IRREDUCIBLE MODULE HOMOMORPHISMS OF A
VON NEUMANN ALGEBRA INTO ITS CENTER(1)

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1. Introduction. A von Neumann algebra \( \mathcal{A} \) can be considered as a module over its center \( \mathcal{Z} \). The norm of \( \mathcal{A} \) induces a norm on the module \( \mathcal{A} \). Whenever we talk of the module \( \mathcal{A} \) it will always be this specific module over \( \mathcal{Z} \). In this article we study the set \( \mathcal{A}_\sim \) of bounded module homomorphisms of \( \mathcal{A} \) into \( \mathcal{Z} \). In an earlier article we studied those module homomorphisms of \( \mathcal{A} \) into \( \mathcal{Z} \) which are continuous in the \( \sigma \)-weak topology of \( \mathcal{A} \) and \( \mathcal{Z} \) respectively. In that paper we discovered a specific form for such homomorphisms and showed that a type I algebra could be characterized in terms of such functionals. These results were analogues of results known for factor algebras. For factor algebras multipliers are scalars and the mappings are scalar-valued functionals while in algebras with arbitrary centers the multipliers are central elements and the mappings are module homomorphisms into the center.

There are always module homomorphisms of \( \mathcal{A} \) into \( \mathcal{Z} \). A kind which is particularly simple although fundamental may be constructed as follows. Let \( \mathcal{Z}' \) be the commutator of \( \mathcal{Z} \) and let \( E \) be an abelian projection in \( \mathcal{Z}' \) with central support \( P \). There is an isomorphism of \( \mathcal{Z}P \) onto \( E\mathcal{Z}'E \) given by \( A \to AE \). For each \( A \) in \( \mathcal{A} \) we denote the inverse image in \( \mathcal{Z}P \) of \( EAE \) under this isomorphism by \( r_E(A) \). Then the function \( \tau_E \) on \( \mathcal{A} \) is a homomorphism into \( \mathcal{Z} \).

In general \( \mathcal{Z} \) is a most suitable range for module homomorphisms. The following Hahn-Banach type theorem illustrates this. Let \( \mathcal{E} \) be a normed space which is a module over a commutative \( AW^* \)-algebra \( \mathcal{Z} \). Let \( E \) be any submodule of \( \mathcal{E} \) and let \( \phi \) be a bounded module homomorphism of \( \mathcal{E} \) into \( \mathcal{Z} \). There is a bounded module homomorphism \( \psi \) of \( \mathcal{E} \) into \( \mathcal{Z} \) such that \( \psi(C) = \phi(C) \) for every \( C \) in \( \mathcal{E} \) and such that \( \|\psi\| = \|\phi\| \) [19], [24]. From this theorem many homomorphisms may be constructed.

A module homomorphism \( \phi \) of \( \mathcal{A} \) into \( \mathcal{Z} \) will be called a functional of the module \( \mathcal{A} \). A functional \( \phi \) of the module \( \mathcal{A} \) is said to be hermitian if \( \phi(A^*) = \phi(A)^* \) for every \( A \) in \( \mathcal{A} \). Every bounded functional of the module \( \mathcal{A} \) can be written as a linear combination of two bounded hermitian functionals. A functional \( \phi \) of the module \( \mathcal{A} \) is said to be positive if \( \phi \) maps \( \mathcal{A}^+ \) into \( \mathcal{Z}^+ \). Since

\[ |\phi(A)|^2 = \phi(A)^*\phi(A) \leq \phi(A^*A)\phi(1), \]

every positive functional \( \phi \) is bounded with bound \( \|\phi(1)\| \). Every bounded hermitian

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functional of the module $\mathcal{A}$ may be written as the difference of two positive functionals of the module $\mathcal{A}$ [19], [24].

In this paper we study the positive functionals of the module $\mathcal{A}$. The set $\mathcal{I}$ of positive functionals of $\mathcal{A}^\sim$ of norm not exceeding 1 is compact in a naturally defined topology in $\mathcal{A}^\sim$. The set $\mathcal{I}$ has extreme points and $\mathcal{I}$ is the closure (in this topology) of the convex hull of its extreme points. Here though the convexity is expressed in terms of multiplication by elements in $\mathcal{Z}$. We show that every linear functional $f$ on $\mathcal{A}$ which is $\sigma$-weakly continuous when restricted to $\mathcal{Z}$ can be expressed as the composition of $f$ with an element of $\mathcal{A}^\sim$.

A positive functional $\phi$ in $\mathcal{A}^\sim$ normalized so that $\phi(1) = 1$ gives rise to a representation of $\mathcal{A}$ as a $\ast$-subalgebra of the algebra of all bounded linear operators on an $AW^\ast$-module $M_\phi$ over the center $\mathcal{Z}$ ([6], [17], [28]). We study the representations that arise from an extreme point $\phi$ of $\mathcal{I}$. By presenting a specific form for the representation we are able to obtain the analogue of Kadison’s theorem on strict irreducibility. If $A \rightarrow A^\sim$ denotes the Gelfand transform of $\mathcal{Z}$ onto the algebra of all continuous complex-valued functions on the spectrum $Z$ of $\mathcal{Z}$, then the analogue of Kadison’s theorem allows us to conclude that $A \rightarrow \phi(A)^\sim(\zeta)$ is a pure state of $\mathcal{A}$ for every $\zeta$ in $\mathcal{Z}$. In a certain sense this result illustrates the advantage of a global theory over a decomposition theory. By an additional construction we are able to find an extreme point $\phi$ such that the kernel of the canonical representation of $\mathcal{A}$ on a Hilbert space induced by $A \rightarrow \phi(A)^\sim(\zeta)$ ($\zeta$ fixed but arbitrary in $Z$) is the smallest closed two-sided ideal $[\zeta]$ in $\mathcal{A}$ containing $\zeta$. So $[\zeta]$ is a minimal primitive ideal of $\mathcal{A}$.

We then define a vector state of the $\mathcal{A}$ as a module. This definition comes from ideas in a previous paper [12]. The set of elements in $\mathcal{A}^\sim$ obtained as pointwise limits of these vector states is called the vector state space. The set of pointwise limits in $\mathcal{A}^\sim$ of the set of extreme points $\phi$ of the positive elements of the unit sphere of $\mathcal{A}^\sim$ which satisfy $\phi(1) = 1$ is called the pure state space of the module $\mathcal{A}$. We then compare the set of all $\phi$ in the unit sphere of $\mathcal{A}^\sim$ such that $\phi(1) = 1$ with the pure state space and the vector state space of the module $\mathcal{A}$. These structures have exactly the same relations as the corresponding structures of scalar functionals as given by Glimm ([3], [4]). Here the ideal of completely continuous operators is replaced by the ideal generated by the abelian projections of $\mathcal{A}$.

2. Existence of extreme points. Let $\mathcal{A}$ be a von Neumann algebra with center $\mathcal{Z}$ and let $\mathcal{A}^\sim$ be the space of bounded functionals of the module $\mathcal{A}$. Let $\mathcal{Z}_\ast$ be the set of all $\sigma$-continuous functionals on $\mathcal{Z}$. For each $f$ in $\mathcal{Z}_\ast$ and $A$ in $\mathcal{A}$ define the seminorm $p_{f,A} = p$ of $\mathcal{A}^\sim$ by $p(\phi) = |f(\phi(A))|$. The family $\{p_{f,A} \mid f \in \mathcal{Z}_\ast, A \in \mathcal{A}\}$ of seminorms of $\mathcal{A}^\sim$ defines a topology on $\mathcal{A}^\sim$ under which $\mathcal{A}^\sim$ is a locally convex Hausdorff topological linear space. We call this topology the weak-$\ast$ topology of $\mathcal{A}^\sim$. If $f$ is a weak-$\ast$ continuous functional on $\mathcal{A}^\sim$, there are functionals $f_1, f_2, \ldots, f_n$ in $\mathcal{Z}_\ast$ and $A_1, A_2, \ldots, A_n$ in $\mathcal{A}$ such that
for every $\psi \in \mathcal{A}$.

Proposition 2.1. Let $\mathcal{A}$ be a von Neumann algebra. Let $\mathcal{A}_1$ be the unit sphere of the set $\mathcal{A}$ of bounded functionals of the module $\mathcal{A}$ and let $\mathcal{P}$ be the set of positive elements of $\mathcal{A}_1$. The sets $\mathcal{A}_1$ and $\mathcal{P}$ are compact in the weak-* topology of $\mathcal{A}$.

Proof. Let $\mathcal{L}_A = \mathcal{X}$ for every $A \in \mathcal{A}$. Let $\prod \{\mathcal{L}_A \mid A \in \mathcal{A}\}$ be the product space of $\{\mathcal{L}_A \mid A \in \mathcal{A}\}$ supplied with the product topology induced by the $\sigma$-weak topology on each $\mathcal{L}_A$. Let $\Phi$ be a function of $\mathcal{A}$ into $\mathcal{L}_A$ given by $\Phi(\phi)_A = \phi(A)$. The function $\Phi$ is an isomorphism of $\mathcal{A}$ onto $\Phi(\mathcal{A})$ which is bicontinuous when $\mathcal{A}$ is supplied with the weak-* topology. Let $\mathcal{N} = \prod \{\mathcal{L}_A \mid A \in \mathcal{A}\}$ be the subset of $\prod \mathcal{L}_A$ defined by the relation

$$(\mathcal{L}_A)_A = \{B \in \mathcal{L}_A \mid \|B\| \leq \|A\|\}.$$ 

The set $\mathcal{N}$ is compact in $\prod \mathcal{L}_A$. Since $\|\Phi(\phi)_A\| \leq \|A\|$ whenever $\phi \in \mathcal{A}_1$, it is sufficient to show that $\Phi(\mathcal{A}_1)$ is closed in $\mathcal{N}$ in order to show $\mathcal{A}_1$ is compact in the weak-* topology. Let $\{\phi_n\}$ be a net in $\mathcal{A}_1$ such that $\{\Phi(\phi_n)\}$ converges to an element $\rho$ in $\mathcal{N}$. Let $f$ be an element of $\mathcal{L}_1$, $A_1$ and $A_2$ be elements of $\mathcal{A}$, and $C_1$ and $C_2$ be elements of $\mathcal{X}$. Since the nets

$$f(\psi_n(C_1 A_1)), \quad f(\psi_n(C_2 A_2)) \quad \text{and} \quad f(\psi_n(C_1 A_1 + C_2 A_2))$$ 

converge to

$$f(C_1 \rho A_1), \quad f(C_2 \rho A_2) \quad \text{and} \quad f(\rho(C_1 A_1 + C_2 A_2))$$

respectively, we have that

$$f(C_1 \rho A_1 + C_2 \rho A_2) = f(\rho(C_1 A_1 + C_2 A_2)).$$

Because $f$ is arbitrary, we have that

$$C_1 \rho A_1 + C_2 \rho A_2 = \rho(C_1 A_1 + C_2 A_2).$$

Therefore, the function $A \rightarrow \rho_A$ is a module homomorphism $\phi$ of $\mathcal{A}$ into $\mathcal{X}$. But $\|\phi(A)\| \leq \|A\|$ and therefore $\phi$ is an element of $\mathcal{A}_1$. This proves $\Phi(\mathcal{A}_1)$ is closed in $\mathcal{N}$.

Now we show that $\mathcal{P}$ is weak-* compact in $\mathcal{A}_1$. Let $\{\psi_n\}$ be a net in $\mathcal{P}$ converging in the weak-* topology to a point $\psi$ in $\mathcal{A}_1$. But if $A$ is a positive element of $\mathcal{A}$, then

$$f(\psi(A)) = \lim_n f(\psi_n(A)) \geq \inf_n f(\psi_n(A)) \geq 0$$

for every positive $\sigma$-weakly continuous $f$ functional of $\mathcal{X}$. Thus $\psi(A) \geq 0$ for every
$A \geq 0$. This proves that $S$ is closed in $A_*^\prime$. So $S$ is compact in the weak-* topology. Q.E.D.

Let $\mathcal{A}$ be a von Neumann algebra with center $Z$. The space $\mathcal{A}^\sim$ of bounded functionals on the module $\mathcal{A}$ is a locally convex linear topological space with the weak-* topology. A linear functional $f$ on $\mathcal{A}^\sim$ is said to be hermitian if $f(\phi)$ is real for every hermitian functional $\phi$ in $\mathcal{A}^\sim$. If $\mathcal{X}$ is a nonvoid convex weak-* closed subset of $\mathcal{A}^\sim$ and if $\phi$ is an element of the complement of $\mathcal{X}$, there is a weak-* continuous functional $f$ of $\mathcal{A}^\sim$ such that

$$\text{lub} \{\text{Re} f(\psi) \mid \psi \in \mathcal{X}\} < \text{Re} f(\phi).$$

Here Re $\alpha$ denotes the real part of the number $\alpha$. Suppose $\phi$ is hermitian and the elements of $\mathcal{X}$ are hermitian. Let $f(\phi) = \sum (\phi(A_j)x_j, y_j)$ where $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ are vectors of the Hilbert space of $\mathcal{A}$ and $A_1, A_2, \ldots, A_n$ are elements of $\mathcal{A}$. Let $g(\phi) = \sum (\phi(A_j^*)y_j, x_j)$. The functional $h(\phi) = (f(\phi) + g(\phi))/2$ is a weak-* continuous hermitian functional on $\mathcal{A}^\sim$ which coincides with $\text{Re} f$ on $\mathcal{X} \cup \{\phi\}$. This means that there is a weak-* continuous hermitian functional $h$ of $\mathcal{A}^\sim$ such that

$$\text{lub} \{h(\psi) \mid \psi \in \mathcal{X}\} < h(\phi).$$

Let $\mathcal{L}$ be a commutative von Neumann algebra and let $Z$ be the spectrum of $\mathcal{L}$. If $C$ is an element of $\mathcal{L}$ whose Gelfand transform $C^\sim$ on $Z$ has range contained in the open interval $(0, 1)$, then $C$ is said to lie strictly between $0$ and $1$. If $C$ lies strictly between $0$ and $1$ we write $0 < C < 1$. If $M$ is a $\mathcal{L}$-module, a subset $\mathcal{H}$ of $M$ will be called $\mathcal{L}$-convex if $CA + (1 - C)B$ is in $\mathcal{H}$ whenever $A$ and $B$ are in $\mathcal{H}$ and $C$ is an $\mathcal{L}$-convex subset of $M$ is said to be an extreme point of $\mathcal{H}$ if $CB + (1 - C)D = A$ implies $B = D = A$ whenever $B$ and $D$ are elements of $\mathcal{H}$ and $C$ is an element of $\mathcal{L}$ strictly between $0$ and $1$.

