CATEGORICAL HOMOTOPY AND FIBRATIONS

BY

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Let $\mathscr{C}$ be an arbitrary category, and let $\mathcal{M}$ be any family of its morphisms. It is known [3] that there is a category $\mathscr{C}/\mathcal{M}$ (the Gabriel-Zisman "category of fractions of $\mathscr{C}$ by $\mathcal{M}$") having the same objects as $\mathscr{C}$, and a covariant functor $\eta: \mathscr{C} \to \mathscr{C}/\mathcal{M}$ which is the identity on objects, such that $\eta(f)$ is invertible in $\mathscr{C}/\mathcal{M}$ for each $f \in \mathcal{M}$.

We will use each class $\mathcal{M}$ to determine a notion of homotopy in $\mathscr{C}$, by defining two morphisms $f$, $g$ to be $\mathcal{M}$-homotopic if $\eta(f) = \eta(g)$. This notion has the usual properties expected of a homotopy notion; moreover, in the category $\text{Top}$ of topological spaces and continuous maps, suitable choices of $\mathcal{M}$ (for example, $\mathcal{M} =$ all homotopy equivalences) reveal $\mathcal{M}$-homotopy equivalent to the usual notion of homotopy.

Each class $\mathcal{M}$ determines a notion of fibration in $\mathscr{C}$. In the category $\text{Top}$, a suitable choice of $\mathcal{M}$ determines both the usual homotopy and the Hurewicz fibrations; however, different classes $\mathcal{M}$ may yield the usual notion of homotopy and distinct notions of fibration. More generally, in an arbitrary category $\mathscr{C}$, it is the given class $\mathcal{M}$ itself, rather than the homotopy notion induced by $\mathcal{M}$, that determines the concept of fibration; from this viewpoint, it turns out, surprisingly, that the notion of a Hurewicz fibration is not a homotopy notion. By "reversing arrows," $\mathcal{M}$ determines also a concept of cofibration; and again there is a splitting: in fact, two classes may determine the same notion of homotopy but distinct notions of cofibration.

In the last section, we introduce the concept of a weak $\mathcal{M}$-fibration. This notion does not, in general, possess all the advantages of the previous one; however, it reduces to the previous notion under suitable restrictions on the class $\mathcal{M}$. Moreover, there are classes, $\mathcal{M}$, $\mathcal{R}$ in the category $\text{Top}$ such that $\{\text{weak $\mathcal{M}$-fibrations}\} = \{\text{Hurewicz fibrations}\}$ and $\{\text{weak $\mathcal{R}$-fibrations}\} = \{\text{Dold fibrations}\}$.

Each covariant functor $\Phi: \mathscr{C} \to \mathcal{L}$ determines a $\Phi$-homotopy in $\mathscr{C}$, by choosing $\mathcal{M} = \{f | \Phi(f) \text{ is invertible}\}$. Most of the homotopy notions encountered in various categories turn out to stem from this general construction. For example: if $\mathbb{G}^\mathbb{Z}$ is the category of graded groups and degree zero homomorphisms, and if $\pi(X)$ is the total homotopy of a space $X$, then in the category $\mathscr{C}$ of spaces dominated by CW-complexes the functor $\pi: \mathscr{C} \to \mathbb{G}^\mathbb{Z}$ determines the usual notion of homotopy;

Received by the editors May 9, 1968.

(1) This research was partially supported by an NSF grant.

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if $\mathcal{K}$ is the category of Kan-complexes, then $\pi$-homotopy is the same as Kan-homotopy; and if $H: \mathcal{K} \to \mathcal{B}$ is the total homology functor on the category of chain-complexes, then $H$-homotopy is the usual chain-homotopy.

In the Appendix, we give a construction of $\mathcal{K}/\mathcal{M}$ that differs slightly from that of Gabriel-Zisman, and contains slightly more detail; one advantage of this procedure is that some results in [3] are seen to hold without the additional requirement that "$\mathcal{M}$ admits a calculus of left fractions".

1. The quotient category $\mathcal{K}/\mathcal{M}$. Let $\mathcal{K}$ be any category, and let $\mathcal{M}$ be any family of its morphisms. By a quotient category we shall mean a pair $(\mathcal{K}/\mathcal{M}, \eta)$, where $\mathcal{K}/\mathcal{M}$ is a category with the same objects as $\mathcal{K}$ and $\eta: \mathcal{K} \to \mathcal{K}/\mathcal{M}$ is a covariant functor that preserves objects, having the following two properties:

C1. If $\alpha \in \mathcal{M}$, then $\eta(\alpha)$ is invertible in $\mathcal{K}/\mathcal{M}$.

C2. [Universality]: If $F: \mathcal{K} \to \mathcal{L}$ is any covariant functor to any category $\mathcal{L}$ such that $F(\alpha)$ is invertible for each $\alpha \in \mathcal{M}$, then there exists a unique covariant functor $\Delta: \mathcal{K}/\mathcal{M} \to \mathcal{L}$ such that $F = \Delta \circ \eta$.

1.1. Theorem. Let $\mathcal{K}$ be any category, and let $\mathcal{M}$ be any family of its morphisms. Then a quotient category $(\mathcal{K}/\mathcal{M}, \eta)$ exists.

This theorem is mentioned in [3]. However, no detailed construction of $(\mathcal{K}/\mathcal{M}, \eta)$ having the generality we require appears in the literature, so we will give a proof of 1.1 in the Appendix. It follows directly from our construction, and we will need this in the sequel, that

1.2. Proposition. Each morphism $G$ in $\mathcal{K}/\mathcal{M}$ has a factorization $G = f_n \circ \hat{a}_n \circ \cdots \circ f_1 \circ \hat{a}_1$, where $f_i = \eta(f_i)$ for some $f_i$ in $\mathcal{K}$, and $\hat{a}_i$ is the inverse in $\mathcal{K}/\mathcal{M}$ for some $\eta(\alpha_i)$, $\alpha_i \in \mathcal{M}$.

It is immediate that

1.3. If $\mathcal{M}$ is the class of all invertible morphisms in $\mathcal{K}$, or if $\mathcal{M}$ is the class of all identities in $\mathcal{K}$, then $\eta: \mathcal{K} \to \mathcal{K}/\mathcal{M}$ is an equivalence.

Proof. Since 1: $\mathcal{K} \to \mathcal{K}$ sends each $\alpha \in \mathcal{M}$ to an invertible morphism, there is, by C2, a factorization $1 = \Delta \circ \eta$ where $\Delta: \mathcal{K}/\mathcal{M} \to \mathcal{K}$. This implies $\eta = \eta \circ (\Delta \circ \eta) = (\eta \circ \Delta) \circ \eta$ and since the factorization through $\eta$ is unique, we must have $\eta \circ \Delta = 1$. Thus $\eta: \mathcal{K} \approx \mathcal{K}/\mathcal{M}$ is an equivalence(2).

Whenever $\mathcal{M} \subset \mathcal{M}'$, the canonical projection $\eta': \mathcal{K} \to \mathcal{K}/\mathcal{M}'$ sends each $\alpha \in \mathcal{M}$ to an invertible morphism, so there is a unique covariant functor $\Delta: \mathcal{K}/\mathcal{M} \to \mathcal{K}/\mathcal{M}'$ such that $\eta' = \Delta \circ \eta$.

(2) The unique factorization property characterizes $(\eta, \mathcal{K}/\mathcal{M})$ up to an equivalence; for the same technique can be used to prove: Let $\mathcal{A}$ be any category, and let $\lambda: \mathcal{K} \to \mathcal{A}$ be a covariant functor such that $\lambda(\alpha)$ is invertible for each $\alpha \in \mathcal{M}$. Assume that for each category $\mathcal{L}$, every covariant functor $T: \mathcal{K} \to \mathcal{L}$ such that $T(\alpha)$ is invertible for each $\alpha \in \mathcal{M}$ factors uniquely through $\lambda$. Then there is a unique equivalence $\Delta: \mathcal{K}/\mathcal{M} \approx \mathcal{A}$ with $\lambda = \Delta \circ \eta$. 

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Proposition 1.3 is a special case of

1.4. Theorem. Let \( \eta: \mathcal{K} \to \mathcal{K}/\mathcal{M} \) be a canonical projection, and let \( \mathcal{M} = \{ f \mid \eta(f) \) is invertible\}. Then there exists a unique equivalence \( \Delta: \mathcal{K}/\mathcal{M} \approx \mathcal{K}/\overline{\mathcal{M}} \) satisfying \( \Delta \circ \eta = \overline{\eta} \). Moreover, \( \overline{\mathcal{M}} \) is the largest class containing \( \mathcal{M} \) for which such an equivalence holds.

