, F) \quad (r_m < r \leq 2r_m, r \notin \mathcal{E}, m > m_0);
\]

the constant \( K_0 \) depends only on \( \kappa_1, \kappa_2 \) and \( \mu \).

Using (8.7) in (8.6) we obtain

\[
m\left( r, \frac{1}{F - \tau_k}; J \right) \leq A_0 K_0 T(r, F) e \left( 1 + \log^+ \frac{1}{\varepsilon} \right)
\]

\((r_m < r \leq 2r_m, r \notin \mathcal{E}, m > m_0; k = 1, 2),\)

and choose \( \varepsilon \) \((0 < \varepsilon < \frac{1}{4} \min(\delta(0), \delta(\infty)))\) so small that the right-hand side of (8.8) is less than

\[
\frac{1}{4} \min \{ \delta(\tau_1, F), \delta(\tau_2, F) \} T(r, F).
\]

We use this value of \( \varepsilon \) in Theorem 2 and select a sequence \( \{ \tilde{r}_m \} \) such that

\[
r_m < \tilde{r}_m \leq 2r_m, \quad \tilde{r}_m \notin \mathcal{E}, \quad \tilde{r}_m \notin \mathcal{E}_m \quad (m > m_0).
\]

This is certainly possible because \( \mathcal{E} \) is of finite measure and

\[
\text{meas} \mathcal{E}_m \leq a_m^{-2} r_m = o(r_m) \quad (m \to \infty).
\]

The set

\[
J_m = C \setminus \{ \Gamma(\omega_m, s(\infty)/2 - \varepsilon) \cup \Gamma(\pi + \omega_m, s(\infty)/2 - \varepsilon) \}
\]

is of measure \( 4\varepsilon \) and hence (8.8) and our choice of \( \varepsilon \) and \( \tilde{r}_m \) imply

\[
m(\tilde{r}_m, 1/(F - \tau_k); J_m) < \frac{1}{4} \delta(\tau_k, F) T(\tilde{r}_m, F) \quad (k = 1, 2).
\]
Now (8.1), the first relation (1.7) and the elementary inequality
\[ \log^+ |1/(F - \tau_k)| \leq \log^+ |f'(F - \tau_k)| + \log^+ |1/f|, \]
yield
\[ (8.10) \quad m(r_m, 1/(F - \tau_k); \Gamma(\omega_m, \omega(\infty)/2 - \epsilon)) = o(T(r_m, F)) \quad (m \to \infty, k = 1, 2). \]
If we consider the inequalities (8.4) with \( r = r_m \), and compare them with (8.9) and (8.10), we see that for \( m \) large enough, there will exist points
\[ z_{1m} = r_m \exp (i\theta_{1m}), \quad z_{2m} = r_m \exp (i\theta_{2m}), \]
such that \( \theta_{1m}, \theta_{2m} \in \Gamma(\omega_m + \pi, \omega(0)/2 - \epsilon) \),
\[ (8.11) \quad |F(z_{1m}) - \tau_1| < \frac{1}{3} |\tau_2 - \tau_1|, \quad |F(z_{2m}) - \tau_2| < \frac{1}{3} |\tau_2 - \tau_1|. \]
Let \( \mathcal{E}_m \) denote the subinterval of \( \Gamma(\omega_m + \pi, \omega(0)/2 - \epsilon) \) having end points \( \theta_{1m}, \theta_{2m} \).
Then, the obvious relation
\[ |F(z_{1m}) - F(z_{2m})| = \left| \int_{\mathcal{E}_m} f(r_m e^{i\theta}) r_m e^{i\theta} d\theta \right|, \]
the second relation (1.7), and the fact that \( \log r_m = o(T(r_m, F)) \), imply
\[ (8.12) \quad |F(z_{1m}) - F(z_{2m})| < \frac{1}{3} (\tau_2 - \tau_1) \quad (m > m_0). \]
The inequalities (8.11) and (8.12) are clearly incompatible. This contradiction shows that \( F(z) \) cannot have the finite, distinct, deficient values \( \tau_1, \tau_2 \), and hence proves Theorem 3.

SYRACUSE UNIVERSITY,
SYRACUSE, NEW YORK