**Theorem 2.2.** Let $\mathcal{A}$ be a von Neumann algebra with center $Z$ and let $S$ be the set of positive functionals of norm not exceeding $1$ of the module $\mathcal{A}$. If $\mathcal{K}$ is a nonvoid $\mathcal{L}$-convex weak-* compact subset of $S$, then $\mathcal{K}$ is the weak-* closure of the smallest $\mathcal{L}$-convex subset of $S$ containing the extreme points of $\mathcal{K}$.

**Proof.** Let $B$ be an element of $\mathcal{A}^\ast$. The set $\{\phi(B) \mid \phi \in \mathcal{K}\}$ is a monotonely increasing net in $\mathcal{L}^\ast$ which is bounded above. Let $B_0 = \text{lub} \{\phi(B) \mid \phi \in \mathcal{K}\}$. The $\mathcal{L}$-convex set $S(B) = \{\phi \in \mathcal{K} \mid \phi(B) = B_0\}$ is nonvoid and contains an extreme point of $\mathcal{K}$. This was demonstrated in Theorem 7 [12] for an analogous situation and virtually the same demonstration applies here.

Let $\mathcal{K}^\prime$ be the weak-* closure of the smallest $\mathcal{L}$-convex subset of $\mathcal{K}$ containing the set of extreme points of $\mathcal{K}$. We show that $\mathcal{K}^\prime = \mathcal{K}^\prime$ by arguing by contradiction. Suppose there is an element $\phi$ in the complement of $\mathcal{K}^\prime$ with respect to $\mathcal{K}$. There is a weak-* continuous hermitian functional $f$ of $\mathcal{A}^\sim$ such that

$$\text{lub} \{f(\psi) \mid \psi \in \mathcal{K}^\prime\} < f(\phi).$$
Let 
\[ T = \{ \theta \in \mathcal{N} \mid f(\theta) = \text{lub} \{ f(\phi) \mid \phi \in \mathcal{N} \} \}. \]

Since \( \mathcal{N} \) is a weak-* compact set and since \( f \) is weak-* continuous, the set \( T \) is a nonvoid weak-* compact subset of \( \mathcal{N} \). We show that \( T \) is \( Z \)-convex. Let \( P \) be a projection in \( Z \). We have that 
\[ (\phi(A)x, y) = (P\phi(A)x, y) + ((1-P)\phi(A)x, y) \]
for every \( \phi \in \mathcal{A}^\ast, A \in \mathcal{A}, \) and \( x \) and \( y \) in the Hilbert space of \( \mathcal{A} \). Thus \( f(\phi) = f(P\phi) + f((1-P)\phi) \) for every \( \phi \) in \( \mathcal{A}^\ast \). Now let \( \theta \) be an element of \( T \). We have that 
\[ f(P\phi + (1-P)\phi) = f(P\phi) + f((1-P)\phi) > f(P\theta) + f((1-P)\phi) = f(\theta). \]
However, the function at \( P\phi + (1-P)\phi \) is an element of \( \mathcal{A}^\ast \). We have reached a contradiction. So,
\[ f(P\theta) = \text{lub} \{ f(P\phi) \mid \phi \in \mathcal{N} \}. \]

This means that \( f(P\theta) = f(P\phi) \) for any two elements \( \theta \) and \( \phi \) in \( T \) and any central projection \( P \). Now let \( C \) be any element in \( Z \). Let \( \epsilon > 0 \) be given; let \( \{ P_j \mid 1 \leq j \leq n \} \) be mutually orthogonal projections of \( Z \) and \( \{ \alpha_j \mid 1 \leq j \leq m \} \) be nonnegative scalars such that \( \| C - \sum \alpha_j P_j \| < \epsilon \). If \( \theta \) and \( \phi \) are elements of \( T \), then
\[ |f(C\theta) - f(C\phi)| \leq |f(C\theta) - f(\sum \alpha_j P_j \theta)| + |f(\sum \alpha_j P_j \phi) - f(C\phi)| \leq 2\epsilon \| f \|. \]

Since \( \epsilon \) is arbitrary, we see that \( f(C\theta) = f(C\phi) \). Thus the set \( T \) is \( Z \)-convex. Now by the remarks made at the beginning of this proof we can conclude that \( T \) has an extreme point \( \phi_0 \). We show that \( \phi_0 \) is an extreme point of \( \mathcal{N} \). Indeed, let \( \phi_1 \) and \( \phi_2 \) be elements of \( \mathcal{N} \) such that \( C\phi_1 + (1-C)\phi_2 = \phi_0 \) for some central element \( C \) strictly between 0 and 1. Let \( D \) be a positive central element; let \( \epsilon > 0 \) be given and let \( \{ P_j \mid 1 \leq j \leq n \} \) be mutually orthogonal central projections such that \( \| D - \sum \alpha_j P_j \| \leq \epsilon \) for suitable nonnegative scalars \( \{ \alpha_j \mid 1 \leq j \leq n \} \). Because
\[ f(P_j\phi_0) = \text{lub} \{ f(P_j\theta) \mid \theta \in \mathcal{N} \} \quad \text{for } j = 1, 2, \ldots, n \]
we have that
\[ f(\sum \alpha_j P_j \phi_1) = \sum \alpha_j f(P_j\phi_1) \leq \sum \alpha_j f(P_j\phi_0) = f(\sum \alpha_j P_j \phi_0). \]

So we have that
\[ f(D\phi_1) \leq f(\sum \alpha_j P_j \phi_1) + \epsilon \| f \| \leq f(\sum \alpha_j P_j \phi_0) + \epsilon \| f \| \leq f(D\phi_0) + 2\epsilon \| f \|. \]

Since \( \epsilon > 0 \) is arbitrary, we have that \( f(D\phi_1) \leq f(D\phi_0) \). So for every central projection \( Q \) we may conclude that
\[ f(CQ\phi_0) = f(CQ\phi_1) \quad \text{and} \quad f((1-C)Q\phi_0) = f((1-C)Q\phi_2), \]
since the sum of the two positive numbers

\[ f(CQ\phi_0) - f(CQ\phi_1) \quad \text{and} \quad f((1 - C)Q\phi_0) - f((1 - C)Q\phi_2) \]

is zero. The elements \( C \) and \( 1 - C \) are invertible in \( \mathcal{S}^+ \). Given \( \varepsilon > 0 \), there are mutually orthogonal central projections \( \{Q_j \mid 1 \leq j \leq n\} \) and nonnegative numbers \( \{\alpha_j \mid 1 \leq j \leq n\} \) such that \( \|C^{-1} - \sum \alpha_j Q_j\| \leq \varepsilon \). Therefore,

\[
|f(\phi_1) - f(\phi_0)| \leq \left| f\left((1 - \left(\sum \alpha_j Q_j\right)C\phi_1\right)\right| + \left| f\left(\left(\sum \alpha_j Q_j\right)C - 1\phi_0\right)\right|
\leq 2\|f\| \left\| 1 - \left(\sum \alpha_j Q_j\right)C \right\|
\leq 2\|f\| \|C\| \left\| C^{-1} - \sum \alpha_j Q_j\right\| \leq 2\|f\| \|C\| \varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary, we have that \( f(\phi_1) = f(\phi_0) \). Similarly we find that \( f(\phi_2) = f(\phi_0) \). This proves that both \( \phi_1 \) and \( \phi_2 \) are elements of \( T \). Because \( \phi_0 \) is an extreme point of \( T \), the element \( \phi_0 \) is equal to \( \phi_1 \) and \( \phi_2 \). Hence \( \phi_0 \) is an extreme point of \( \mathcal{K} \). However, \( \phi_0 \) cannot be in the set \( \mathcal{K}' \). This is a contradiction. Therefore, we must have that \( \mathcal{K} = \mathcal{K}' \). Q.E.D.

In the final section of this paper we shall present some facts about the closure of the smallest \( \mathcal{S} \)-convex subset of \( \mathcal{S} \) containing the extreme points of \( \mathcal{S} \) in the topology of pointwise convergence on \( \mathcal{S} \) where \( \mathcal{S} \) is taken with the uniform topology.

Let \( \mathcal{A} \) be a von Neumann algebra with center \( \mathcal{S} \). A positive functional \( \phi \) of the module \( \mathcal{A} \) is said to majorize a positive functional \( \psi \) if \( \phi - \psi \) is a positive functional of the module \( \mathcal{A} \). If \( \phi \) majorizes \( \psi \), we write \( \phi \geq \psi \). A positive functional \( \phi \) is said to be \( \mathcal{S} \)-irreducible if given any positive functional \( \psi \) majorized by \( \phi \) then there is a positive element \( C \) in \( \mathcal{S} \) such that \( C\phi = \psi \). In [12] we proved the following theorem:

Let \( \mathcal{A} \) be a von Neumann algebra with center \( \mathcal{S} \). Let \( \mathcal{S} \) be the set of all positive functionals of the module \( \mathcal{A} \) with norm not exceeding 1. Let \( \phi \in \mathcal{S} \). The following are equivalent:

1. \( \phi \) is an extreme point of \( \mathcal{S} \); and
2. \( \phi(1) \) is a projection and \( \phi \) is \( \mathcal{S} \)-irreducible.

3. Functionals \( \alpha \)-weakly continuous on the center. In this section we examine the positive functionals of a von Neumann algebra which are \( \alpha \)-weakly continuous when restricted to the center.

If \( \phi \) is a positive functional on a \( C^* \)-algebra \( \mathcal{A} \), let \( L_f \) be the closed left-ideal of \( \mathcal{A} \) given by

\[ L_f = \{ A \in \mathcal{A} \mid f(A^*A) = 0 \}. \]

The space \( \mathcal{A} - L_f \) is a prehilbert space with the inner product

\[ (A - L_f, B - L_f) = f(B^*A). \]

Let \( H(f) \) be the completion of \( \mathcal{A} - L_f \). The representation \( \pi \) of \( \mathcal{A} \) on \( H(f) \) which extends the left multiplication of \( \mathcal{A} \) on \( \mathcal{A} - L_f \) is called the canonical representation.
of $\mathcal{A}$ induced by $f$. There is a vector $x$ in $H(f)$ which is cyclic under $\pi(\mathcal{A})$ such that $(\pi(A)x, x) = f(A)$ for every $A$ in $\mathcal{A}$.

**Theorem 3.1.** Let $f$ be a positive functional of a von Neumann algebra $\mathcal{A}$. Suppose that the restriction $g$ of $f$ to the center $\mathcal{Z}$ of $\mathcal{A}$ is $\sigma$-weakly continuous. There is a unique positive functional $\phi$ of module $\mathcal{A}$ such that $f = g \cdot \phi$ and such that $\phi(1)$ is equal to the support of $g$.

**Proof.** Let $P$ be the support of $g$. Let $\pi$ be the canonical representation of $\mathcal{A}$ on a Hilbert space $H$ induced by $f$. Let $x$ be an element of $H$ cyclic under $\pi(\mathcal{A})$ such that $f(A) = (\pi(A)x, x)$ for every $A$ in $\mathcal{A}$. The representation $\pi$ restricted to $\mathcal{Z}$ is $\sigma$-weakly continuous. Indeed, let $\{A_n\}$ be a monotonely increasing net in $\mathcal{Z}$ with least upper bound $A$. Then $\{A_n - A\} \geq 0$ converges $\sigma$-weakly to $0$. So

$$\{g(A_n - A)\} \geq 0$$

converges to $0$. This means that $\lim_{n \to \infty} \pi(A_n - A)x = 0$. Therefore, $\lim_{n \to \infty} \pi(A_n)x = \pi(A)x$ for every $B$ in $\mathcal{Z}$. Since the net $\{\pi(A_n)\}$ is bounded, the net $\{\pi(A_n)\}$ converges strongly to $\pi(A)$. This proves $\pi$ is $\sigma$-weakly continuous on $\mathcal{Z}$. This shows that $\pi(\mathcal{Z})$ is a von Neumann algebra on $H$ [1, Chapter I, §3, Theorem 2, Corollary 2].

The algebra $\mathcal{Z}P$ is isomorphic to $\pi(\mathcal{Z})$ under the map $\pi$. Let $\pi^{-1}$ denote the inverse of this map. Now let $E$ be the abelian projection of the commutator $\pi(\mathcal{Z})'$ of $\pi(\mathcal{Z})$ on $H$ corresponding to the subspace

$$\text{closure} \{Ax \mid A \in \pi(\mathcal{Z})\}.$$

We have that

$$f(A) = (\pi(A)x, x) = (\pi(\pi(A))x, x)$$

for every $A$ in $\mathcal{A}$. Then define $\phi(A) = \pi^{-1}(\pi(\pi(A)))$. We have that $\phi$ is a positive functional of the module $\mathcal{A}$ such that $\phi(1) = P$. Also we see that

$$g(\phi(A)) = (\pi(\phi(A))x, x) = (\pi(A)x, x) = f(A)$$

for every $A$ in $\mathcal{A}$.