Proof. Since \( \mathcal{M} \subset \overline{\mathcal{M}} \), we have a unique \( \Delta: \mathcal{K}/\mathcal{M} \to \mathcal{K}/\overline{\mathcal{M}} \) with \( \overline{\eta} = \Delta \circ \eta \). Since \( \eta(f) \) is invertible for each \( f \in \mathcal{M} \), there is also a \( \overline{\Delta}: \mathcal{K}/\overline{\mathcal{M}} \to \mathcal{K}/\mathcal{M} \) such that \( \eta = \overline{\Delta} \circ \overline{\eta} \). Thus, \( \eta = \Delta \circ \overline{\Delta} \circ \overline{\eta} \) and \( \eta = \Delta \circ \Delta \circ \eta \), so by the unique factorization property we find \( \Delta \circ \Delta = 1 \), \( \Delta \circ \Delta = 1 \), therefore \( \Delta \) is an equivalence. To see that \( \overline{\mathcal{M}} \) is maximal, note that if \( \mathcal{M}_0 \supset \mathcal{M} \), there is only one \( \Delta' \) such that \( \eta_0 = \Delta' \circ \overline{\eta} \), and if \( f \in \mathcal{M}_0 - \overline{\mathcal{M}} \), then \( \eta_0(f) \) is invertible whereas \( \overline{\eta}(f) \) is not, so that \( \Delta' \) cannot be an equivalence.

2. Homotopy. Let \( \mathcal{M} \) be an arbitrary class of morphisms in an arbitrary category \( \mathcal{K} \), and let \( \eta: \mathcal{K} \to \mathcal{K}/\mathcal{M} \) be the canonical projection.

2.1. Definition. Two morphisms \( f, g: X \to Y \) in \( \mathcal{K} \) are called \( \mathcal{M} \)-homotopic (written \( f \approx g \mod \mathcal{M} \)) if \( \eta(f) = \eta(g) \).

This has the standard properties required of a homotopy notion:

2.2. Theorem. (a) \( \mathcal{M} \)-homotopy is an equivalence relation in each \( \mathcal{K}(X, Y) \).
(b) Let \( f_0, f_1: X \to Y \) be \( \mathcal{M} \)-homotopic. If \( g: W \to X \) and \( h: Y \to Z \) are any two morphisms, then \( f_0 \circ g \approx f_1 \circ g \mod \mathcal{M} \) and \( h \circ f_0 \approx h \circ f_1 \mod \mathcal{M} \).
(c) Each invertible morphism in \( \mathcal{K} \) is an \( \mathcal{M} \)-homotopy equivalence.\(^{(*)}\)
(d) If \( f, g: X \to Y \) have an equalizer (or a coequalizer) \( c \in \mathcal{M} \), then \( f \approx g \mod \mathcal{M} \).

Proof. (a)–(c) are trivial, since \( \eta \) is a covariant functor. For (d): Assume, say, \( c \circ f = c \circ g \); then \( \eta(c)\eta(f) = \eta(c)\eta(g) \) and, since \( c \in \mathcal{M} \) so that \( \eta(c) \) is invertible, we conclude that \( \eta(f) = \eta(g) \).

Denote the \( \mathcal{M} \)-homotopy class of \( f \in \mathcal{K}(X, Y) \) by \( [f]_{\mathcal{M}} \) and the set of all \( \mathcal{M} \)-homotopy classes in \( \mathcal{K}(X, Y) \) by \( [X, Y]_{\mathcal{M}} \). From 2.2 it follows in the usual way that (1) there is a category \( \mathcal{K}([\mathcal{M}] \) whose objects are those of \( \mathcal{K} \) and whose morphisms are the \( \mathcal{M} \)-homotopy classes, and (2) the function \( p: \mathcal{K} \to \mathcal{K}([\mathcal{M}] \) which is the identity on objects and sends each \( f \) to \( [f]_{\mathcal{M}} \), is a covariant functor\(^{(*)}\).

This method of defining a homotopy notion in a category is reasonable, in that by a suitable choice (in fact, many choices) of \( \mathcal{M} \), one gets the usual notion of homotopy in the category Top of all topological spaces and continuous maps. This will follow from the general

\(^{(4)}\) Precisely, \( f: X \to Y \) is an \( \mathcal{M} \)-homotopy equivalence if there exists a \( g: Y \to X \) such that \( f \circ g \approx 1 \mod \mathcal{M} \) and \( g \circ f \approx 1 \mod \mathcal{M} \).

\(^{(*)}\) In particular, if \( c: A \to A \) is such that \( c \circ f = c \circ g \) for every \( f, g: Z \to A \), then whenever \( c \in \mathcal{M} \), the set \([Z, A]_{\mathcal{M}}\) consists of a single element.
2.3. Theorem. Let $\mathcal{K}$ be a category and ~ an equivalence relation in each $\mathcal{K}(A, B)$ such that the transformation $h: \mathcal{K} \to \mathcal{K}_h$ taking each object $A$ to itself and each $f$ to its ~-class is a covariant functor.

Assume that, if $f_0 \sim f_1: X \to Y$ then there exists an object $I_X \in \mathcal{K}$, and morphisms, $r: I_X \to X, \ i_0, i_1: X \to I_X, \ F: I_X \to Y$, such that $r \circ i_j = 1$ and $f_j = F \circ i_j$ ($j = 0, 1$).

Finally, let $\mathcal{M}$ be a class of morphisms in $\mathcal{K}$ and $\eta: \mathcal{K} \to \mathcal{K}_h/\mathcal{M}$ the projection. Then

(a) If $\eta(r)$ is invertible in $\mathcal{K}_h/\mathcal{M}$, then $f_0 \sim f_1$ implies $f_0 \sim f_1$ mod $\mathcal{M}$.

(b) If $h(\alpha)$ is invertible in $\mathcal{K}_h$ for each $\alpha \in \mathcal{M}$, then $f_0 \sim f_1$ mod $\mathcal{M}$ implies $f_0 \sim f_1$.

Proof. (a) Assume $f_0 \sim f_1$. Since $\eta(r) \circ \eta(i_j) = 1$ ($j = 0, 1$) and $\eta(r)$ is invertible, we find that each $\eta(i_j)$ is invertible, and that $\eta(i_0) = \eta(r)^{-1} = \eta(i_1)$. Thus $\eta(f_0) = \eta(F)\eta(i_0) = \eta(F)\eta(i_1) = \eta(f_1)$ so $f_0 \sim f_1$ mod $\mathcal{M}$.

(b) Because of C2, the functor $h$ has a factorization $h = \Delta \circ \eta$; therefore $\eta(f_0) = \eta(f_1)$ implies $h(f_0) = h(f_1)$, i.e., that $f_0 \sim f_1$. This completes the proof.

Let $\mathcal{K} = \text{Top}$ and ~ the usual homotopy in Top; then by taking $\mathcal{M}$ to be any one of the four classes:

1. $\mathcal{M}_1$ = all homotopy equivalences,
2. $\mathcal{M}_2$ = all maps $r: X \times I \to X$, where $r(x, t) = x$,
3. $\mathcal{M}_3$ = all maps $i: X \to X \times I$, where $i(x) = (x, 0)$,
4. $\mathcal{M}_4$ = all inclusion maps $j: W \to X$, where $j(W)$ is a zero-set(5) and a strong deformation retract of $X$, it follows immediately from 2.3 that $\mathcal{M}$-homotopy is exactly the same as the usual homotopy.

We shall say that two classes $\mathcal{M}, \mathcal{M}$ in a category $\mathcal{K}$ determine the same notion of homotopy, and write $[\mathcal{M}] = [\mathcal{N}]$, whenever $f \sim g$ mod $\mathcal{M}$ if and only if $f \sim g$ mod $\mathcal{N}$. As the examples in Top show, distinct classes may yield the same homotopy notion. We now examine briefly some questions that arise, such as: to find conditions assuring that the $\mathcal{M}$-homotopy equivalences determine the same homotopy notion as $\mathcal{M}$ itself.

Let $\mathcal{M}$ be fixed, let $\eta: \mathcal{K} \to \mathcal{K}/\mathcal{M}$ be the canonical map, and let

- $\mathcal{M}_0 = \{f|\eta(f)\text{ is invertible}\}$,
- $E(\mathcal{M}) = \{f|\eta(f)\text{ is invertible and has some }\eta(g)\text{ as inverse}\}$
- $\mathcal{M}_1 = \{f|f\text{ is an }\mathcal{M}\text{-homotopy equivalence}\}$.

It is clear that, if $\mathcal{M} \subset \mathcal{M}$, then $\mathcal{M} \subset \mathcal{M}_0$ and $E(\mathcal{M}) \subset E(\mathcal{M})$. The general relation of these concepts is

2.4. (a) $[\mathcal{M}] = [\mathcal{M}_0]$ and $E(\mathcal{M}) = E(\mathcal{M}_0)$; (b) $\mathcal{M} = \mathcal{M}_0 \Rightarrow [\mathcal{M}] = [\mathcal{M}_0] \Rightarrow E(\mathcal{M}) = E(\mathcal{M}_0)$.