Assume that $\psi$ is a positive functional of the module $\mathcal{A}$ such that $g \cdot \psi = f$. If $P\psi \neq \phi$, then there is an element $A$ in $\mathcal{A}^+$ such that $P\psi(A) \neq \phi(A)$. There is a nonzero projection $Q$ in $\mathcal{Z}P$ and an $\epsilon > 0$ such that either

$$Q\psi(A) + \epsilon Q \leq \phi(A) \quad \text{or} \quad Q\psi(A) + \epsilon Q \leq Q\psi(A).$$

However, we have that $g(Q\psi(A)) = g(Q\psi(A))$ and so $g(\epsilon Q) = 0$ in either case. This means $Q = 0$. This is a contradiction. Therefore $P\psi = \phi$. Q.E.D.

A positive functional $f$ of a C*-algebra $\mathcal{A}$ with center $\mathcal{Z}$ is said to be centrally reducible if for every positive functional $g$ of $\mathcal{A}$ majorized by $f$ there is an element $C$ in $\mathcal{Z}^+$ such that $f(CA) = g(A)$ for every $A$ in $\mathcal{A}$. These centrally reducible functionals have been the object of much study ([15], [8], [25], [26]). The next theorem concerns these functionals.
Theorem 3.2. Let $\mathcal{A}$ be a von Neumann algebra with center $\mathcal{Z}$. Let $f$ be a positive functional on $\mathcal{A}$ whose restriction $g$ to the center $\mathcal{Z}$ is $\sigma$-weakly continuous. The functional $f$ is centrally reducible if and only if the unique positive functional $\phi$ of the module $\mathcal{A}$ with $g \cdot \phi = f$ and with $\phi(1)$ equal to the support $P$ of $g$ is $\mathcal{Z}$-irreducible.

Proof. Suppose $f$ is centrally reducible. Let $\psi$ be a positive functional of the module $\mathcal{A}$ which is majorized by $\phi$. Then $g \cdot \psi$ is majorized by $g \cdot \phi$. There is a $C$ in $\mathcal{Z}^+$ with $g(C \phi(A)) = g(\psi(A))$ for every $A$ in $\mathcal{A}$. By the same argument as contained in Theorem 3.1, we find that $C \phi(A) = P \psi(A)$ for every $A$ in $\mathcal{A}$. Because $0 \leq \psi(1-P) \leq \phi(1-P) = 0$ we have that $P \psi = \psi$. Therefore $C \phi = \psi$. This proves $\phi$ is $\mathcal{Z}$-irreducible.

Conversely, let $\phi$ be $\mathcal{Z}$-irreducible. Let $h$ be a positive functional on $\mathcal{A}$ majorized by $f$. The restriction of $h$ to the center of $\mathcal{A}$ is majorized by $g$. Therefore, $h$ is weakly continuous on $\mathcal{Z}$. By the Radon-Nikodym theorem there is a positive element $B$ in $\mathcal{Z}^P$ such that $g(BA) = h(A)$ for every $A$ in $\mathcal{Z}$. There is by Theorem 3.1 a positive functional of the module $\mathcal{A}$ such that $h \cdot \psi = h$. Thus $g(B \phi(A)) = h(A)$ for every $A$ in $\mathcal{A}$. Hence, for every $A$ in $\mathcal{Z}^+$ we find that $\phi(A) - B \psi(A) \geq 0$. This means that $\phi$ majorizes $B \psi$. There is a $C$ in $\mathcal{Z}^+$ such that $C \phi = B \psi$. Thus we find that $f(CA) = h(A)$ for every $A$ in $\mathcal{A}$. This proves $f$ is centrally reducible. Q.E.D.

Now let $f$ be a positive functional of the von Neumann algebra with center $\mathcal{Z}$. Suppose the restriction $g$ of $f$ to $\mathcal{Z}$ is weakly continuous. Let $\nu$ be the so-called spectral measure on the spectrum $Z$ of $\mathcal{Z}$ such that $g(A) = \int A^\wedge(\zeta) \, d\nu(\zeta)$ for every $A \in \mathcal{Z}$. Here $A^\wedge$ denotes the Gelfand transform of $A$. Let $\phi$ denote the unique positive functional of the module $\mathcal{A}$ such that $\phi(1)$ is the support $P$ of $g$ and such that $f = g \cdot \phi$. Then $f(A) = \int \phi(A)^\wedge(\zeta) \, d\nu(\zeta)$. We note that

1. $\{ \zeta \in Z \mid P^\wedge(\zeta) = 1 \}$ is the support of the spectral measure $\nu$;
2. $f_\zeta(A) = \phi(A)^\wedge(\zeta)$ is a positive functional of $\mathcal{A}$ whose kernel contains $[\zeta]$;
3. for each fixed $A$ in $\mathcal{A}$, the function $\zeta \to f_\zeta(A)$ is continuous on $\mathcal{Z}$.

In §4 we shall show that

4. $f_\zeta$ is irreducible if $\phi$ is $\mathcal{Z}$-irreducible.

If $\nu$ is a spectra measure and $\{ f_\zeta \mid \zeta \in Z \}$ is a family of functions satisfying properties (1)-(3) (respectively, (1)-(4)) then the relation $f(A) = \int f_\zeta(A) \, d\nu(\zeta)$ defines a positive functional (respectively, a centrally reducible functional) which is weakly continuous on $\mathcal{Z}$ [26].

4. Representations on $AW^*$-modules. In this section we study the representations induced by positive module homomorphisms. Our main result will be an analogue of Kadison's Theorem [13] on strictly irreducible representations.

Let $\mathcal{A}$ be a von Neumann algebra. A positive functional $\phi$ of the module $\mathcal{A}$ will be called a state (or expectation) of the module $\mathcal{A}$ if $\phi(1) = 1$. Then if $\psi$ is a positive functional of the module $\mathcal{A}$, there is a state $\phi$ of the module $\mathcal{A}$ such that $\psi = \psi(1) \phi$ [19], [24]. A state $\phi$ of the module $\mathcal{A}$ is said to be a pure state if it is an extreme point of the set of positive functionals of norm not exceeding 1 of the module $\mathcal{A}$.
Proposition 4.1. Let $\mathcal{A}$ be a von Neumann algebra. Let $E$ be a projection in $\mathcal{A}$ and let $P$ be the central support of $E$. There is a pure state of the module $\mathcal{A}$ such that $\phi(E) = P$.

Proof. Let $\mathcal{Z}$ be the center of $\mathcal{A}$. The set $\mathcal{K}$ of states $\phi$ of the module $\mathcal{A}$ such that $\phi(E) = P$ is a $*$-convex weak*-compact subset of the set $\mathcal{S}$ of positive functionals of norm not exceeding 1 of the module $\mathcal{A}$. The set $\mathcal{K}$ is nonvoid. Indeed, let $F_1$ be an abelian projection in the commutator $\mathcal{Z}'$ of $\mathcal{Z}$ with central support $P$ which is majorized by $E$. Let $F_2$ be an abelian projection in $\mathcal{Z}'$ of central support $1 - P$. Then $F = F_1 + F_2$ is an abelian projection of central support 1. This means that $\tau_F$ restricted to $\mathcal{A}$ is a state. Also $\tau_F(E) = P$, i.e. $\tau_F$ is an element of $\mathcal{K}$.

Let $\phi$ be an extreme point of $\mathcal{K}$ (Theorem 2.2). We show $\phi$ is an extreme point of $\mathcal{K}$. Let $\phi_1$ and $\phi_2$ be two functionals in $\mathcal{K}$ and let $C$ be a central element strictly between 0 and 1 such that

$$C\phi_1 + (1 - C)\phi_2 = \phi.$$

We have that $\phi_j(1) \leq 1$ and thus, $\phi_j(E) \leq \phi_j(P) \leq P$ $(j = 1, 2)$. Therefore,

$$C\phi_1(1) + (1 - C)\phi_2(1) = 1 \quad \text{and} \quad C\phi_1(E) + (1 - C)\phi_2(E) = P$$

imply that $\phi_1(1) = \phi_2(1) = 1$ and $\phi_1(E) = \phi_2(E) = P$. Thus, both $\phi_1$ and $\phi_2$ are elements of $\mathcal{K}$. Because $\phi$ is an extreme point of $\mathcal{K}$, we have that $\phi_1 = \phi_2 = \phi$. Q.E.D.

Let $\mathcal{A}$ be a von Neumann algebra and let $\phi$ be a state of $\mathcal{A}$. Let

$$L_\phi = \{A \in \mathcal{A} \mid \phi(A^*A) = 0\}.$$

The factor set $\mathcal{A} - L_\phi$ is a module over $\mathcal{Z}$ which is supplied with an inner product $(A - L_\phi, B - L_\phi) = \phi(B^*A)$ with values in $\mathcal{Z}$. The space $\mathcal{A} - L_\phi$ can then be embedded in a faithful $AW^*$-module $M_\phi$ over $\mathcal{Z}$ obtained by completing $\mathcal{A} - L_\phi$ in the following way. The set $M_\phi$ is the norm completion of the set of all $\{A_n - L_\phi, P_n\}$, where $\{P_n\}$ is a set of orthogonal central projections of sum 1 and $\{A_n - L_\phi\}$ is a set of elements of $\mathcal{A} - L_\phi$ with $\{\phi(A_n^*A_n)\}$ bounded in $\mathcal{Z}$, supplied with the norm induced by the inner product

$$\langle\{A_n - L_\phi, P_n\}, \{B_m - L_\phi, Q_m\} \rangle = \sum_{m,n} \phi(B_m^*A_n)P_nQ_m.$$

There is a bounded homomorphism $\pi_\phi$ of $\mathcal{A}$, which is also a module homomorphism over $\mathcal{Z}$, into the algebra $L(M_\phi)$ of all bounded module homomorphisms of $M_\phi$ onto itself that extends the left multiplication representation of $\mathcal{A}$ on $\mathcal{A} - L_\phi$. This map $\pi_\phi$ is called the canonical representation of $\mathcal{A}$ on $M_\phi$ induced by $\phi$. For the operators $T$ in $L(M_\phi)$ an involution $T \mapsto T^*$ of $L(M_\phi)$ is defined. We have the relation $\langle TA, B \rangle = \langle A, T^*B \rangle$ for $A$ and $B$ in $M_\phi$. The involution also satisfies the relation $||T^*T|| = ||T||^2$. Finally, the representation $\pi_\phi$ preserves adjoints in the sense that $\pi_\phi(A^*) = \pi_\phi(A)^*$ for every $A$ in $\mathcal{A}$ ([17], [6], [28]).

If $\mathcal{Z}$ is a commutative von Neumann algebra on a Hilbert space $H$ and if $\mathcal{Z}'$ is the commutator of $\mathcal{Z}$ on $H$, then for any abelian projection $E$ of $\mathcal{Z}'$ of central
support 1 the module $\mathcal{L}'E$ is an $AW^*$-module over $\mathcal{L}$. The inner product is defined to be $\langle A, B \rangle = \tau_E(B^*A)$ for $A$ and $B$ in $\mathcal{L}'E$ [17].

A specific form for $M_\phi$ is now obtained.

**Proposition 4.2.** Let $\mathcal{A}$ be a von Neumann algebra with center $\mathcal{L}$ and let $\phi$ be a state of the module $\mathcal{A}$. There is a Hilbert space $H$ and a representation $\pi$ of $\mathcal{A}$ on $H$ with the following properties:

1. $\pi$ is faithful on $\mathcal{L}$;
2. $\pi(\mathcal{L})$ is a von Neumann algebra on $H$;
3. there is an abelian projection $E$ in the commutator $\pi(\mathcal{L})'$ of $\pi(\mathcal{L})$ on $H$ such that $\pi(\phi(A)) = \tau_E(\pi(A))$; and
4. there is a function $\Phi$ of $M_\phi$ onto the completion of the module $\pi(\mathcal{A})E$ in $\pi(\mathcal{L})'E$ such that

$$\Phi(A_1B_1 + A_2B_2) = \pi(A_1)\Phi(B_1) + \pi(A_2)\Phi(B_2)$$

for every $A_1$ and $A_2$ in $\mathcal{L}$ and every $B_1$ and $B_2$ in $M_\phi$;

$$\pi(\langle A, B \rangle) = \langle \Phi(A), \Phi(B) \rangle$$

for every $A$ and $B$ in $M_\phi$; and $\Phi(\phi(A)B) = \pi(A)\Phi(B)$ for every $A$ in $\mathcal{A}$ and $B$ in $M_\phi$.

**Proof.** Let $\{P_n\}$ be a set of nonzero mutually orthogonal projections of $\mathcal{A}$ with sum equal to 1 such that each algebra $\mathcal{L}P_n$ is $\sigma$-finite. Let $x_n$ be a unit vector of the Hilbert space of $\mathcal{L}P_n$ which separates $\mathcal{L}P_n$ [1, I, §2, No. 1]. Let $\pi_n$ be the canonical representation of $\mathcal{A}$ on the Hilbert space $H_n$ induced by the positive functional $w_{x_n}\phi$ of $\mathcal{A}$. Here $w_{x_n}(A) = \langle Ax, x \rangle$ for any vector $x$. Let $y_n$ be a vector in $H_n$ cyclic under $\pi_n(A)$ such that

$$(\pi_n(A)y_n, y_n) = w_{x_n}(\phi(A)).$$

Let $\pi$ be the representation $\pi = \sum \pi_n$ on the Hilbert space $H = \sum H_n$.