Proof. (a) According to 1.4, there are equivalences $\Delta: \mathcal{K}/\mathcal{M} \cong \mathcal{K}/\mathcal{M}_0$, $\Delta: \mathcal{K}/\mathcal{M}_1 \cong \mathcal{K}/\mathcal{M}$ such that $\eta = \Delta \circ \eta_0$. This implies that $\eta(f) = \eta(g)$ if and only if

(5) I.e., there exists a continuous $\phi: X \to I$ with $\phi^{-1}(0) = j(W)$.
It also implies that \( E(\mathfrak{M}) = \mathfrak{M} \); and the definition shows \( E(\mathfrak{M}) \subseteq \mathfrak{M} \).

(b) If \( \mathfrak{M} = \mathfrak{M} \) then obviously \( \mathfrak{M} = \mathfrak{M} \) and, from (a), also \( \mathfrak{M} = \mathfrak{M} \). Now let \( f \in E(\mathfrak{M}) \); then there is a \( g \) with \( f \equiv g \) mod \( \mathfrak{M} \), \( g \equiv f \) mod \( \mathfrak{M} \); since \( \mathfrak{M} = \mathfrak{M} \), we find \( f \equiv g \) mod \( \mathfrak{M} \), \( g \equiv f \) mod \( \mathfrak{M} \) and so \( f \in E(\mathfrak{M}) \). Similarly, \( f \in E(\mathfrak{M}) \) implies \( f \in E(\mathfrak{M}) \) and the proof is complete.

In general, none of the implications in 2.4(b) are reversible.

(A) Let \( \mathcal{K} \) consist of two objects \( A, B \) together with the identity morphisms and a single morphism \( f: A \to B \). Let \( \mathfrak{M} = \{f, 1_A, 1_B\} \) and \( \mathfrak{N} = \{1_A, 1_B\} \); then \( [\mathfrak{M}] = [\mathfrak{N}] \) yet \( \mathfrak{M} \neq \mathfrak{N} \) (see 1.3).

(B) Let \( \mathcal{K} \) consist of three intervals \( A_1 = [0, r], r = 1, 2, 3 \), with the identity maps, the inclusion maps \( j_{1r}: A_r \to A_s, r < s \), and the map \( h: A_2 \to A_3 \) given by \( h(x) = x \), \( 0 \leq x \leq 1 \), \( h(x) = 1, x > 1 \), so that \( j_{13} = j_{23} \circ j_{12} = h \circ j_{12} \). Let \( \mathfrak{M} = \{1_{11}, 1_{12}, 1_{13}, j_{12}\} \) and \( \mathfrak{N} = \{1_{11}, 1_{12}, 1_{13}\} \); then \( [\mathfrak{M}] \neq [\mathfrak{N}] \) since \( h \) is \( \mathfrak{M} \)-homotopic to \( j_{23} \); however \( E(\mathfrak{M}) = E(\mathfrak{N}) = \mathfrak{N} \) since there is no map in \( \mathcal{K} \) going backwards. Note that this shows also that \( [\mathfrak{M}] \neq [\mathfrak{N}] = [E(\mathfrak{M})] \).

2.5. Definition. The class \( \mathfrak{M} \) is called stable if \( E(\mathfrak{M}) = \mathfrak{M} \); i.e., if the class of \( \mathfrak{M} \)-homotopy equivalences is exactly \( \mathfrak{M} \).

If \( \mathfrak{M} \) is stable, then \( E(\mathfrak{M}) \) and \( \mathfrak{M} \) determine the same notion of homotopy, since \( E(\mathfrak{M}) = \mathfrak{M} = \mathfrak{M} \). Furthermore, distinct stable classes determine distinct notions of homotopy, because, from 2.4(b) and stability, we have

2.6. If \( \mathfrak{M}, \mathfrak{N} \) are stable classes, then \( \mathfrak{M} = \mathfrak{N} \Rightarrow [\mathfrak{M}] = [\mathfrak{N}] \Rightarrow E(\mathfrak{M}) = E(\mathfrak{N}) \).

The stable classes are characterized by

2.7. Theorem. A class \( \mathfrak{M} \) in \( \mathcal{K} \) is stable if and only if \( \gamma: \mathcal{K} \to \mathcal{K}/\mathfrak{M} \) is epic(\( ^{(*)} \)).

Proof. “If”: We need show only that \( \mathfrak{M} \subseteq E(\mathfrak{M}) \). Let \( f \in \mathfrak{M} \); then \( \gamma(f) \) has an inverse \( G \) in \( \mathcal{K}/\mathfrak{M} \); since \( \gamma \) is epic, \( G = \gamma(g) \) for some \( g \in \mathcal{K} \), so \( f \in E(\mathfrak{M}) \). “Only if”: We have seen in 1.2 that each given \( G \) in \( \mathcal{K}/\mathfrak{M} \) has a factorization as \( G = f_{\alpha_1} \circ \alpha_1 \circ \cdots \circ f_1 \circ \alpha_1 \), where each \( f_1 = \gamma(f) \) for some \( f_1 \) in \( \mathcal{K} \), and each \( \alpha_1 \) is the inverse of some \( \gamma(\alpha_1) \). In particular, each \( \alpha_1 \in \mathfrak{M} = E(\mathfrak{M}) \), so \( \gamma(\alpha_1) \) has some \( \gamma(\beta_1) \) for inverse and, since inverses are unique, \( \alpha_1 = \gamma(\beta_1) \). Thus, \( G = \gamma(f_\alpha \circ \beta_\alpha \circ \cdots \circ f_1 \circ \beta_1) \) and \( \gamma \) is epic.

2.8. Corollary. Let \( \mathfrak{M} \) be any class. Then there exists a unique maximal stable class \( \mathfrak{M} \subseteq \mathfrak{M} \) with the following property: if \( \mathfrak{M} \subseteq \mathfrak{M} \) is stable, then \( \mathfrak{M} \subseteq \mathfrak{M} \).

Proof. In view of our constructions, the empty class \( \emptyset \subseteq \mathfrak{M} \) is (1.3) clearly stable. It is therefore enough to show that the union of any family of stable classes is stable. Let \( \{\mathfrak{M}_\beta \mid \beta \in B\} \) be the family of all stable classes in \( \mathfrak{M} \), so that each \( \eta_\beta: \mathcal{K} \to \mathcal{K}/\mathfrak{M}_\beta \) is epic; letting \( \mathfrak{M} = \bigcup \mathfrak{M}_\beta \), we will prove that \( \eta: \mathcal{K} \to \mathcal{K}/\mathfrak{M} \) is also epic. Each morphism \( G \) in \( \mathcal{K}/\mathfrak{M} \) has a factorization \( G = f_\alpha \circ \alpha_\alpha \circ \cdots \circ f_1 \circ \alpha_1 \),

\( ^{(\star)} \) I.e., for each \( A, B \) and \( f \in \text{hom}(A, B) \), there exists an \( f: A \to B \) with \( \gamma(f) = f \). Note that the stability of \( \mathfrak{M} \) implies that each \( \text{hom}(A, B) \) is in fact a set, cf. footnote 14, 15.
where each \( f_i = \eta(f_i) \) for some \( f_i \) in \( \mathcal{K} \). Each \( \alpha_i \) lies in some \( \mathfrak{M}_i \) so there is a \( \beta_i \) in \( \mathcal{K} \) with \( \eta(\beta_i) = \alpha_i \) in \( \mathcal{K}/\mathfrak{M} \). Since (1.4) there is an \( \eta_i : \mathcal{K}/\mathfrak{M}_i \to \mathcal{K}/\mathfrak{M} \) with \( \eta = \eta_i \circ \eta \), it follows that \( \eta(\beta_i) = \alpha_i \) in \( \mathcal{K}/\mathfrak{M} \) and therefore that \( \eta(f_n \circ \beta_n \circ \cdots \circ f_1 \circ \beta_1) = G \). This completes the proof.

The maximal stable class \( \mathfrak{M} \) may not give the same notion of homotopy as \( \mathfrak{M} \).

In fact, there are notions of \( \mathfrak{M} \)-homotopy that cannot be obtained from any stable class: in example (B) above, \( \mathfrak{M} = \mathfrak{N} \) and, indeed, \( \mathfrak{M} \)-homotopy cannot be obtained from a stable class. Moreover, \( E(\mathfrak{M}) \neq E(\mathfrak{N}) \) in general (cf. Appendix A5).

The machinery developed so far can be applied to any covariant functor \( \Phi : \mathcal{K} \to \mathcal{L} \) to introduce into \( \mathcal{K} \) a notion of homotopy induced (or adapted to) the functor \( \Phi \).

2.9. Definition. Let \( \Phi : \mathcal{K} \to \mathcal{L} \) be any covariant functor. Let

\[ \mathfrak{M}(\Phi) = \{ f \in \mathcal{K} \mid \Phi(f) \text{ is invertible} \} \]

The homotopy in \( \mathcal{K} \) determined by the class \( \mathfrak{M}(\Phi) \) is called \( \Phi \)-homotopy.

The category \( \mathcal{K}/\mathfrak{M}(\Phi) \) is written \( \mathcal{K}^* \) and is essentially the category first studied by Bauer in [1]. Observe that, because of C2, the functor \( \Phi \) has a unique factorization \( \Phi = \Delta \circ \eta \) through \( \mathcal{K}^* \). It follows from this that \( \mathfrak{M}(\Phi) = [\mathfrak{M}(\Phi)]^- \) always:

for, if \( f \in [\mathfrak{M}(\Phi)]^- \), then \( \eta(f) \) is invertible, therefore so also is \( \Delta(\eta(f)) = \Phi(f) \), consequently \( f \in \mathfrak{M}(\Phi) \); thus, \( [\mathfrak{M}(\Phi)]^- \subset \mathfrak{M}(\Phi) \) and the opposite inclusion is trivial.

We give some applications of \( \Phi \)-homotopy.

(1) Let \( \Phi : \text{Top} \to \text{Ens} \) be the forgetful functor. Then \( f \) is \( \Phi \)-homotopic to \( g \) if and only if \( f = g \). For, consider the factorization \( \Phi = \Delta \circ \eta \) through \( \mathcal{K}^* \). If \( \eta(f) = \eta(g) \), then \( \Phi(f) = \Phi(g) \) so that \( f = g \) as maps of sets; and the converse is trivially true. The class \( \mathfrak{M} = \mathfrak{M}(\Phi) \) consists of all bijective continuous maps, whereas \( E(\mathfrak{M}) \) is the class of all bijective bicontinuous maps (i.e., homeomorphisms).

(2) Let \( \mathcal{K} \) be the category of spaces dominated by CW-complexes, let \( \mathcal{G}^2 \) be the category of graded groups and degree zero homomorphisms, and let \( \pi : \mathcal{K} \to \mathcal{G}^2 \) be the total homotopy functor. Then two maps are \( \pi \)-homotopic if and only if they are homotopic. For, we apply 2.3 because (a) each projection \( r : X \times I \to X \) is such that \( \pi(r) \) is invertible, therefore \( r \in \mathfrak{M}(\pi) \), consequently \( \eta(r) \) is invertible, and (b) if \( f \in \mathfrak{M}(\pi) \), i.e., if \( \pi(f) \) is invertible, then, by Whitehead's theorem [6], \( f \) is a homotopy equivalence and therefore \( h(f) \) is invertible. Note that, again using Whitehead's theorem, the class \( \mathfrak{M}(\pi) \) is stable, so that \( \eta : \mathcal{K} \to \mathcal{K}^* \) is epic. Similarly, if \( \mathcal{K} \) is the category of Kan-complexes, then \( \pi \)-homotopy in \( \mathcal{K} \) is exactly Kan-homotopy.

(3) Let \( \mathcal{K} \) be the category of all simply connected CW-complexes, and \( H : \mathcal{K} \to \mathcal{G}^2 \) the total homology functor. Exactly the same considerations as in (2) reveal \( H \)-homotopy to be the usual homotopy. Using \( H : \mathcal{K} \to \mathcal{G}^2 \) on the category of css complexes yields a homotopy notion extending the usual one for Kan-complexes. And on the category \( \mathcal{K}^* \) of chain-complexes, \( H \)-homotopy yields the usual notion of chain-homotopy.

Thus, the various notions of homotopy encountered, and usually defined...
differently, in various categories turn out to be essentially the homotopy relation determined in the above manner by a suitable covariant functor. For arbitrary contravariant functors, one turns to the dual category.

3. **Fibrations.** Each class $\mathcal{M}$ of morphisms in a category $\mathcal{K}$ determines a concept of fibration in $\mathcal{K}$:

3.1. **Definition.** A morphism $p: E \rightarrow B$ in $\mathcal{K}$ is called an $\mathcal{M}$-fibration if for each diagram

$$
\begin{array}{ccc}
W & \xrightarrow{\mu} & X \\
\downarrow f & & \downarrow p \\
B & \xrightarrow{g} & E
\end{array}
$$

in which $\mu \in \mathcal{M}$ and $p \circ g \circ \mu = f \circ \mu$, there exists a $g': X \rightarrow E$ in $\mathcal{K}$ with $g \circ \mu = g' \circ \mu$ and $p \circ g' = f$.

We are requiring simply that every triangle involving $p$ which can be equalized by a morphism belonging to $\mathcal{M}$, can be made itself commutative. Note that if $\mathcal{M}$-homotopy is used in $\mathcal{K}$, the conditions $p \circ g \circ \mu = f \circ \mu$ and $g \circ \mu = g' \circ \mu$ imply, by 2.2(d), that $p \circ g \equiv f \mod \mathcal{M}$ and $g \equiv g' \mod \mathcal{M}$. The relations between $\mathcal{M}$-homotopy and $\mathcal{M}$-fibration will be considered after we verify that the class of $\mathcal{M}$-fibrations has the properties usually required of fibrations.

3.2. **Theorem.** Let $\mathcal{M}$ be a fixed class of morphisms in $\mathcal{K}$. Then:

(a) If $p: E \rightarrow B$ is invertible, then $p$ is an $\mathcal{M}$-fibration.

(b) If $p_1: E_1 \rightarrow E_0$ and $p_0: E_0 \rightarrow B$ are $\mathcal{M}$-fibrations, so also is $p_0 \circ p_1: E_1 \rightarrow B$.

(c) If $p: E \rightarrow B$ is an $\mathcal{M}$-fibration, and if

$$
\begin{array}{c}
E' \\
\downarrow p' \\
B'
\end{array} \xrightarrow{\text{h}} \begin{array}{c}
E \\
\downarrow p \\
B
\end{array}
$$

is a cartesian diagram(7), then $p'$ is also an $\mathcal{M}$-fibration; and $h$ can be lifted into $E$ if and only if $p'$ has a section(8).

**Proof.** (a) is trivial.
(b) Given

\[ W \xrightarrow{\mu} X \xrightarrow{g} E_1 \xrightarrow{p_1} E_0 \]

with \( p_0 \circ p_1 \circ g \circ \mu = f \circ \mu \) and \( \mu \in \mathcal{W} \), there exists first a \( G : X \to E_0 \) with \( p_0 \circ G = f \), \( G \circ \mu = p_1 \circ g \circ \mu \); and then a \( g' : X \to E_1 \) with \( p_1 \circ g' = G \), \( g' \circ \mu = g \circ \mu \); since \( p_0 \circ p_1 \circ g = p_0 \circ G = f \), the required morphism is \( g' \).

(c) We are given

\[ W \xrightarrow{\mu} X \xrightarrow{g} E' \xrightarrow{h} E \]

where the square is cartesian and \( f \circ \mu = p' \circ g \circ \mu \). Thus, \( h \circ f \circ \mu = p \circ h \circ g \circ \mu \) so, because \( p \) is an \( \mathcal{W} \)-fibration, there is a \( G : X \to E \) such that \( G \circ \mu = h \circ g \circ \mu \) and \( p \circ G = h \circ f \). Because the square is cartesian and \( p \circ G = h \circ f \), there exists a unique \( g' : X \to E' \) such that \( h \circ g' = G \) and \( p' \circ g' = f \). Since also \( h \circ (g' \circ \mu) = G \circ \mu = h \circ (g \circ \mu) \) and \( p' \circ (g \circ \mu) = f \circ \mu = p \circ (g' \circ \mu) \), the uniqueness of a morphism \( W \to E' \) satisfying these two conditions in a cartesian diagram shows \( g' \circ \mu = g \circ \mu \) and therefore we find that \( p' \) is an \( \mathcal{W} \)-fibration. The second part is proved in a similar manner.

We now examine the notion of \( \mathcal{W} \)-fibration in Top, using the classes, and the symbols for those classes, that are listed following Theorem 2.3. We have

3.3 Theorem. Let \( \mathcal{W} = \mathfrak{S} \) in Top. Then \( \mathfrak{S} \)-homotopy is the classical notion, and \( p : E \to B \) is a \( \mathfrak{S} \)-fibration if and only if it is a Hurewicz fibration.

Proof. The first part was established in 2.3. Let now \( p : E \to B \) be a \( \mathfrak{S} \)-fibration; we show it has the covering homotopy property, i.e., that it is a Hurewicz fibration. We start with the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & E \\
\downarrow{i} & & \downarrow{p} \\
X \times I & \xrightarrow{F} & B \\
\end{array}
\]

where \( i(x) = (x, 0) \). Let \( r : X \times I \to X \) be the map \( r(x, t) = x \); we then have

\[
\begin{array}{ccc}
X & \xrightarrow{i} & X \times I \\
\downarrow{F} & & \downarrow{p} \\
B & \xrightarrow{r} & E \\
\end{array}
\]
where \( F \circ i = p \circ f = p \circ f \circ r \circ i \). Since \( p \) is a \( \mathcal{A} \)-fibration, there is an \( F: X \times I \rightarrow E \) with \( p \circ F = F \) and \( F \circ i = f \circ r \circ i = f \); so that \( F \) is a homotopy of \( f \) covering \( F \), and therefore \( p \) is a Hurewicz fibration.