We show that $\pi$ is faithful on $\mathcal{L}$. Indeed, if $A \in \mathcal{L}$ and $\pi(A) = 0$, then $\pi(AP_n) = 0$ for every $n$. This means $\pi_n(AP_n) = 0$. However, the representation $\pi_n$ is faithful on $\mathcal{L}P_n$; hence $AP_n = 0$ for every $n$. This means $A = 0$. Thus $\pi$ is faithful on $\mathcal{L}$.

We prove now that $\pi$ is $\sigma$-weakly continuous when restricted to $\mathcal{L}$. Let $\{A_m\}$ be a monotonely increasing net $\mathcal{L}^+$ with least upper bound $A$. We have (Proposition 3.1) that $\{\tau_n(A_m)\}_m$ converges strongly to $\pi_n(A)$ for each $n$. Now let $x$ be an element in $H$ and let $\epsilon > 0$ be given. There is a finite subset $P_1, P_2, \ldots, P_k$ of $\{P_n\}$ of sum $P$ such that $\|x - \pi_n(x)\| \leq \epsilon$ because each $\pi_n(P_n)$ is the projection of $H$ on $H_n$. Suppose that for $m \geq m_0$ we have that

$$\|\pi_j(A_m) - \pi_j(A)\| \leq \epsilon k^{-1} \quad \text{for } j = 1, \ldots, k.$$
Then
\[
\|\pi(A)x - \pi(A_m)x\| \leq \|(\pi(A) - \pi(A_m))(1 - \pi(P))x\| + \|(\pi(A) - \pi(A_m))\pi(P)x\|
\]
\[
\leq 2\|A\|e + \sum_{1 \leq j \leq k} (\|\pi_j(A) - \pi_j(A_m)\|\pi_j(P)x\|
\]
\[
\leq (2\|A\| + 1)e.
\]

This proves that \(\pi\) is a \(\sigma\)-weakly continuous isomorphism of \(A\).

By the proof of Theorem 3.1 there is for each \(n\) an abelian projection \(E'_n\) in the commutator \(\pi_n(AF_P)\)' in \(H_n\) associated with the subspace closure \(\{\pi(A)y_n \mid A \in \mathcal{A}\}\)
such that
\[
\tau_{E'_n}(\pi_n(A)) = \pi_n(\phi(AP_n)).
\]

Since \(\pi(\mathcal{A})'\pi(P_n)\) is the commutator of \(\pi(\mathcal{A})\pi(P_n)\) in \(H_n\), we have that there is an abelian projection \(E_n\) in the von Neumann algebra \(\pi(\mathcal{A})'\) on \(H\) majorized by \(\pi(P_n)\) such that
\[
\tau_{E_n}(\pi(AP_n)) = \pi(\phi(AP_n)).
\]

Let \(E\) be the abelian projection in \(\pi(\mathcal{A})'\) given by \(E = \sum E_n\). Then
\[
\tau_{E_n}(\pi(A))\pi(P_n) = \tau_{E_n}(\pi(AP_n)) = \pi(\phi(AP_n))
\]
\[
= \pi(\phi(A))\pi(P_n) \quad \text{for every } n.
\]

This proves that \(\tau_{E_n}(\pi(A)) = \pi(\phi(A))\) for every \(A\) in \(\mathcal{A}\).

Let \(\{A_n - L_\phi \mid n \in N\}\) and \(\{B_m - L_\phi \mid m \in N'\}\) be two bounded sets in \(\mathcal{A} - L_\phi\) and let \(\{Q_n \mid n \in N\}\) and \(\{R_m \mid m \in N'\}\) be two sets of mutually orthogonal central projections of sum 1 respectively. Then \(\sum \pi(Q_n)\pi(A_n)E\) and \(\sum \pi(R_m)\pi(B_m)E\) are elements of the \(AW^*\)-module \(\pi(\mathcal{A})'E\). We have that
\[
\pi\left(\sum Q_n(A_n - L_\phi), \sum R_m(B_m - L_\phi)\right) = \pi\left(\sum_{m,n} Q_n R_m \phi(B^*_m A_n)\right)
\]
\[
= \left< \sum \pi(Q_n)\pi(A_n)E, \sum \pi(R_m)\pi(B_m)E \right>
\]
in the respective inner products of \(M_\phi\) and \(\pi(\mathcal{A})'E\). Therefore,
\[
\Phi\left(\sum Q_n(A_n - L_\phi)\right) = \sum \pi(Q_n)\pi(A_n)E
\]
defines a function of a uniformly dense submodule
\[
M_1 = \left\{ \sum Q_n(A_n - L_\phi) \mid \{Q_n\} \text{ is a set of mutually orthogonal } \right.
\]
central projections of sum 1;
\[
\{A_n - L_\phi\} \text{ is a bounded set in } \mathcal{A} - L_\phi\}
\]
of the module \(M_\phi\) into the submodule.
\[ M_2 = \left\{ \sum \pi(Q_n)\pi(A_n)E \mid \{\pi(Q_n)\} \text{ is a set of mutually orthogonal projections of } \pi(\mathcal{F}) ; \right. \]
\[ \{\pi(A_n)E\} \text{ is a bounded subset of } \pi(\mathcal{A})E \}\]

of the module \( \pi(\mathcal{F})'E \).

We have that \( \Phi \) is a linear function of \( M_1 \) into \( M_2 \) such that \( \Phi(AB) = \pi(A)\Phi(B) \)
for every \( A \in \mathcal{F} \) and \( B \in M_1 \). The range of \( \Phi \) is \( M_2 \). There is a unique extension of \( \Phi \) to a map which we again call \( \Phi \) of the norm completion \( M_\phi \) of \( M_1 \) onto the closure of \( M_2 \) in \( \pi(\mathcal{F})'E \) such that

\[ \Phi(A_1B_1 + A_2B_2) = \pi(A_1)\Phi(B_1) + \pi(A_2)\Phi(B_2) \]

for every \( A_1 \) and \( A_2 \) in \( \mathcal{F} \) and \( B_1 \) and \( B_2 \) in \( M_\phi \) and such that

\[ \langle \Phi(A), \Phi(B) \rangle = \pi(\langle A, B \rangle) \]

for every \( A \) and \( B \) in \( M_\phi \). Since the closure of \( M_2 \) is precisely the \( AW^* \)-module generated by \( \pi(\mathcal{A})E \) in \( \pi(\mathcal{F})'E \) [6, Lemma 4.1], we have that the range of \( \Phi \) is the \( AW^* \)-module generated by \( \pi(\mathcal{A})E \).

Finally, let \( \{A_n - L_\phi\} \) be a bounded set in \( \mathcal{A} - L_\phi \) and let \( \{Q_n\} \) be a set of mutually orthogonal central projections of sum 1. Then

\[ \Phi(\pi_\phi(A)(\sum Q_n(A_n - L_\phi))) = \Phi(\sum Q_n(AA_n - L_\phi)) \]
\[ = \sum \pi(Q_n)\pi(AA_n)E = \pi(A) \sum \pi(Q_n)\pi(A_n)E \]
\[ = \pi(A)\Phi(\sum Q_n(A_n - L_\phi)) \]

for every \( A \in \mathcal{A} \). Thus we have that \( \Phi(\pi_\phi(A)B) = \pi(A)\Phi(B) \) for every \( A \in \mathcal{A} \) and \( B \) in \( M_\phi \). This completes the proof of (4).

Now assume \( \phi \) is a pure state. Let \( \eta \) be the inverse of \( \pi \) restricted to \( \mathcal{F} \). Let \( A \) be a positive element in the unit sphere of the commutator, \( \pi(\mathcal{A})' \) of \( \pi(\mathcal{A}) \) on \( H \). Let \( \tau = \tau_\phi \). The relation

\[ \eta(\tau(A\pi(B))) = \psi(B) \]

defines a functional of the module \( \mathcal{A} \). For every \( B \) in \( \mathcal{A} \) we have that

\[ \psi(B^*B) = \eta(\tau(1/2\pi(B^*B)A^{1/2})) \geq 0 \]

and

\[ \psi(B^*B) = \eta(\tau(\pi(B^*B)^{1/2}A\pi(B^*B)^{1/2})) \]
\[ \leq \eta(\tau(\pi(B^*B)))\|A\| \leq \phi(B^*B). \]

So \( \psi \) is a positive functional majorized by \( \phi \). There is a \( C \) in \( \mathcal{F}^+ \) such that \( C\phi = \psi \) (cf. §2). So for every \( B_1 \) and \( B_2 \) in \( \mathcal{A} \) we have that

\[ \tau(\pi(B_2)^*(A - \pi(C))\pi(B_1)) = 0. \]
This means that

\[(A - \pi(C))\pi(B_1)y_n, \pi(B_2)y_m = 0\]

for every \(y_n\) and \(y_m\). However, the closure of the linear span of

\[\{\pi(B)y_n \mid B \in \mathcal{A}, \text{all } y_n\}\]

is \(H\). Thus \(A = \pi(C)\). Therefore \(\pi(\mathcal{A})'\) is equal to \(\pi(\mathcal{X})\). Q.E.D.

Before continuing we present a brief discussion of a certain trace that is particularly useful. Let \(\mathcal{A}\) be a type I algebra with center \(\mathcal{X}\). There is a locally compact space \(X\) and a positive measure \(\nu\) on \(X\) of support \(X\) such that \(\mathcal{Z}\) is isometric *-isomorphic to the algebra \(L^\infty(X, \nu)\) of all essentially bounded complex-valued measurable functions on \(X\). Identify \(\mathcal{Z}\) with \(L^\infty(X, \nu)\). There is a function \(\text{Tr}\) of \(\mathcal{A}^+\) into the set of all positive finite or infinite valued measurable functions on \(X\) with the following properties:

1. \(\text{Tr}(C_1A_1 + C_2A_2) = C_1\text{ Tr}(A_1) + C_2\text{ Tr}(A_2)\) for \(C_1, C_2 \in \mathbb{R}^+\) and \(A_1, A_2 \in \mathcal{A}^+\);
2. \(\text{Tr}(U^*AU) = \text{Tr}(A)\) for every \(A \in \mathcal{A}^+\) and every unitary \(U \in \mathcal{A}\);
3. if \(\{A_n\}\) is a monotonely increasing net in \(\mathcal{A}^+\) with least upper bound \(A\), then \(\{\text{Tr}(A_n)\}\) has least upper bound \(\text{Tr}(A)\);
4. \(\text{Tr}(E) = \tau_E(E)\) for every abelian projection \(E\) in \(\mathcal{A}\).

If \(\mathcal{P} = \{A \in \mathcal{A}^+ \mid \text{ Tr}(A) \in \mathcal{Z}^+\}\), then \(\mathcal{P}\) is the set of all positive elements of a two-sided ideal \(\mathcal{I}\) in \(\mathcal{A}\) called the trace class of \(\mathcal{A}\). In particular every abelian projection is a member of \(\mathcal{I}\). The function \(\text{Tr}\) on \(\mathcal{P} = \mathcal{I} \cap \mathcal{A}^+\) may be extended to a linear function \(\text{Tr}\) of \(\mathcal{I}\) into \(\mathcal{Z}\) which is also a module homomorphism. For every \(A \in \mathcal{I}\) the function \(B \rightarrow \text{Tr}(AB)\) is a function of \(\mathcal{A}^+\) which is also continuous in the respective \(\sigma\)-weak topologies. We have that \(\text{Tr}(BA) = \text{Tr}(AB)\) for every \(A \in \mathcal{I}\) and \(B \in \mathcal{A}\). Also we have that \(\|B\|^2 \leq \|\text{Tr}(B^*B)\|\) for every \(B \in \mathcal{I}\) [9, §4].

Let \(M\) be an \(\mathcal{A}^*\)-module over the commutative \(\mathcal{A}^*\)-algebra \(\mathcal{X}\) and let \(\mathcal{B}\) be a subalgebra of the algebra \(L(M)\) of all bounded linear operators on \(M\). The algebra \(\mathcal{B}\) is said to be irreducible on \(M\) if given \(A \in L(M)\) and \(C_1, C_2, \ldots, C_n\) in \(M\) then there is a \(B \in \mathcal{B}\) such that \(BC_j = AC_j\) for every \(j = 1, 2, \ldots, n\).

**Theorem 4.3.** Let \(\mathcal{A}\) be a von Neumann algebra with center \(\mathcal{X}\). Let \(\phi\) be a pure state of the module \(\mathcal{A}\). Then the module \(M_\phi\) induced by \(\phi\) is equal to \(\mathcal{A} - L_\phi\) and \(\pi_\phi(\mathcal{A})\) is irreducible on \(M_\phi\).

**Proof.** Let \(\pi\) be the representation relative to \(\phi\) of \(\mathcal{A}\) on the Hilbert space \(H\) constructed in Proposition 4.2. Then \(\pi\) enjoys properties (1)-(5) of this proposition. Let \(E\) be an abelian projection of the commutator \(\mathcal{B}\) of \(\pi(\mathcal{X})\) on \(H\) such that \(\tau_\mathcal{B}(\pi(A)) = \pi(\phi(A))\). We show that \(\pi(\mathcal{A})E = BE\). This means that the module \(M_\phi\) is \(\mathcal{A} - L_\phi\). The algebra of all bounded linear operators on \(BE\) is identified with \(\mathcal{B}\) acting on \(BE\) by left multiplication [17, Theorem 8]. Given \(B_1, B_2, \ldots, B_m\) and \(B\) in \(\mathcal{B}\) we show that there is an \(A \in \mathcal{A}\) with \(\pi(A)B_jE = BB_jE\) for \(j = 1, 2, \ldots, m\). We...
may also show that \( A \) can be chosen to be self-adjoint if \( B \) is self-adjoint. The proof essentially consists of showing that \( E \) is a regular projection with respect to \( \pi(\mathcal{A}) [27] \) using a construction known for pure states (cf. [2, §2.8]).

As a preliminary step assume that \( B_1 E, B_2 E, \ldots, B_m E \) are partial isometric operators \( V_1, V_2, \ldots, V_m \) respectively. Assume also that the range projections \( F_1, F_2, \ldots, F_m \) of the \( V_1, V_2, \ldots, V_m \) are mutually orthogonal. We show that there is an element \( B' \) in \( \mathcal{B} \) such that \( B' V_i = B V_i \) \((1 \leq i \leq m)\) and such that \( \|B'\|^2 \leq 2 \sum \|V_i^* B^* B V_i\| \). We show that \( B' \) may be chosen to be self-adjoint if \( B \) is self-adjoint. Let \( G_i \) be the range projection of \( B V_i \) \((1 \leq i \leq m)\). Since \( G_i \) is equivalent to the domain projection of \( B V_i \), which is majorized by \( E \), the projection \( G_i \) is abelian (cf. [1, III, §1]). Let \( G \) be the least upper bound of the set

\[
\{ F_i \mid 1 \leq i \leq m \} \cup \{ G_i \mid 1 \leq i \leq m \}.
\]

The projection \( G - \sum F_i \) may be written as the sum of mutually orthogonal abelian projections \( F_{m+1}, F_{m+2}, \ldots, F_p \) (cf. [9, Theorem 2.1]). Let

\[
B' = \sum \{ F_i B F_i \mid 1 \leq i \leq m; 1 \leq j \leq p \}
\]

if \( B \) is not self-adjoint and let

\[
B' = \sum \{ F_i B F_i \mid 1 \leq i \leq p; 1 \leq j \leq m \} + \sum \{ F_i B F_i \mid 1 \leq i \leq m; m+1 \leq j \leq p \}
\]

if \( B \) is self-adjoint. In this case \( B' \) is self-adjoint. In either case

\[
B' V_i = \sum \{ F_i B V_i \mid 1 \leq j \leq p \} = B V_i
\]

for \( i = 1, 2, \ldots, m \). In the first case

\[
\text{Tr} (B'^* B') = \sum \{ \text{Tr} (F_i B'^* B' F_i) \mid 1 \leq i \leq m \}
\]

\[
= \sum \{ \tau_{F_i} (B'^* B') \mid 1 \leq i \leq m \},
\]

In the second case we have that

\[
\text{Tr} (B'^* B') = \text{Tr} (B'^2) = \sum \{ \text{Tr} (F_i B'^2 F_i) \mid 1 \leq i \leq m \}
\]

\[
+ \sum \{ \text{Tr} (F_i B' F_i B' F_i) \mid m+1 \leq i \leq m; 1 \leq j \leq p \}
\]

\[
= \sum \{ \text{Tr} (F_i B'^2 F_i) \mid 1 \leq i \leq m \}
\]

\[
+ \sum \{ \text{Tr} (F_i B' F_i B' F_i) \mid m+1 \leq i \leq m; 1 \leq j \leq m \}
\]

\[
\leq 2 \sum \{ \tau_{F_i} (B'^2) \mid 1 \leq i \leq m \}
\]

since \( F_i B' (\sum F_i \mid m+1 \leq i \leq p) B' F_i \leq F_i B'^2 F_i \).

We have that

\[
\|\tau_{F_i} (B'^* B')\| = \|F_i B'^* B' F_i\| = \|V_i^* B'^* B V_i\|.
\]

Thus in either case we conclude that

\[
\|B'^* B'\| \leq \|\text{Tr} (B'^* B')\| \leq 2 \sum \|V_i^* B^* B V_i\|.
\]
This verifies the existence of $B'$ in $\mathcal{B}$. So we may assume that

$$\|B\| \leq (2m)^{1/2} \alpha \quad \text{where} \quad \alpha = \max \{\|BV_i\| \mid 1 \leq i \leq m\}.$$ 

By an application of Tomita's results [27, Theorem 6] we may find a nonzero projection $F$ in $\mathcal{B}$ majorized by $E$ and an element $A$ in $\pi(\mathcal{A})$ such that $\|A\| \leq 2(2m)^{1/2} \alpha$ and $AV_jF = BV_jF$ for $j = 1, 2, \ldots, m$. Indeed given a unit vector $x$ in the Hilbert space of $\mathcal{B}$ such that $Ex = x$, then we may construct by induction a decreasing sequence $\{F'_n\}$ of abelian projections and a sequence of elements $\{A_n\}$ in $\pi(\mathcal{A})$ such that

1. $\|F_n x - F'_{n+1} x\| \leq 4^{-n+1}$ and $\|x - F'_n x\| \leq 4^{-1}$;
2. $\|A_n\| \leq 2^{-n+1}(2m)^{1/2} \alpha$; and
3. $\max \{\|\sum A_j : 1 \leq j \leq n\| - B\|F'_n\| : 1 \leq i \leq m\| \leq 2^{-\alpha}$ for every $n = 1, 2, \ldots$.

Then $A = \sum A_n$ and $F = \text{glb} F'_n \neq 0$. If $B$ is self-adjoint then $A$ may be chosen self-adjoint. Let $\{P_n \mid n \in D\}$ be a maximal set of mutually orthogonal nonzero projections in $\pi(\mathcal{D})$ with the property: for each $P_n$ there is an element $A_n$ in $\pi(\mathcal{A})P_n$ such that $\|A_n\| \leq 2(2m)^{1/2} \alpha$ and such that $A_nV_jE = BV_jEP_n$. We see that $\sum P_n = 1$; otherwise, the projection $P = 1 - \sum P_n$ is nonzero. There is a nonzero projection $F$ majorized by $EP$ and an element $A$ in $\pi(\mathcal{A})$ such that $\|A\| \leq 2(2m)^{1/2} \alpha$ and $AV_jF = BV_jF$. But there is a nonzero projection $Q$ in $\pi(\mathcal{D})$ majorized by $P$ such that $QE = F$. This contradicts the maximality of the set $\{P_n\}$. Therefore, the least upper bound of the set $\{P_n\}$ is 1. There is a set $\{Q_n \mid n \in D\}$ of mutually orthogonal projections in $\mathcal{D}$ such that $\pi(Q_n) = P_n$ for each $n \in D$. Since $\pi$ is norm decreasing, there is for each $A_n$ an element $B_n$ in $\mathcal{A}P_n$ of norm not exceeding $3(2m)^{1/2} \alpha$ such that $\pi(B_n) = A_n$. There is an $A$ in $\mathcal{A}$ such that $AQ_n = B_n$ for each $n \in D$. For each $j = 1, 2, \ldots, m$ we have $\pi(A)V_jE = BV_jE$ because $\pi(A)V_jEP_n = BV_jEP_n$ for every $n$ in $D$.

Let us now assume that $B_1E, B_2E, \ldots, B_mE$ are arbitrary. Let $F_1, F_2, \ldots, F_m$ be the range projections of $B_1E, B_2E, \ldots, B_mE$ respectively. Let $F$ be the least upper bound of $F_1, F_2, \ldots, F_m$. There are mutually orthogonal abelian projections $G_1, G_2, \ldots, G_p$ of sum $F$. Let $V_1, V_2, \ldots, V_p$ be partial isometries with range $G_1, G_2, \ldots, G_p$ respectively and domain support majorized by $E$ (cf. [1, Chapter III, §3, Lemma 1]). By the first part of the proof there is an element $A$ in $\mathcal{A}$, which may be chosen to be self-adjoint if $B$ is self-adjoint, such that $\pi(A)V_j = BV_j$ (1 $\leq j \leq p$). We have that $GB_jE = B_jE$ (1 $\leq j \leq p$).

Thus, we obtain

$$BB_jE = \sum \tau_k(V_k^*B_j)V_k \mid 1 \leq k \leq p\}.$$ 

Thus, we obtain

$$BB_jE = \sum \tau_k(V_k^*B_j)BV_k = \sum \tau_k(V_k^*B_j)\pi(A)V_k = \pi(A)B_jE$$

for $j = 1, 2, \ldots, m$.

Q.E.D.

In the corollary we use the following ideas. Let $\mathcal{A}$ be a von Neumann algebra with center $\mathcal{D}$; let $\zeta$ be a maximal ideal of $\mathcal{D}$. The smallest closed two-sided ideal of
\( \mathcal{A} \) containing \( \zeta \) is denoted by \( [\zeta] \). Then \( [\zeta] \) is the closure of the set

\[ \{ \sum_{1 \leq i \leq n} (A_i B_i) \mid A_i, B_i \in \mathcal{A} \} \quad (1 \leq i \leq n) ; n = 1, 2, \ldots \}. \]

Let \( \mathcal{A}(\zeta) \) be the factor \( C^* \)-algebra \( \mathcal{A}/[\zeta] \) and let \( A(\zeta) \) denote the image of \( A \) in \( \mathcal{A}(\zeta) \). Then J. Glimm proved that for each fixed \( A \) in \( \mathcal{A} \) the function \( \zeta \to ||A(\zeta)|| \) is continuous on the spectrum of \( \mathcal{Z} \) [3, Lemma 10]. If \( P \) is a projection of \( \mathcal{Z} \), then

\[ ||A P|| = \text{lub} \{ ||A(\zeta)|| \mid \zeta \text{ in the spectrum of } \mathcal{Z} \text{ and } P(\zeta) = 1 \}. \]

The least upper bound is attained. If \( A(\zeta) \) is a positive element in \( \mathcal{A}(\zeta) \) for each \( \zeta \), then \( A \) is positive in \( \mathcal{A} \).

Now assume \( \mathcal{A} \) is a type I algebra. Let the notation be the same as the preceding paragraph. Let \( \zeta \) be a fixed maximal ideal of \( \mathcal{Z} \). Suppose \( E \) is an abelian projection in \( \mathcal{A} \) such that \( E(\zeta) \neq 0 \). The space \( H(\zeta) = \mathcal{A}E(\zeta) \) is a Hilbert space with the inner product \( \langle AE(\zeta), BE(\zeta) \rangle = \tau_\psi(B^* A)E(\zeta) \). The algebra \( \mathcal{A} \) has a representation \( \Psi \) with kernel \( [\zeta] \) on the algebra of all bounded operators on \( H(\zeta) \) given by \( \Psi(A)BE(\zeta) = AEB(\zeta) \), for every \( A \) and \( B \) in \( \mathcal{A} \). The closed two-sided ideal \( I_\zeta \) of \( \mathcal{A} \) generated by the abelian projections of \( \mathcal{A} \) maps onto the ideal of completely continuous operators of \( H(\zeta) \). In particular if \( x \) is an arbitrary vector in \( H(\zeta) \) there is an abelian projection \( F \) in \( \mathcal{A} \) such that \( \Psi(F)x = x \). The images of abelian projections under \( \Psi \) have dimension not exceeding 1 [3, §4].

**Corollary.** Let \( \mathcal{A} \) be a von Neumann algebra with center \( \mathcal{Z} \) and let \( \phi \) be a \( \mathcal{Z} \)-irreducible functional of the module \( \mathcal{A} \). For every \( \zeta \) in the spectrum of \( \mathcal{Z} \) the functional \( \phi(A)E(\zeta) \) of \( \mathcal{A} \) is irreducible. In particular if \( \phi \) is an extreme point of the set of positive functionals of norm not exceeding 1 of the module \( \mathcal{A} \), then \( \phi(A)E(\zeta) \) is irreducible on \( \mathcal{A} \).

**Proof.** We may assume that \( \phi(1)E(\zeta) \neq 0 \). There is a projection \( P \) in \( \mathcal{Z} \) which does not lie in the maximal ideal \( \zeta \) of \( \mathcal{Z} \) and a number \( \alpha > 0 \) such that \( \phi(1)P \geq \alpha P \). Let \( C \) be a positive element in \( \mathcal{Z}P \) such that \( C\phi(1) = P \). The functional \( \psi = C\phi \) is a \( \mathcal{Z} \)-irreducible functional of the module \( \mathcal{A} \). Indeed, if \( \psi \) majorizes the positive functional \( \theta \) of the module \( \mathcal{A} \), then \( P\psi \) majorizes \( P\phi(1)\theta \) and so \( \phi \) majorizes \( P\phi(1)\theta \). There is a \( D \) in \( \mathcal{Z}^+ \) such that \( D\phi = P\phi(1)\theta \). Thus \( D\psi = CD\phi = C\phi(1)\theta = \theta \). This proves that \( \psi \) is \( \mathcal{Z} \)-irreducible. Since the functional \( \phi(A)E(\zeta) \) is equal to a nonzero scalar multiple \( \phi(A)E(\zeta) \), it is sufficient to prove that \( \phi(A)E(\zeta) \) is irreducible.