Conversely, let \( p: E \rightarrow B \) be a Hurewicz fibration. The morphisms in \( \mathcal{A} \) being simply the maps \( i: X \rightarrow X \times I \), we consider any diagram

\[
\begin{array}{ccc}
X & \overset{i}{\rightarrow} & X \times I \\
\downarrow{F} & & \downarrow{p} \\
\multicolumn{3}{c}{E} \\
\end{array}
\]

in which \( p \circ f \circ i = F \circ i \); since \( p \) is a Hurewicz fibration, there is a homotopy \( F \) of \( f \circ i \) covering \( F \), so that \( p \circ F = F \) and \( f \circ i = F \circ i \). Thus, \( p \) is a \( \mathcal{M} \)-fibration, and the proof is complete.

If \( \mathcal{M} \subset \mathcal{R} \), it is clear that \( \{ \mathcal{M} \text{-fibrations} \} \subset \{ \mathcal{M} \text{-fibrations} \} \). Moreover,

(a) Two classes may determine the same notion of homotopy, but distinct notions of fibration.

Example in Top: We have seen that \( \mathcal{B} \) determines the same homotopy notion as \( \mathcal{A} \), that is, the usual notion of homotopy. However, every continuous map is a \( \mathcal{B} \)-fibration: for, clearly, \( \mathcal{B} \subset \) (all continuous surjections), and, if \( \mu \) is surjective, the condition \( p \circ g \circ \mu = f \circ \mu \) of Definition 3.1 implies that \( p \circ g = f \); thus, every continuous \( p: E \rightarrow B \) belongs to \( \{ \text{surjective map fibrations} \} \subset \{ \mathcal{B} \text{-fibrations} \} \).

(b) Two classes may determine the same notion of fibration but distinct notions of homotopy.

Example in Top: Let \( \mathcal{M} = \) class of all identity maps \( 1_A: A \rightarrow A \). Then every continuous map is an \( \mathcal{M} \)- and also a \( \mathcal{B} \)-fibration; yet \( \mathcal{B} \)-homotopy is the usual notion of homotopy, whereas we have seen that \( \mathcal{M} \)-homotopy is simply equality.

From this viewpoint, the notions of homotopy and of fibration are independent. In the category Top, the Hurewicz fibrations appear as a concept dependent on a particular class of morphisms that happens to yield the usual homotopy notion, rather than as a concept dependent on the homotopy notion itself.

Returning to 3.1, we shall establish a simple criterion for two classes to determine the same notion of fibration. This is based on

3.4. Theorem. Let \( \mathcal{M}, \mathcal{Q} \), be two classes of morphisms in \( \mathcal{K} \). Assume that for each \( \mu: W \rightarrow X, \mu \in \mathcal{M} \), there exist a \( \lambda: Y \rightarrow Z, \lambda \in \mathcal{Q} \) and morphisms in \( \mathcal{K} \) such that

\( (*) \) We denote the class of \( \mathcal{M} \)-fibrations by \( \{ \mathcal{M} \text{-fibrations} \} \).

\( (\dagger) \) The class \( \mathcal{K} \) of Hurewicz fibrations does not determine the usual homotopy: in fact, since every constant map \( f: E \rightarrow e \) is a Hurewicz fibration, it follows (cf. footnote 4) that for each \( X, Y \), all \( f, g: X \rightarrow Y \) are \( \mathcal{K} \)-homotopic.
is commutative and $k \circ j = 1$. Then $\{s\text{-fibrations}\} \subset \{\mathcal{M}\text{-fibrations}\}$.

**Proof.** Let $p: E \to B$ be an $s$-fibration. Given a diagram

$$
\begin{array}{ccc}
W & \xrightarrow{i} & Y \xrightarrow{s} W \\
\mu \downarrow & & \mu \downarrow \\
X & \xrightarrow{j} & Z \xrightarrow{k} X
\end{array}
$$

with $p \circ g \circ \mu = f \circ \mu$ and $\mu \in \mathcal{M}$, we find that $p \circ g \circ k \circ \lambda = p \circ g \circ \mu \circ s = f \circ \mu \circ s = f \circ k \circ \lambda$ so, since $p$ is an $s$-fibration, there is a $\Gamma: Z \to E$ with $p \circ \Gamma = f \circ k$ and $\Gamma \circ \lambda = g \circ k \circ \lambda$. Letting $G = \Gamma \circ j$, we find $p \circ G = p \circ \Gamma \circ j = f \circ k \circ j = f$, and $G \circ \mu = \Gamma \circ j \circ \mu = g \circ k \circ \mu \circ i = g \circ k \circ j \circ \mu = g \circ \mu$, so that $p$ is an $\mathcal{M}$-fibration.

As an application,

3.5. In Top, $\{\mathcal{X}\text{-fibrations}\} = \{$Hurewicz fibrations$\}$.

**Proof.** Clearly, $\mathcal{S} \subset \mathcal{X}$, so $\{\mathcal{X}\text{-fibrations}\} \subset \{\mathcal{S}\text{-fibrations}\}$. For the converse, let $\mu: W \to X$, $\mu \in \mathcal{X}$, be given; we identify $W$ with a subset of $X$ in order to cut down excessive notation, and let $\Phi: r \simeq 1$ be a strong deformation retraction. Choose $\Phi: X \to I$ vanishing exactly on $W$, and define $\Phi: X \times I \to X$ by

$$
\Phi(x, t) = \begin{cases} 
\Phi(x, t/\phi(x)), & x \notin W, \\
\Phi(x, 1), & x \in W,
\end{cases}
$$

which is easily verified to be continuous. Then the diagram

$$
\begin{array}{ccc}
W & \xrightarrow{\mu} & X \xrightarrow{r} W \\
\mu \downarrow & & \mu \downarrow \\
X & \xrightarrow{j} & X \times I \xrightarrow{\Phi} X
\end{array}
$$

where $r =$ retraction, $\mu =$ inclusion, $\lambda(x) = (x, 0)$ and $j(x) = (x, \phi(x))$ is easily seen to be commutative. Thus, $\{\mathcal{S}\text{-fibrations}\} \subset \{\mathcal{X}\text{-fibrations}\}$ and, with 3.3, the theorem is proved.

We recall that, in Top, a fibration is called regular whenever a covering homotopy can always be chosen stationary at all the points where the given homotopy is stationary\(^{(1)}\). Let $\mathcal{F}$ be the family of all inclusions $\mu: A \to X$, where $A$ is a

\(^{(1)}\) An example of a nonregular Hurewicz fibration, as well as a general condition on a topological space $B$ that assures the regularity of every Hurewicz fibration over $B$, is given in [5].
strong deformation retract of \(X\), and let \(\mathcal{S}\) be the class of all maps \(\lambda_B: X \to X_B = (X \times I) \cup_r B, r \in \mathcal{S}\), where \(B \subset X\) is arbitrary, and \(\lambda_B = \rho_B \circ i\) with \(i \in \mathcal{S}\) and \(\rho_B: X \times I \to X_B\) the identification. Clearly, \(\mathcal{X} \subset \mathcal{K}\) and \(\mathcal{S} \subset \mathcal{S}\) so that every \(\mathcal{X}\)- and every \(\mathcal{S}\)-fibration is a Hurewicz fibration.

3.7. Theorem \((12)\). In Top \(\{\mathcal{X}\text{-fibrations}\} = \{\mathcal{S}\text{-fibrations}\} = \{\text{regular Hurewicz fibrations}\}\).

Proof. The proof that \(\{\mathcal{S}\text{-fibrations}\} = \{\text{regular Hurewicz fibrations}\}\) is a repetition of that for 3.3, with \(X \times I\) replaced by suitable \(X_B\) and \(\mathcal{S}\) replaced by \(\mathcal{S}\). Because \(\mathcal{S} \subset \mathcal{K}\), we have \(\{\mathcal{X}\text{-fibrations}\} \subset \{\mathcal{S}\text{-fibrations}\}\). To prove the converse inclusion, let \(\mu: A \to X, \mu \in \mathcal{K}\), be given, let \(\Phi: r \simeq 1\), be a strong deformation retraction of \(X\) onto \(A\); then a continuous \(\Phi: X_A \to X\) such that \(\Phi \circ \rho_A = \Phi\) exists. Let \(i_1: X \to X \times I\) be the map \(x \to (x, 1)\); thus, the diagram

\[
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow & & \downarrow \\
X & \longrightarrow & X_A \\
\mu & \downarrow & \mu \\
& & \Phi
\end{array}
\]

with \(j = \rho_A \circ i_1\), is commutative and \(\Phi \circ j = 1\). Thus, by 3.5, the proof is complete.