Now let \( \psi \), be any pure state of the module \( \mathcal{A} \). The functional \( P\psi + (1 - P)\psi = \psi' \) is a \( \mathcal{Z} \)-irreducible state of the module \( \mathcal{A} \). Indeed, if \( \theta \) is a positive functional of the module \( \mathcal{A} \) majorized by \( \psi' \), then \( P\theta \) majorizes \( \theta \) and \( (1 - P)\psi \) majorizes \( (1 - P)\theta \). There are elements \( C_1 \) and \( C_2 \) in \( \mathcal{Z}^+ \) with \( C_1\psi = P\theta \) and \( C_2\psi = (1 - P)\theta \). We may assume that \( PC_1 = C_1 \) and \( (1 - P)C_2 = C_2 \). Setting \( C = C_1 + C_2 \) we have that \( C\psi = \theta \). So \( \psi' \) is a \( \mathcal{Z} \)-irreducible state of the module \( \mathcal{A} \), i.e. \( \psi' \) is a pure state of \( \mathcal{A} \). Since \( \phi(A)E(\zeta) = \psi(A)E(\zeta) \) for every \( A \) in \( \mathcal{A} \), there is no loss of generality in assuming that \( \psi \) is a pure state of the module \( \mathcal{A} \).
Let \( \pi \) be a representation of \( \mathcal{A} \) on a Hilbert space \( H \) constructed in Proposition 4.2 relative to \( \phi \). Let \( E \) be a maximal abelian projection of the von Neumann algebra \( \mathcal{B} \) generated by \( \pi(\mathcal{A}) \) on \( H \) such that \( \tau_E(\pi(A)) = \pi(\phi(A)) \) for every \( A \) in \( \mathcal{A} \). There is a homeomorphism \( \eta \) of the spectrum \( Z \) of the center \( \mathcal{Z} \) of \( \mathcal{A} \) onto the spectrum of \( \pi(\mathcal{Z}) \) such that \( \pi(A)^* \eta(\xi) = A^* \eta(\xi) \) for every \( \xi \in Z \). Let \( \zeta \) be a fixed element in \( Z \) and let \( \eta(\zeta) = \zeta' \). Then

\[
\phi(A)^* \zeta = \tau_E(\pi(A))^* \zeta' = (\Psi(\pi(A)))x, x
\]

for every \( A \) in \( \mathcal{A} \). This proves that \( \phi(A)^* \zeta \) is irreducible on \( \mathcal{A} \). Q.E.D.

We now record some facts about the kernel of \( \pi_\phi \).

**Proposition 4.4.** Let \( \mathcal{A} \) be a von Neumann algebra and let \( \phi \) be a state of the module \( \mathcal{A} \). The kernel of \( \pi_\phi \) is contained in the strong radical (viz, the intersection of all two-sided maximal ideals) of \( \mathcal{A} \). In particular, if \( \mathcal{A} \) is finite or if \( \mathcal{A} \) is \( \sigma \)-finite and of type III, then \( \pi_\phi \) is faithful.

**Proof.** Let \( A \) be an element of \( \mathcal{A} \). Let \( \mathcal{A}^\prime \) be the uniform closure of the set

\[
\left\{ \sum \alpha_i U_i^* A U_i \mid i = 1, 2, \ldots, n \middle| \alpha_1, \alpha_2, \ldots, \alpha_n \text{ are positive of sum 1}; \ U_1, U_2, \ldots, U_n \text{ are unitary in } \mathcal{A}; n = 1, 2, \ldots \right\}.
\]

Then \( \mathcal{A}^\prime \cap \mathcal{Z} = \mathcal{A} \) is nonvoid for every \( A \) in \( \mathcal{A} \). If \( \mathcal{A} \) is finite, then \( \mathcal{A}^\prime \) contains a single element \( A^\# \). In this case if \( A \in \mathcal{A}^+ \) and \( A^\# = 0 \), then \( A = 0 \) [1, III, §5].

Assume first that \( \mathcal{A} \) is finite. Set \( \pi_\phi = \pi \) and let \( A \) be an element of \( \mathcal{A} \) such that \( \pi(A) = 0 \); then \( \pi(A^* A) = 0 \). Since

\[
\pi\left( \sum \alpha_i U_i^* A^* A U_i \right) = \sum \alpha_i \pi(U_i^* \pi(A^* A) \pi(U_i)) = 0
\]

and since \( \pi \) is uniformly continuous, we have that \( \pi((A^* A)^\#) = 0 \). This means

\[
0 = \phi((A^* A)^\#) = (A^* A)^\#.
\]

Therefore, \( A^* A = 0 \) and thus \( \pi \) is faithful.

Now assume that \( \mathcal{A} \) is properly infinite. The radical of \( \mathcal{A} \) is the ideal of \( \mathcal{A} \) all of
whose positive elements $A$ satisfy the relation $\mathcal{X}_A = \{0\}$, [10, Proposition 2.4]. Therefore we readily conclude that $\pi(A) = 0$ implies that $A$ is in the radical of $\mathcal{A}$.

Now in the general case there is a projection $P$ in the center of $\mathcal{A}$ such that $\mathcal{A}P$ is finite and $\mathcal{A}(1-P)$ is properly infinite. If $A$ is an element in the kernel of $\pi$, then $AP = 0$ and $A(1-P)$ is in the radical of $\mathcal{A}(1-P)$. But the radical of $\mathcal{A}(1-P)$ is the radical of $\mathcal{A}$. So the kernel of $\pi$ is contained in the radical of $\mathcal{A}$. Q.E.D.

We now show that there are states which have faithful representations.

**Proposition 4.5.** Let $\mathcal{A}$ be a von Neumann algebra. There is a projection $E$ in $\mathcal{A}$ of central support 1 such that every state $\phi$ of the module $\mathcal{A}$ with the property $\phi(E) = 1$ has a faithful representation $\pi_\phi$.

**Proof.** First let $\mathcal{A}$ be semifinite. Let $E$ be any finite projection of $\mathcal{A}$ of central support 1. Then let $\phi$ be a state of $\mathcal{A}$ such that $\phi(E) = 1$. Let $F$ be a projection of $\mathcal{A}$ with $\pi(F) = 0$ where $\pi = \pi_\phi$. Suppose $F$ has central support $P$. First assume that $F \leq EP$. Since $EP$ is finite, there is a set $\{P_i\}$ of mutually orthogonal central projections of sum 1 such that for each $P_i$ there is a set

$$\{F_{ij} \mid 1 \leq j \leq n_i < +\infty\}$$

of mutually orthogonal projections with the properties:

$$F_{ij} \sim FP_{ij} \quad \text{and} \quad F_i' = EP_i - \sum_j F_{ij} < FP_{ij} \quad [1, III, \S 1].$$

Since $\pi(F_{ij}) = 0$ we have that $\pi(F_i) = 0$ (for $j \leq n_i$) and $\pi(F_i') = 0$. Indeed, if $V$ is a partial isometric operator and $\pi(V^*V) = 0$, then $0 = \pi(V^*V) = \pi(V^* \pi(V))$ implies $\pi(V) = 0$. So $\pi(VV^*) = 0$. Then we conclude that $\pi(EF_i) = 0$ for every $P_i$. This means $P_i = 0$ and thus $P = 0$. So $F = 0$.

In the general case there is a central projection $P$ such that $FP < EP$ and $E(1-P) < F(1-P)$. We have that $FP = 0$ from the first part of the proof since we may assume $FP \leq EP$. Also $\pi(E(1-P)) = 0$. So $1-P = 0$. Thus, $F = 0$.

Now let $A$ be any element of $\mathcal{A}$ such that $\pi(A) = 0$. Suppose $\varepsilon > 0$ is given; let $F_1, F_2, \ldots, F_m$ be orthogonal projections and let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be positive numbers with $0 \leq \sum \alpha_i F_i \leq A^*A$ and $\|A^*A - \sum \alpha_i F_i\| \leq \varepsilon$. Then $\pi(F_i) = 0$ and so $F_i = 0$ (for $i = 1, 2, \ldots, m$). We obtain this from the first part. This shows $\|A^*A\| \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have that $A = 0$. This shows that $\pi$ is faithful if $\mathcal{A}$ is semifinite.

Now let $\mathcal{A}$ be a purely infinite von Neumann algebra with no nonzero $\sigma$-finite central projections. There is a net $\{P_i\}$ of orthogonal central projections of sum 1 such that each $P_i$ is least upper bound of a set $S_i$ of equivalent mutually orthogonal $\sigma$-finite projections $[1, III, \S 1, Lemma 7]$. For each $i$ let $E_i \in S_i$ and let $E = \sum E_i$. Then $E$ is a projection of central support 1. If $F$ is a projection of central support $Q$ then $EQP_i < FP_i$ for each $P_i$ $[1, III, \S 8, Corollary 5]$. So $EQ < F$.

Let $\phi$ be a state of the module $\mathcal{A}$ such that $\phi(E) = 1$. We show that the kernel of $\pi_\phi = \pi$ is 0. It is sufficient to show that $\pi(F) = 0$ implies $F = 0$ whenever $F$ is a pro-
jection. However, if $F$ has central support $Q$ then $EQ \prec F$. So $\pi(EQ) = 0$ and thus $\phi(EQ) = Q = 0$. This proves $F = 0$.

Now let $\mathcal{A}$ be a purely infinite algebra. There is a projection $P$ of $\mathcal{A}$ such that $P$ is the least upper bound of $\sigma$-finite central projections and such that $1 - P$ majorizes no nonzero $\sigma$-finite central projections. Now let $F$ be any projection in $\mathcal{A}P$ of central support 1 and let $E$ be a projection previously constructed for a purely infinite von Neumann algebra with no nonzero $\sigma$-finite central projections. Let $\phi$ be a state of $\mathcal{A}$ such that $\phi(E + F) = 1$. The canonical representation $\pi$ induced by $\phi$ has kernel equal to $(0)$ [Proposition 4.4].

The general result for an arbitrary von Neumann $\mathcal{A}$ algebra now follows from the fact that there is a central projection $P$ such that $\mathcal{A}P$ is semifinite and $\mathcal{A}(1 - P)$ is purely infinite. Q.E.D.

**Theorem 4.6.** Let $\mathcal{A}$ be a von Neumann algebra. There is a pure state of the module $\mathcal{A}$ whose canonical representation is faithful.

**Proof.** There is a projection $E$ of $\mathcal{A}$ of central support 1 such that the canonical representation $\pi_\phi$ induced by a state $\phi$ of the module $\mathcal{A}$ is faithful whenever $\phi(E) = 1$ (Proposition 4.5). By Proposition 4.1 there is a pure state $\phi$ of the module $\mathcal{A}$ such that $\phi(E) = 1$. Q.E.D.

**Theorem 4.7.** Let $\mathcal{A}$ be a von Neumann algebra and let $\zeta$ be a maximal ideal of the center of $\mathcal{A}$. The smallest closed two-sided ideal $[\zeta]$ in $\mathcal{A}$ containing $\zeta$ is a primitive ideal.

**Proof.** Let $\phi$ be a pure state of $\mathcal{A}$ whose canonical representation $\pi_\phi$ is faithful. The representation $\pi$ of $\mathcal{A}$ on the space $H$ satisfying properties (1)–(5) of Proposition 4.2 constructed relative to $\phi$ is faithful. Let $\zeta' = \pi(\zeta)$ and let $[\zeta']$ be the smallest closed two-sided ideal in the von Neumann algebra $\mathcal{B}$ generated by $\pi(\mathcal{A})$ on $H$ which contains $\zeta'$. There is an irreducible representation $\Psi$ of $\pi(\mathcal{A})$ with kernel $\pi(\mathcal{A}) \cap [\zeta']$ (corollary, Theorem 4.3). However $\pi(\mathcal{A}) \cap [\zeta']$ is the smallest closed two-sided ideal $J$ of $\pi(\mathcal{A})$ which contains $\zeta'$. Indeed, if $E$ is a projection in $\pi(\mathcal{A}) \cap [\zeta']$ then the Gelfand transform $P^\sim$ of the central support $P$ of $E$ vanishes at the point $\zeta'$. Thus the projection $P$ is in the maximal ideal $\zeta'$ and so $E$ is in the ideal $J$. Because $J$ contains all projections of $\pi(\mathcal{A}) \cap [\zeta']$, the ideal $\pi(\mathcal{A}) \cap [\zeta']$ is contained in $J$. Therefore we have that $J = \pi(\mathcal{A}) \cap [\zeta']$. However, $\pi^{-1}(J) = [\zeta]$ since $\pi$ is faithful. So the kernel of $\Psi^{-1}$ is $[\zeta]$. Q.E.D.

The set $\text{Prim}(\mathcal{A})$ of all primitive ideals of $\mathcal{A}$ supplied with the hull-kernel topology is called structure space of $\mathcal{A}$.

**Proposition 4.7.** Let $\mathcal{A}$ be a von Neumann algebra with center $\mathcal{Z}$. Let $Z$ be the spectrum of $\mathcal{Z}$. The set $\{[\zeta] \mid \zeta \in Z\}$ is dense in the structure space of $\mathcal{A}$.

**Proof.** Let $X$ be a nonvoid open set in $\text{Prim}(\mathcal{A})$. There is an ideal $I$ in $\mathcal{A}$ such that

$$X = \{J \in \text{Prim}(\mathcal{A}) \mid J \neq I\}.$$
Let $J$ be an ideal in $X$ and let $J \cap \mathcal{Z} = \zeta$. The ideal $\zeta$ is maximal in $\mathcal{Z}$. We have that $[\zeta] \not\supset J$ since $[\zeta] \not\subset J$. This proves that $[\zeta] \in X$. Thus $\{[\zeta] : \zeta \in \mathbb{Z}\}$ is dense in $\text{Prim } (\mathcal{A})$. Q.E.D.

The set $\mathcal{A}^\wedge$ of unitary equivalence classes of irreducible representations of $\mathcal{A}$ with the topology induced by the map $\pi \mapsto \text{kernel } \pi$ of $\mathcal{A}^\wedge$ into $\text{Prim } (\mathcal{A})$ is known to be a Baire space [2, 3.4.13]. A proof of this fact is obtainable from the preceding proposition.

The next theorem characterizes a pure state in terms of its kernel. It is the analogue of a theorem of Kadison [13].