As in the previous section, a covariant functor \(\Phi: \mathcal{K} \to \mathcal{S}\) gives rise to a notion of \(\Phi\)-fibration, i.e., \(\mathcal{M}(\Phi)\)-fibration. Observe that, if \(\pi: \mathcal{K} \to \mathcal{S}\) is the total homotopy functor on the category of spaces dominated by CW-complexes, then the corresponding notion of \(\Phi\)-fibration is not that of Hurewicz fibration, but of a certain subset: for, \(\mathcal{M}(\Phi)\) = set of all homotopy equivalences and \(\mathcal{S} \subset \mathcal{M}(\Phi)\) so that \(\{\mathcal{M}(\Phi)\text{-fibrations}\} \subset \{\mathcal{S}\text{-fibrations}\} = \{\text{Hurewicz fibrations}\}\).


4.1. Definition. Let \(\mathcal{R}\) be a class of morphisms in a category \(\mathcal{K}\). A morphism \(j: B \to E\) in \(\mathcal{K}\) is called an \(\mathcal{R}\)-cofibration if for each diagram

\[
\begin{array}{ccc}
E & \longrightarrow & X \\
\downarrow & \searrow \Phi & \mu \downarrow \\
A & \longrightarrow & W \\
\downarrow \mu & \downarrow j & \downarrow f \\
B & \longrightarrow & B
\end{array}
\]

in which \(\mu \in \mathcal{R}\) and \(\mu \circ g \circ j = \mu \circ f\), there exists a \(g': E \to X\) with \(g' \circ j = f\) and \(\mu \circ g' = \mu \circ g\).

This concept depends on the class \(\mathcal{R}\) itself rather than on the homotopy notion that \(\mathcal{R}\) determines. In fact, the corresponding homotopy notions may be the same, and the associated cofibrations different, since

4.2. Theorem. In the category Top, \((12)\) For \(\mathcal{K}\)-fibrations, this theorem was suggested by the referee.

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(a) \( \{3\text{-cofibrations}\} = \{3\text{-cofibrations}\} = \text{all morphisms in Top}; \)
(b) \( \{3\text{-cofibrations}\} = \text{class of all } j: B \to E \text{ such that every continuous } \phi: B \to I \text{ has a factorization } \phi = \psi \circ j \text{ where } \psi: E \to I. \)

**Proof.** (a) is trivial. (b) Assume that \( j: B \to E \) is a \( 3\text{-cofibration}. \) Let \( \phi: B \to I \) be given and consider

\[
\begin{array}{c}
E \xrightarrow{g} x_0 \times I \xrightarrow{r} x_0 \\
\downarrow j \quad \quad \quad \downarrow f \\
B
\end{array}
\]

where \( f(b) = (x_0, \phi(b)) \) and \( g(e) = (x_0, 0) \). By hypothesis, there exists a \( g': E \to x_0 \times I \) such that \( r \circ g' = r \circ g \) and \( g' \circ j = f \), i.e., \( g'(e) = (x_0, \psi(e)) \) and \( \psi \circ j = \phi \).

Conversely, assume that the property is satisfied for \( j: B \to E \), and consider any diagram

\[
\begin{array}{c}
E \xrightarrow{g} X \times I \xrightarrow{r} X \\
\downarrow j \quad \quad \downarrow f \\
B
\end{array}
\]

where \( r \circ f = r \circ g \circ j \). Using the projections on each factor, write \( f(b) = (f_x(b), f_1(b)) \) and \( g(e) = (g_x(e), g_1(e)) \); the commutativity requirement assures \( f_x(b) = g_x(jb) \); but \( f_1(b) = \psi(jb) \) for suitable \( \psi: E \to I \), so that \( g'(e) = (g_x(e), \psi(e)) \) is the desired map.

In particular, if \( j: B \to E \) is inclusion onto a retract of \( E \), then \( j \) is a \( 3\text{-cofibration}. \) Moreover,

4.3. If \( B \) is a functional Hausdorff space\(^{(13)} \), then each \( 3\text{-cofibration } j: B \to E \) is injective.

**Proof.** Let \( j: B \to E \) be a map such that \( j(b_0) = j(b_1) \) for some \( b_0 \neq b_1 \). There exists a map \( \phi: B \to I \) with \( \phi(b_0) = 0, \phi(b_1) = 1 \); and \( \phi \) cannot factor as \( \phi = \psi \circ j \).

It is interesting to note that, within this framework, \( \{3\text{-fibrations}\} \equiv \{\text{Hurewicz fibrations}\} \) has \( \{3\text{-cofibrations}\} \equiv \{\text{all morphisms}\} \) for dual; and that \( \{3\text{-fibrations}\} \equiv \{3\text{-cofibrations}\} \equiv \{\text{all morphisms}\} \) for dual.

For a covariant functor \( \Phi: \mathcal{X} \to \mathcal{L} \) the notion of \( \Phi\text{-cofibration} \) is defined, in the customary manner, to be that of \( \mathcal{M}(\Phi)\text{-cofibration}. \) For contravariant functors, one uses the dual category, in which fibrations are exchanged against cofibrations.

5. **Weak \( \mathcal{M}\text{-fibrations}.** We recall that, in the category Top, a map \( p: E \to B \) is called a Dold fibration (= a map with the WCHP in [2]) if for each commutative diagram

\[(13) \text{I.e., for each pair } b_0 \neq b_1 \text{ of points in } B, \text{ there exists a continuous } \phi: B \to I \text{ such that } \phi(b_0) = 0 \text{ and } \phi(b_1) = 1.\]
there exists a map \( F : X \times I \to E \) such that \( p \circ F = f \) and \( F \circ i_0 \) is fiber-homotopic to \( f \).

For any \( X \in \text{Top} \), let \( p = p_X : X \times I \to X \times I \) denote the map

\[
\rho(x, t) = \begin{cases} 
(x, 0), & 0 \leq t \leq \frac{1}{2}, \\
(x, 2t - 1), & \frac{1}{2} \leq t \leq 1,
\end{cases}
\]

and let \( \mathcal{R} = \{\rho_X \mid X \in \text{Top}\} \). Then

5.1. \textbf{Lemma.} A map \( p : E \to B \) is a Dold fibration if and only if for each diagram

\[
\begin{array}{ccc}
X & \xrightarrow{i_0} & X \times I \\
\downarrow & & \downarrow p \\
X \times I & \xrightarrow{F} & B
\end{array}
\]

with \( p \circ g \circ i_0 = F \circ i_0 \), there exists a \( g' : X \times I \to E \) such that \( g \circ i_0 = g' \circ i_0 \) and \( p \circ g' = F \circ \rho \).

Because the proof of 5.1 is entirely analogous to that of 3.3, it is omitted.

Let \( \mathcal{X} \) be an arbitrary category and \( \mathcal{R} \) be any family of its morphisms. With 5.1 in mind, we make the following

5.2. \textbf{Definition.} A morphism \( f : X \to B \) is called a weak \( \mathcal{R} \)-fibration if for each diagram

\[
\begin{array}{ccc}
W & \xrightarrow{\mu} & X \\
\downarrow & & \downarrow g \\
\downarrow f & & \downarrow p \\
& & B
\end{array}
\]

in which \( \mu \in \mathcal{R} \) and \( p \circ g \circ \mu = f \circ \mu \), there exists a \( g' : X \to E \) in \( \mathcal{X} \), and an \( r : X \to X \) in \( \mathcal{R} \) such that \( r \circ \mu = \mu \), \( g \circ \mu = g' \circ \mu \), and \( p \circ g' = f \circ r \).

\textbf{Remarks.} (1) If \( \mathcal{R} \) is such that \( \mathcal{R} \cap \mathcal{X}(X, X) = \{1_X : X \to X\} \) for each \( X \in \mathcal{X} \), then \{weak \( \mathcal{R} \)-fibrations\} = \{\( \mathcal{R} \)-fibrations\}.

(2) If \( \mathcal{R} \) is any class that contains all the identity morphisms, then each \( \mathcal{R} \)-fibration is also a weak \( \mathcal{R} \)-fibration.

(3) Unlike the \( \mathcal{R} \)-fibrations, it is not in general true that \( (\mathcal{R} \subseteq \mathcal{R}) = (\{\text{weak } \mathcal{R} \text{-fibrations}\} \subseteq \{\text{weak } \mathcal{R} \text{-fibrations}\}) \).

5.3. \textbf{Definition.} For any family \( \mathcal{R} \) of morphisms in \( \text{Top} \), we let \( \mathcal{R}_* = \mathcal{R} \cup \{\text{all identities}\} \) and \( \mathcal{R}^* = \mathcal{R}_* \cup \mathcal{R} \). Then

5.4. \textbf{Theorem.} In the category \( \text{Top} \),
(a) \{weak \mathcal{X}_e\text{-fibrations}\} = \{weak \mathcal{X}_c\text{-fibrations}\} = \{Hurewicz fibrations\};

(b) \{weak \mathcal{X}^*\text{-fibrations}\} = \{weak \mathcal{X}^c\text{-fibrations}\} = \{Dold fibrations\}.