**Theorem 4.8.** Let $\mathcal{A}$ be a von Neumann algebra. A state $\phi$ of the module $\mathcal{A}$ is a pure state if and only if the kernel of $\phi$ is the sum of the sets

$$L_\phi = \{ A \in \mathcal{A} : \phi(A^*A) = 0 \} \quad \text{and} \quad L^*_\phi = \{ A \in \mathcal{A} : A^* \in L_\phi \}.$$

**Proof.** Suppose $\phi$ is a pure state of the module $\mathcal{A}$. Let $\pi$ be a representation of $\mathcal{A}$ on a Hilbert space $H$ which satisfies properties (1)–(5) of Proposition 4.2 with respect to $\phi$. Let $E$ be the abelian projection of the von Neumann algebra $\mathcal{B}$ generated by $\pi(\mathcal{A})$ such that $\pi(A) = \tau_E(\pi(A))$ for every $A$ in $\mathcal{A}$. Suppose $A$ is a point of the kernel of $\phi$. The range projection $F$ of $\pi(A)E$ in $\mathcal{B}$ is an abelian projection orthogonal to $E$. There is a hermitian element $C$ in $\mathcal{A}$ such that $\tau_E(C)E = 0$ (Theorem 4.3). Thus, $A - CA \in L_\phi$ and $A^*C \in L_\phi$. So $A = (A - CA) + CA$ is an element of $L_\phi + L^*_\phi$. This proves that $L_\phi + L^*_\phi$ contains the kernel of $\phi$. Because $|\phi(A)|^2 \leq \phi(A^*A)$ for every $A$ in $\mathcal{A}$, the kernel of $\phi$ contains $L_\phi + L^*_\phi$. So the kernel of $\phi$ is equal to $L_\phi + L^*_\phi$.

Conversely, let $L_\phi + L^*_\phi$ be the kernel of $\phi$. Let $C$ be a central element of $\mathcal{A}$ strictly between 0 and 1 and let $\phi_1$ and $\phi_2$ be two positive functionals of the module $\mathcal{A}$ of norm not exceeding 1 such that $C \phi_1 + (1-C) \phi_2 = \phi$. First notice that $\phi_1$ and $\phi_2$ are states of $\mathcal{A}$. Then if $\phi(A) = 0$, there are elements $B_1$ and $B_2$ in $L_\phi$ and $L^*_\phi$ respectively such that $A = B_1 + B_2$. Because $\phi(B_1^*B_2) = 0$, we have that $\phi_1(B_j) = \phi_2(B_j) = 0$ for $j = 1, 2$. Thus $\phi_1(A) = \phi_2(A) = 0$. Now for arbitrary $A$ in $\mathcal{A}$ there is a central element $B$ in $\mathcal{A}$ such that $\phi(A - B) = 0$. Thus $\phi_1(A - B) = \phi_2(A - B) = 0$ and so $\phi_1(A) = \phi_2(A) = B = \phi(A)$. This proves $\phi$ is a pure state. Q.E.D.

### 5. Pointwise convergence of states

Let $\mathcal{A}$ be a von Neumann algebra. A net of states $\{\phi_n\}$ of the module $\mathcal{A}$ is said to converge pointwise to a state $\phi$ if $\{\phi_n(A)\}$ converges uniformly to $\phi(A)$ for every $A$ in $\mathcal{A}$. The set $E(\mathcal{A})$ of states of the module $\mathcal{A}$ taken with the topology of pointwise convergence is called the state space of $\mathcal{A}$. The closure in the state space of the module $\mathcal{A}$ of the set of pure states in $\mathcal{A}$ is called the pure state space of the module $\mathcal{A}$. It is denoted by $P(\mathcal{A})$. An element $\phi$ in $E(\mathcal{A})$ is said to be a vector state if there is an abelian projection $E$ in the commutator of the center of $\mathcal{A}$ such that $\phi(A) = \tau_E(A)$ for every $A$ in $\mathcal{A}$. The closure in the space $E(\mathcal{A})$ of the set of vector states is called the vector state space of $\mathcal{A}$. It is denoted by $V(\mathcal{A})$.  

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We now study the structure of $P(\mathcal{A})$ and $V(\mathcal{A})$ using the theorems of Glimm [3, §4] as our guide.

**Theorem 5.1.** If $\mathcal{A}$ is a continuous von Neumann algebra, the state space, the pure state space, and the vector state space of the module $\mathcal{A}$ coincide.

**Proof.** First we show that the vector state space $V(\mathcal{A})$ of the module $\mathcal{A}$ coincides with the state space $E(\mathcal{A})$ of the module $\mathcal{A}$. Let $\phi$ be an element of $E(\mathcal{A})$ and let $A_1, A_2, \ldots, A_n$ be elements of $\mathcal{A}$. Assume $A_1 = 1$. Let $\mathcal{Z}$ be the commutator of the center $\mathcal{Z}$ of $\mathcal{A}$ and let $[\xi]$ denote the smallest closed two-sided ideal in $\mathcal{Z}$ which contains the maximal ideal $\xi$ of $\mathcal{Z}$. There is for each ideal $\xi$ an irreducible representation $\Psi_\xi$ of $\mathcal{Z}$ with kernel $[\xi]$ on the algebra of bounded linear operators of a Hilbert space $H(\xi)$ such that $\Psi_\xi(\mathcal{Z})$ contains the ideal $C(H(\xi))$ of completely continuous operators on $H(\xi)$. Since $\mathcal{A}$ is a continuous algebra, the image $\Psi_\xi(\mathcal{A})$ of $\mathcal{A}$ contains no minimal projections. So $\Psi_\xi(\mathcal{A}) \cap C(H(\xi)) = (0)$. There is unit vector $x_\xi$ in $H(\xi)$ such that
\[
\langle \Phi(A), (\Psi_\xi(A)x_\xi, x_\xi) \rangle < \frac{1}{2} \quad \text{for } j = 1, 2, \ldots, n.
\]
Indeed, the kernel of the functional $A \mapsto \langle \Phi(A), x_\xi \rangle$ contains the ideal $\mathcal{A} \cap [\xi]$. So there is a functional $\phi_\xi$ of $\Psi_\xi(\mathcal{A})$ such that
\[
\phi_\xi(\Psi_\xi(A)) = \phi(A)_{\xi}.
\]
Then the statement in question simply states that the functional $\phi_\xi$ is the pointwise limit of vector states of $\Psi_\xi(\mathcal{A})$ [2, 11.2.1]. There is an abelian projection $E_\xi$ of $\mathcal{Z}$ such that
\[
\langle \Psi_\xi(B)x_\xi, x_\xi \rangle = (\tau_{E_\xi}(B)x_\xi, x_\xi)
\]
for every $B$ in $\mathcal{Z}$ (cf. [7, Theorem 1]). This means that there is a central projection $P$ with $P_{\xi} = 1$ such that $E_\xi P$ has central support $P$ and such that
\[
\|\phi(A) - \tau_{E_\xi}(A)\| < 1
\]
for $j = 1, 2, \ldots, n$. Thus there is an abelian projection $E$ of central support 1 such that $E(P_{\xi}A)P$ is central support $P$. This shows that $\phi$ is the pointwise limit of vector states. Thus $E(\mathcal{A}) = V(\mathcal{A})$.

We now show that $E(\mathcal{A})$ is equal to the pure state space of the module $\mathcal{A}$. First let $\psi$ be any pure state of the module $\mathcal{A}$ whose canonical representation $\pi_\psi$ is faithful (Theorem 4.6). Let $\pi$ be a faithful representation of $\mathcal{A}$ on a Hilbert space $H$ such that the commutator $\pi(\mathcal{A})' \cap \pi(\mathcal{Z})$ on $H$ is equal to $\pi(\mathcal{Z})$ and such that there is an abelian projection $E$ in $\pi(\mathcal{Z})'$ of central support 1 with the property $\tau_{E}(\pi(A)) = \pi(\psi(A))$ for every $A$ in $\mathcal{A}$ (Proposition 4.2). Let $\phi$ be an element of $E(\mathcal{A})$ and let $A_1, A_2, \ldots, A_m$ be elements of $\mathcal{A}$. There is an abelian projection $F$ of central support 1 in $\pi(\mathcal{Z})'$ such that
\[
\|\pi \cdot \phi \cdot \pi^{-1}(\pi(A)) - \tau_{E}(\pi(A))\| < 1
\]
for $j = 1, 2, \ldots, m$. Indeed, if $\zeta$ is a maximal ideal in $\pi(\mathcal{L})$, there is an irreducible representation $\Psi_\zeta$ of $\pi(\mathcal{L})'$ with kernel $[\zeta]$ on a Hilbert space such that the image of $\pi(\mathcal{A})$ contains no completely continuous operators. The same reasoning as in the previous paragraph therefore is applicable. So it is sufficient to show that $\pi^{-1} \cdot \tau_F \cdot \pi$ is a pure state of $\mathcal{A}$. We do this by showing that it is $\mathcal{L}$-irreducible. Let $\theta$ be a positive functional of the module $\mathcal{A}$ majorized by $\pi^{-1} \cdot \tau_F \cdot \pi$. Then $\theta' = \pi \cdot \theta \cdot \pi^{-1}$ on $\pi(\mathcal{A})$ is majorized by $\tau_F$ on $\pi(\mathcal{A})$. Let $\zeta$ be a maximal ideal in $\pi(\mathcal{L})$. There are positive functionals $f$ and $g$ on $\Psi_\zeta(\pi(\mathcal{A}))$ such that

$$f(\Psi_\zeta(A)) = \theta'(A)^{\zeta} \quad \text{and} \quad g(\Psi_\zeta(A)) = \tau_F(A)^{\zeta}$$

for $A$ in $\pi(\mathcal{A})$. Then $g$ majorizes $f$ on $\Psi_\zeta(\pi(\mathcal{A}))$. However $g$ is irreducible on $\Psi_\zeta(\pi(\mathcal{A}))$ and so there is an $a_\zeta$ in the complex field such that $f(A) = a_\zeta g(A)$ for all $A$ in $\Psi_\zeta(\pi(\mathcal{A}))$. But $a_\zeta = \theta'(1)^{\zeta}$. Since $\zeta$ is arbitrary we have that $\theta' = \theta'(1) \tau_F$ on $\pi(\mathcal{A})$. This proves that $\pi^{-1} \cdot \tau_F \cdot \pi$ is $\mathcal{L}$-irreducible. Q.E.D.

We see that if $\pi$ is a faithful representation of the continuous algebra $\mathcal{A}$ on a Hilbert space $H$ with the property that the commutator of $\pi(\mathcal{A})$ is $\pi(\mathcal{L})$ and that there is an abelian projection $E$ with central support 1 in the commutator $\pi(\mathcal{L})'$ of $\pi(\mathcal{L})$ such that $\pi^{-1} \cdot \tau_E \cdot \pi$ is a pure state of $\mathcal{A}$, then the set

$$\{\pi^{-1} \cdot \tau_F \cdot \pi \mid F \text{ is an abelian projection of central support 1 in } \pi(\mathcal{L})'\}$$

is pointwise dense in $E(\mathcal{A})$.

We now identify the pure state and vector state spaces of a type I algebra. We begin with the following theorem.

**Theorem 5.2.** If $\mathcal{A}$ is a type I von Neumann algebra, the vector state space $V(\mathcal{A})$ of the module $\mathcal{A}$ is equal to the pure state space $P(\mathcal{A})$ of the module $\mathcal{A}$.

**Proof.** Since every vector state of the module $\mathcal{A}$ is a pure state of the module $\mathcal{A}$, we have that $V(\mathcal{A}) = P(\mathcal{A})$ [12, Remark, Theorem 9].

Now let $\phi$ be a pure state of the module $\mathcal{A}$. Let $A_1, A_2, \ldots, A_m$ be elements of $\mathcal{A}$. For each maximal ideal $\zeta$ of the center of $\mathcal{A}$ there is an irreducible representation $\Psi_\zeta$ of $\mathcal{A}$ with kernel $[\zeta]$ on a Hilbert space $H(\zeta)$ such that $\Psi_\zeta$ contains the completely continuous operators on $H(\zeta)$. The kernel of the function $A \rightarrow \phi(A)^{\zeta}$ on $\mathcal{A}$ contains the ideal [7]. There is thus a functional $\phi_\zeta$ of $\Psi_\zeta$ such that $\phi_\zeta(\Psi_\zeta(A)) = \phi(A)^{\zeta}$ for every $A$. Since $\phi(A)^{\zeta}$ is a pure state of $\mathcal{A}$ (corollary, Theorem 4.3), the functional $\phi_\zeta$ is a pure state of $\Psi_\zeta$. The pure state space of $\Psi_\zeta$ is equal to the vector space of $\Psi_\zeta$ (cf. [2, 3.4.1] due to R. V. Kadison). There is a unit vector $x_\zeta$ in $H(\zeta)$ such that

$$|\phi_\zeta(\Psi_\zeta(A_j)) - (\Psi_\zeta(A_j)) x_\zeta, x_\zeta| < 1$$

for $j = 1, 2, \ldots, m$. There is an abelian projection $E_\zeta$ in $\mathcal{A}$ such that

$$(\Psi_\zeta(A_j)) x_\zeta = \tau_{E_\zeta}(A_j)^{\zeta}$$
for every $A$ in $\mathcal{A}$. By the same reasoning as Theorem 5.1 we obtain an abelian projection $E$ in $\mathcal{A}$ of central support 1 such that
$$
\|\phi(A_j) - \tau_E(A_j)\| < 1
$$
for $j=1, 2, \ldots, m$. This means that $\phi \in V(\mathcal{A})$. Therefore, $P(\mathcal{A}) \subset V(\mathcal{A})$. This completes the proof.