**Proof.** (a) follows immediately from Remark (1) above and Theorems 3.3, 3.5. To prove (b): Observe that we have

\[ \{weak \mathcal{X}^*\text{-fibrations}\} \subseteq \{weak \mathcal{X}^c\text{-fibrations}\} \]

because \mathcal{X} \subseteq \mathcal{X}^* and all morphisms \( r \) that can possibly appear in Definition 5.2 are in \( \mathcal{X}_e \). The converse inclusion follows by an argument entirely analogous to that in the proof of 3.5. Finally, \{weak \mathcal{X}^c\text{-fibrations}\} = \{Dold fibrations\} by 5.1. This completes the proof (b).

Everything which has been said for weak \( \mathcal{M} \)-fibrations can be dualized in the usual fashion, to yield weak \( \mathcal{M} \)-cofibrations.

**Appendix.** In this section, we will give a proof of Theorem 1.1. We also include an example to show that in general \( E(\mathcal{E}(\mathcal{M})) \neq E(\mathcal{M}) \) and \( E(\mathcal{M}) \neq E(\mathcal{E}) \).

Let \( \mathcal{X} \) be an arbitrary category, and let \( \mathcal{M} \) be any subclass of its morphisms; the case that \( \mathcal{M} \) is the family of all morphisms in \( \mathcal{X} \) is not excluded. We denote the elements of \( \mathcal{M} \) by small Greek letters, \( \alpha, \beta, \ldots \).

Given \( A, B \in \text{ob}(\mathcal{X}) \), by an \( \mathcal{M} \)-word \( m = (A, X_1, \ldots, X_{2n-1}, B) \) from \( A \) to \( B \) we mean a finite chain of morphisms and objects

\[
A \xleftarrow{\alpha_1} X_1 \xrightarrow{f_1} X_2 \xleftarrow{\alpha_2} X_3 \xrightarrow{f_2} \cdots \xleftarrow{\alpha_n} X_{2n-1} \xrightarrow{f_n} B
\]

in which the \( \xleftarrow{\cdot} \) and \( \xrightarrow{\cdot} \) alternate, and each morphism \( \alpha_i \) “in the wrong direction” either belongs to \( \mathcal{M} \) or is an identity morphism. The class of all \( \mathcal{M} \)-words from \( A \) to \( B \) is denoted by \( \mathcal{M}(A, B) \).

We introduce an equivalence relation \( \sim \) in \( \mathcal{M}(A, B) \) by

A.1. **Definition.** Let \( m, \hat{m} \in \mathcal{M}(A, B) \). Declare \( m \sim \hat{m} \) if there is a finite sequence \( m = m_1, m_2, \ldots, m_\ell = \hat{m} \) of elements of \( \mathcal{M}(A, B) \) in which each \( m_i \) is obtained from \( m_{i-1} \) (or from \( m_{i+1} \)) by one of the following two operations:

(i) A pushout operation: replacement of a segment

\[
\begin{array}{ccc}
\cdot & \xrightarrow{f} & X \\
\downarrow & \alpha & \downarrow g \\
Y \xleftarrow{\alpha} & \xrightarrow{Y} & Z \\
\end{array}
\]

in \( m_{i-1} \) (or \( m_{i+1} \)) by

\[
\begin{array}{ccc}
W \xleftarrow{\cdot} & \xrightarrow{u \circ f} & Y \\
\downarrow & \downarrow v \circ \beta \\
\cdot & \xrightarrow{\cdot} & Z
\end{array}
\]

if there is a commutative diagram

\[
\begin{array}{ccc}
X & \xleftarrow{\alpha} & Y \\
\downarrow u & \downarrow & \downarrow v \\
W & \xrightarrow{\cdot} & Z
\end{array}
\]
and \( v \circ \beta \in \mathcal{M} \) or is an identity morphism. This operation is denoted by

\[ \text{PO}(X, Y, Z; u, W, v). \]

(ii) A pullback operation: replacement of a segment

\[ \begin{array}{c}
\alpha \\
\downarrow \alpha \circ u \\
X \rightarrow Y \leftarrow Z \rightarrow W
\end{array} \]

in \( m_{i-1} \) (or \( m_{i+1} \)) by

\[ \begin{array}{c}
\alpha \circ u \\
\downarrow \alpha \circ u \\
W \rightarrow W
\end{array} \]

if there is a commutative diagram

\[ \begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
W & \xrightarrow{u} & W
\end{array} \]

and \( \alpha \circ u \in \mathcal{M} \) or is an identity morphism. This operation is denoted by

\[ \text{PB}(X, Y, Z; u, W, v). \]

It is clear that \( \sim \) is an equivalence relation in \( \mathcal{M}(A, B) \); the equivalence class of \( m \) is denoted by \([m]\). Furthermore, the map \( \mathcal{M}(A, B) \otimes \mathcal{M}(B, C) \rightarrow \mathcal{M}(A, C) \) defined by

\[ (A, X_1, \ldots, X_{2n-1}, B) \otimes (B, Y_1, \ldots, C) = (A, X_1, \ldots, X_{2n-1}, B, Y_1, \ldots, C) \]

is associative, and the equivalence class of \( m \otimes m' \) depends only on that of \( m \) and of \( m' \).

A.2. Definition. The category \( \mathcal{K}/\mathcal{M} \) is that having for objects \( \{A | A \in \text{ob}(\mathcal{X})\} \); for morphisms \( \text{hom}(A, B) \): the equivalence classes\(^{(14)}\) in \( \mathcal{M}(A, B) \), with the composition law \( \text{hom}(B, C) \circ \text{hom}(A, B) \rightarrow \text{hom}(A, C) \) given by \( [m'] \circ [m] = [m \otimes m'] \).

The identity \([e_A]\) in \( \text{hom}(A, A) \) is the morphism \([A \xleftarrow{1} A \xrightarrow{1} A]\): for, given \([m] \in \text{hom}(B, A)\) we find that \( m \otimes e_A \) is a word of form

\[ \cdots \leftarrow X \xrightarrow{f} A \leftarrow 1 \xrightarrow{A} A \rightarrow A \]

and \( \text{PB}(X, A, A; 1, X, f) \) shows \( m \otimes e_A \sim m \); similarly, \( e_A \otimes m' \sim m' \) for \( m' \in \mathcal{M}(A, B) \) so \([e_A]\) is the (unique) identity morphism of \( A \).

A.3. Let each morphism in a word \( m \in \mathcal{M}(A, B) \) be either a member of \( \mathcal{M} \) or an identity. Then \([m]\) is invertible in \( \mathcal{K}/\mathcal{M} \); and its inverse is represented by \( m \) written in reverse order.

\(^{(14)}\) If \( \mathcal{X} \) is not a small category, \( \text{hom}(A, B) \) need not be a set. However, in all the choices of \( \mathcal{X} \) and \( \mathcal{M} \) considered in this paper, the class \( \text{hom}(A, B) \) can be proved to be a set, so that the construction of this category is legitimate.
Proof. Letting \( m' \) be the word in reverse order, we have \( m' \in \mathcal{M}(B, A) \), and

\[
m \otimes m' = A \xleftarrow{\alpha} X \xrightarrow{\beta} Y \xleftarrow{\gamma} B \xrightarrow{\beta} Y \xleftarrow{\gamma} B \xrightarrow{\alpha} A.
\]

Starting from the middle with \( PB(Y, B, Y; 1, Y, 1) \) and repeating, shows \( m \otimes m' \sim A \xleftarrow{\alpha} X \xrightarrow{\beta} Y \xleftarrow{\gamma} B \xrightarrow{\beta} Y \xleftarrow{\gamma} B \xrightarrow{\alpha} A \).

Inserting \( PO(A, X, A; 1, A, 1) \) we find \( m \otimes m' \sim e_A \). Similarly \( m' \otimes m \sim e_B \) and the proof is complete.

Observe that any word

\[
A \leftarrow X_1 \xrightarrow{f_1} X_2 \leftarrow \cdots \leftarrow X_{2n-2} \xrightarrow{\alpha_n} X_{2n-1} \xrightarrow{f_n} B
\]

is equivalent to

\[
(A \leftarrow X_1 \xrightarrow{1} X_1) \otimes (X_1 \leftarrow X_1 \xrightarrow{f_1} X_2) \otimes \cdots \otimes (X_{2n-2} \leftarrow X_{2n-1} \xrightarrow{1} X_{2n-1} ) \otimes (X_{2n-1} \leftarrow X_{2n-1} \xrightarrow{f_n} B).
\]

Thus, each morphism in \( \mathcal{X}/\mathcal{M} \) can be factored as \( f_n \circ \alpha_n \circ \cdots \circ f_1 \circ \alpha_1 \) where, according to A.3, each \( \alpha_i \) is invertible.