Let $\mathcal{A}$ be a type I von Neumann algebra with center $\mathcal{Z}$. The uniformly closed $\ast$-subalgebra of $\mathcal{A}$ generated by the abelian projections of $\mathcal{A}$ is a two-sided ideal $I_a$ in $\mathcal{A}$ [16]. If $A \in I_a^+$, there is a sequence $\{A_n\}$ of positive central elements and a sequence $\{E_n\}$ of orthogonal abelian projections such that

1. $A_1 \geq A_2 \geq \cdots$;
2. $\lim A_n = 0$ (uniformly);
3. the central support of $E_n$ has Gelfand transform equal to the characteristic function of the support for the Gelfand transform of $A_n$ for each $n=1, 2, \ldots$;
4. $A = \sum A_n E_n$; and
5. the sequence $\{A_n\}$ is uniquely determined.

The sum $\sum A_n E_n$ is called a spectral decomposition of $A$.

Let $\mathcal{F}$ be the trace class of $\mathcal{A}$ and let $\text{Tr}$ be the canonical trace of $\mathcal{A}$ ($\S 4$). For each $A$ in $\mathcal{F}$ define the bounded module homomorphism $\Phi_A$ of $I_a$ into $\mathcal{F}$ by
$$
\Phi_A(B) = \text{Tr}(AB).
$$
Then if $\mathcal{F}$ is given the norm
$$
\|A\|_1 = \|\text{Tr}((A^* A)^{1/2})\|,
$$
the function $A \to \Phi_A$ defines an order preserving isometric isomorphism of the $\mathcal{F}$-module $\mathcal{F}$ onto the set of all bounded module homomorphisms of $I_a$ into $\mathcal{F}$ [9, §4].

**Theorem 5.3.** Let $\mathcal{A}$ be a type I von Neumann algebra. Let $I_a$ be the closed two-sided ideal of $\mathcal{A}$ generated by the abelian projections of $\mathcal{A}$. The vector state space $V(\mathcal{A})$ of the module $\mathcal{A}$ consists of the set of all states of the module $\mathcal{A}$ of the form
$$
C\phi + (1 - C)\tau_E
$$
where $C$ is a central element of $\mathcal{A}$ with $0 \leq C \leq 1$, $\phi$ is a state of the module $\mathcal{A}$ such that $C\phi$ vanishes on $I_a$ and $E$ is a maximal abelian projection of $\mathcal{A}$.

**Proof.** First let $\phi$ be an element of $V(\mathcal{A})$; set $\phi \mid I_a = \theta_1$. There is a positive element $B$ in the trace class of $\mathcal{A}$ such that $\theta_1(A) = \text{Tr}(AB)$ for every $A$ in $I_a$. Let $\theta(A) = \text{Tr}(AB)$ for every $A$ in $\mathcal{A}$. We show that the functional $\phi - \theta$ is positive. Let $A \in \mathcal{A}^+$. There is a monotonely increasing net $\{A_n\}$ in $I_a^+$ which converges strongly to $A$ [1, I, §3, Theorem 2, Corollary 5] because $I_a$ is strongly dense in $\mathcal{A}$. Let $x$ be a vector in the Hilbert space of $\mathcal{A}$. We have that
$$
(\phi(A)x, x) - (\theta(A_n)x, x) \geq (\phi(A_n)x, x) - (\theta(A_n)x, x) = 0
$$
for every $A_n$. Thus
$$
(\phi(A)x, x) - (\theta(A)x, x) = \lim_n ((\phi(A)x, x) - (\theta(A_n)x, x)) \geq 0.
$$
This proves $\phi - \theta$ is a positive functional of the module $\mathcal{A}$. We also have that $\phi(A) - \theta(A) = 0$ for every $A \in I_n$.

Now let $B = \sum B_i E_i$ be a spectral decomposition for $B$. Here $\{E_i\}$ is a sequence of orthogonal abelian projections with $E_1 > E_2 > \cdots$; $\{B_i\}$ is a decreasing sequence of positive central elements with $\lim B_n = 0$ (uniformly); and the support of each $B_i$ is equal to the central support of $E_i$. There is a set of mutually orthogonal central projections $\{P_n\}$ of sum 1 such that for each $P_n$ the series $\sum \{P_n B_i \mid i = 1, 2, \ldots\}$ converges uniformly [9, Theorem 4.1]. Let $n$ be fixed and let $X_n$ be the set of $\xi$ in the spectrum $\mathcal{Z}$ of the center of $\mathcal{A}$ such that $P_n^*(\xi) = 1$. For $\xi \in X_n$ let $\psi_\xi$ be an irreducible representation of $\mathcal{A}$ with kernel $[\xi]$ on a Hilbert space $H(\xi)$. Let $\phi_\xi$ be the positive functional on $\Psi_\xi(\mathcal{A}) = \mathcal{S}(\xi)$ given by $\phi_\xi(A(\xi)) = \phi(A)\tau(A)$. Here $\Psi_\xi(A) = A(\xi)$. Since every functional $f$ having the form $f(A(\xi)) = \tau_f(A)\tau(A)$, where $F$ is an abelian projection of $\mathcal{A}$ of central support 1, is a vector state of $\mathcal{A}(\xi)$, the functional $\phi_\xi$ is in the vector state space of $\mathcal{A}(\xi)$. By Glimm’s theorem [3, Theorem 2], there is an $\alpha_\xi$ in the interval $[0, 1]$, a state $g_\xi$ of $\mathcal{A}(\xi)$ vanishing on the completely continuous operators of $H(\xi)$, and a unit vector $x_\xi$ in $H(\xi)$ such that

$$\phi_{\xi} = \alpha_\xi g_\xi + (1 - \alpha_\xi)w_x.$$

Now we have that

$$\theta(A)\tau(A) = \left(\sum B_i E_i(A)\right)\tau(A) = \sum B_i E_i(A)\tau(A)\tau(A)$$

by the uniform convergence of $\sum B_i E_i$. Since $\Psi_\xi(I_n)$ is precisely the ideal of completely continuous operators on $H(\xi)$, we must have that

$$(1 - \alpha_\xi)w_x(A(\xi)) = \sum B_i E_i(A)\tau(A)\tau(A)$$

for each $A \in I_n$. For each $E_i$ there is a unit vector $y_i$ in $H(\xi)$ such that

$$B_i E_i(A)\tau(A)\tau(A) = B_i E_i(A)y_i, y_i).$$

Indeed, $E_i(\xi)$ is a projection on $H(\xi)$ of dimension not exceeding 1. Therefore, we have that

$$(1 - \alpha_\xi)w_x(A(\xi)) = B_i E_i(A)\tau(A)\tau(A)$$

for every $A \in I_n$. Then $B_1^2 E_1(\xi), B_2^2 E_2(\xi), \ldots$ vanish. Because $\xi$ in $X_n$ is arbitrary, we conclude that $0 = B_2 P_n = B_3 P_n = \cdots$ and thus that $B P_n = (B_1 E_1) P_n$. Because $P_n$ is arbitrary, we find that $B_2, B_3, \ldots$ vanish. Thus $B = B_1 E_1$ and $\theta(A) = B_1 E_1\tau(A)$ for every $A \in \mathcal{A}$. Since the support of $B_1 E_1$ is equal to that of $E_1$, we may assume $E = E_1$ is a maximal abelian projection and still retain the formula $B_1 E_1\tau(A) = \theta(A)$.

There is a sequence $\{Q_n\}$ of orthogonal central projections of sum equal to the support $Q$ of $C = \phi(1) - \theta(1)$ such that for each $Q_n$ there is a positive central element $D_n$ with $D_n Q_n = D_n$ and $D_n C = Q_n$. The sequence $\|D_n(\phi(A) - \theta(A))\|$ is bounded above by $\|A\|$ for each $A$ in $\mathcal{A}$ since $\phi - \theta$ is a positive functional of the module $\mathcal{A}$. Set $\psi(A) = \sum D_n(\phi(A) - \theta(A))$ for each $A$ in $\mathcal{A}$. Then $\psi$ is a positive functional...
functional of the module $\mathcal{A}$ with the property $\psi_1(1) = Q$. We extend $\psi_1$ to a state $\psi$ on the module $\mathcal{A}$ by setting $\psi = \psi_1 + \psi_2$ where $\psi_2$ is a positive functional of the module $\mathcal{A}$ with $\psi_2(1) = 1 - Q$.

We show that $C\psi + B_1 \tau_B = \phi$. For each $Q_n$, we have that

$$Q_n(C\psi(A) + B_1 \tau_B(A)) = Q_n(\phi(A) - \theta(A) + \theta(A)) = Q_n\phi(A)$$

for every $A$ in $\mathcal{A}$. Also

$$(1 - Q)(C\psi(A) + B_1 \tau_B(A)) = (1 - Q)\theta(A) = (1 - Q)\phi(A).$$

So $C\psi + B_1 \tau_B = \phi$. Since both $\psi$ and $\tau_B$ are states, we have that $C + B_1 = 1$. This completes the first part of the proof.

Conversely, let $\phi$ be a state of the module $\mathcal{A}$ of the form

$$\phi = C\psi + (1 - C)\tau_B,$$

where $C$ is a central element of $\mathcal{A}$ with $0 \leq C \leq 1$, $\psi$ is a state of the module $\mathcal{A}$ such that $C\psi$ vanishes on $I_a$, and $E$ is an abelian projection of central support 1. Let $A_1, A_2, \ldots, A_n$ be elements of $\mathcal{A}$. Let $\zeta$ be a maximal ideal of the center of $\mathcal{A}$ and let $\Psi_\zeta$ be an irreducible representation with kernel $[\zeta]$ of $\mathcal{A}$ on the Hilbert space $H(\zeta)$. Let $\Psi_\zeta(\mathcal{A}) = \mathcal{A}(\zeta)$ and $\Psi_\zeta(A) = A(\zeta)$. The relation

$$\phi(A(\zeta)) = \phi(A)^\zeta$$

defines a functional in the vector state space of $\mathcal{A}(\zeta)$ [3, Theorem 2] since $\Psi_\zeta(I_a)$ is the ideal of completely continuous operators on $H(\zeta)$. There is a unit vector $x_\zeta$ in $H(\zeta)$ such that

$$|\phi(A_j(\zeta)) - (A_j(\zeta)x_\zeta, x_\zeta)| < 1$$

for $j = 1, 2, \ldots, n$. But there is an abelian projection $E_\zeta$ in $A$ such that

$$(A(\zeta)x_\zeta, x_\zeta) = \tau_{E_\zeta}(A)^\zeta$$

for every $A$ in $\mathcal{A}$. By the same procedure as employed in Theorem 5.2, we obtain an abelian projection $F$ of central support 1 in $\mathcal{A}$ such that

$$|\phi(A_j(\zeta)) - \tau_F(A_j)^\zeta| < 1$$

for every $j = 1, 2, \ldots, n$ and every maximal ideal $\zeta$. So

$$\|\phi(A_j) - \tau_F(A_j)\| < 1$$

for $j = 1, 2, \ldots, n$. Thus $\phi$ is in the vector state space of the module $\mathcal{A}$. Q.E.D.

In a type I algebra every state is the pointwise limit of $\sigma$-weakly continuous states.

**Theorem 5.4.** Let $\mathcal{A}$ be a type I von Neumann algebra. Every state of the module $\mathcal{A}$ is the pointwise limit of normal states of the module $\mathcal{A}$.

**Proof.** Let $\phi$ be a state of $\mathcal{A}$. Let $\theta_1$ be the restriction of $\phi$ to $I_a$. There is a positive element $B$ of the trace class of $\mathcal{A}$ such that $\theta_1(A) = \text{Tr} (BA)$ for every $A$ in $I_a$. Let
θ( A ) = \text{Tr} ( B A ) \text{ for every } A \text{ in } \mathcal{A} \text{. Then } \phi - \theta = \psi_1 \text{ is a positive functional on the module } \mathcal{A} \text{ which vanishes on } I_\alpha \text{ (cf. proof of Theorem 5.3). Let the central projection } Q \text{ be the support of } C = \psi_1(1). \text{ There is a positive functional } \psi \text{ of the module } \mathcal{A} \text{ such that } \psi(1) = Q \text{ and such that } C \psi = \psi_1. \text{ Now let } A_1, A_2, \ldots, A_n \text{ be elements of } \mathcal{A}. \text{ The restriction of } \psi \text{ to the } \mathcal{I} Q\text{-module } \mathcal{A} Q \text{ vanishes on the closed two-sided ideal } I_\alpha Q \text{ generated by the abelian projections of } \mathcal{A} Q. \text{ There is an abelian projection } E \text{ in } \mathcal{A} Q \text{ with central support } Q \text{ such that }

\| \psi(A Q) - \tau_E(A_j Q) \| < (\| C \| + 1)^{-1}

\text{for } j = 1, 2, \ldots, n \text{ (Theorem 5.3). This means that }

\| \psi_1(A_j) - C \tau_E(A_j) \| < 1

\text{for } j = 1, 2, \ldots, n. \text{ The functional }

A \mapsto C \tau_E(A) + \text{Tr} ( B A )

\text{is a } \sigma\text{-weakly continuous positive functional of the module } \mathcal{A}. \text{ We have that }

C \tau_E(1) + \text{Tr} ( B ) = \phi(1) - \theta(1) + \theta(1) = \phi(1) = 1.

\text{Also }

\| \psi(A_j) - C \tau_E(A_j) - \text{Tr} ( B A_j ) \| < 1

\text{for } j = 1, 2, \ldots, n. \text{ Thus the state } \phi \text{ is the pointwise limit of positive } \sigma\text{-weakly continuous states. Q.E.D.}

\section*{Bibliography}


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