The transformation \( \eta: \mathcal{X} \rightarrow \mathcal{X}/\mathcal{M} \) given by rules

\[
\eta(A) = A \quad \text{on the objects},
\]

\[
\eta(f) = [A \leftarrow A \xrightarrow{f} B] \quad \text{for each } f \in \mathcal{X}(A, B)
\]

is easily seen to be a covariant functor, called the canonical projection. Because of A.3, \( \eta(\alpha) \) is invertible in \( \mathcal{X}/\mathcal{M} \) for each \( \alpha \in \mathcal{M} \); and the known [3] universality property of \( (\eta, \mathcal{X}/\mathcal{M}) \), in a formulation adapted for our purposes, is

A.4. Theorem. Let \( T: \mathcal{X} \rightarrow \mathcal{L} \) be any covariant functor to any category \( \mathcal{L} \). Then

(a) There exists a covariant functor \( \Delta: \mathcal{X}/\mathcal{M} \rightarrow \mathcal{L} \) with \( T = \Delta \circ \eta \) if and only if \( T(\alpha) \) is invertible in \( \mathcal{L} \) for each \( \alpha \in \mathcal{M} \).

(b) If \( T = \Delta \circ \eta \), then \( \Delta \) is unique.

Proof. (a) If \( T = \Delta \circ \eta \) then, because \( \eta(\alpha) \) is invertible for each \( \alpha \in \mathcal{M} \) and functors preserve invertible morphisms, \( T \) has the required property. Conversely, assume \( T(\alpha) \) invertible for each \( \alpha \in \mathcal{M} \). Define \( \Delta \) on the objects of \( \mathcal{X}/\mathcal{M} \) by \( \Delta(A) = T(A) \). For each word

\[
m = A \leftarrow X_1 \xrightarrow{f_1} X_2 \leftarrow \cdots \leftarrow X_{2n-1} \xrightarrow{f_n} B
\]

in \( \mathcal{M}(A, B) \), let

\[
\Delta(m) = T(f_n) \circ T(\alpha_n)^{-1} \circ \cdots \circ T(f_1) \circ T(\alpha_1)^{-1} \in \mathcal{L}(\Delta(A), \Delta(B))
\]

where \( T(\alpha_i)^{-1} \) is the inverse of \( T(\alpha_i) \) in \( \mathcal{L} \). If \( m \sim m' \), then \( \Delta(m) = \Delta(m') \) because the operations in changing \( m \) to \( m' \) are governed by commutative diagrams, and all

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morphisms “in the wrong direction” belong to $\mathcal{M}$. Thus, $[m] \mapsto \Delta(m)$ is a well-defined\(^{(15)}\) map $\Delta : \text{hom}(A, B) \rightarrow \mathcal{L}(\Delta(A), \Delta(B))$ for each $A, B \in \text{ob}(\mathcal{X}/\mathcal{M})$ which with the above indicated correspondence of the objects, is easily seen to determine a covariant functor $\Delta : \mathcal{X}/\mathcal{M} \rightarrow \mathcal{L}$. Clearly, $\Delta \circ \eta(A) = A$ on objects, and $\Delta \circ \eta(f) = \Delta([A \xleftarrow{\perp} A \rightarrow B]) = \mathcal{T}(f)$ for each $f \in \mathcal{L}(A, B)$, so $\Delta \circ \eta = T$ and the proof is complete.

(b) Assume $T = \Delta \circ \eta = \Gamma \circ \eta$; then $\Delta(A) = T(A) = \Gamma(A)$ for each $A \in \text{ob}(\mathcal{X}/\mathcal{M})$. For each morphism of type $f = [A \xleftarrow{\perp} A \rightarrow B]$ we have $\Delta(f) = \Delta \circ \eta(f) = \mathcal{T}(f) = \Gamma \circ \eta(f) = \Gamma(f)$. Each morphism of type $\hat{\alpha} = [A \xleftarrow{\perp} B \rightarrow A]$ is invertible in $\mathcal{N}/\mathcal{M}$, with inverse $\hat{\alpha}^{-1} = [B \xleftarrow{\perp} B \rightarrow A]$. Thus $\Delta(\hat{\alpha})$, $\Gamma(\hat{\alpha})$ are invertible in $\mathcal{L}$ and, from what we have already shown, $\Delta(\hat{\alpha}) = \Delta(\hat{\alpha}^{-1})^{-1} = \Gamma(\hat{\alpha}^{-1})^{-1} = \Gamma(\hat{\alpha})$. Since each morphism in $\mathcal{X}/\mathcal{M}$ is representable as a composition of morphisms of these two types, we find $\Delta([m]) = \Gamma([m])$ for each morphism $[m]$ and the proof is complete.

**Remark.** If $\mathcal{M}$ is the family of all morphisms in $\mathcal{X}$, then all the morphisms of $\mathcal{X}/\mathcal{M}$ are invertible therefore each hom $(X, X)$ is a group, $\pi(X)$. Furthermore, for each $\phi \in \text{hom}(X, Y)$ there is a homomorphism $\phi^* : \pi(X) \rightarrow \pi(Y)$ given by $\phi^*(f) = \phi \circ f \circ \phi^{-1}$, and clearly $\phi^*$ is an isomorphism. Thus the groups $\pi(X)$ belonging to a “component” of $\mathcal{X}$ are all isomorphic; if $\mathcal{X}$ is “connected” then, as in topology, we can define the fundamental group $\pi(\mathcal{X})$ of a category. From C2 it follows that a covariant functor $\Phi : \mathcal{X} \rightarrow \mathcal{L}$ on “connected” categories induces a homomorphism $\pi(\mathcal{X}) \rightarrow \pi(\mathcal{L})$. It is easy to see that for $\text{Top}_0$, the category of based topological spaces with base-point preserving continuous maps, each word from $(X, x_0)$ to $(X, x_0)$ becomes equivalent to $(X, x_0) \leftrightarrow (x_0, x_0) \rightarrow (X, x_0)$ and therefore that $\pi(\text{Top}_0) = 0$. Similarly, it can be shown that $\pi(\text{Top}) = 0$.

We now show

**A.5. Proposition.** There exists a category $\mathcal{X}$ and a class $\mathcal{M}$ such that $E(E(\mathcal{M})) \neq E(\mathcal{M})$ and $E(\mathcal{M}) \neq E(\mathcal{M})$, where $\mathcal{M}$ is the maximal stable subclass in $\mathcal{M}$.

**Proof.** Consider the following diagram

$$
\begin{array}{c}
*_{A} \xrightarrow{h'} S^2_{\beta} \xrightarrow{\alpha} S^2_{\beta} \xrightarrow{h} *_{B} \\
\end{array}
$$

where $S^2_{\beta}, S^2_{\beta}$ are two distinct 2-spheres, $*_{A}, *_{B}$ are distinct points, $h'(*) = \text{north pole}$ of $S^2_{\beta}$, $\alpha$ is the rotation about the $z$-axis with angle $\pi$, and $\beta$ is the rotation about the same axis with angle $\pi/u$, $u$ an irrational. Let $\mathcal{X}$ be the category with these four distinct objects, and with morphisms generated by $h', \alpha, \beta, h$.

Let $\mathcal{M} = \{h, h'\}$; then it is immediate that $\mathcal{M} = \mathcal{X}$ and $E(\mathcal{M}) = \mathcal{M} - \mathcal{M}$. Moreover

\(^{(15)}\) The set of all morphisms $A \rightarrow B$ in a category $\mathcal{L}$ will be denoted by $\mathcal{L}(A, B)$; if $\mathcal{L} = \mathcal{X}/\mathcal{M}$, this set is denoted by $\text{hom}(A, B)$.
(a) $E(E(\mathcal{W})) \neq E(\mathcal{W})$. Let $\mathcal{L}$ be the smallest subcategory of Top containing $\mathcal{K}$ and the inverses $\alpha, \beta$ of $\alpha, \beta$. Let $\lambda: \mathcal{K} \to \mathcal{L}$ be the inclusion functor. Since $\lambda$ sends every member of $E(\mathcal{W})$ to an isomorphism in $\mathcal{L}$, there exists (A.4) a functor $\Delta: \mathcal{K}/E(\mathcal{W}) \to \mathcal{L}$ such that $\Delta \circ \eta = \lambda$. Now, $\alpha \notin E(\mathcal{W})$: for if $\eta(\alpha)(g) = 1$ for some $g$ in $\mathcal{K}$, then $\lambda(\alpha \circ g) = 1$ in $\mathcal{L}$, i.e., $\alpha \circ g = 1$ in $\mathcal{L}$; however $\alpha$ does not have a right inverse in $\mathcal{K}$, so such a $g$ cannot exist in $\mathcal{K}$. Thus, $E(E(\mathcal{W})) \neq E(\mathcal{W})$. Moreover, since $\alpha \in E(\mathcal{W}) \subset E(\mathcal{W})^{-}$, this shows also that $E(\mathcal{W})$ is not stable; furthermore, $\mathcal{K} = \emptyset$, since $h$ and $h'$ do not have inverses in $\mathcal{K}$, so we also have $E(\mathcal{W}) \neq E(\mathcal{W})$.

Bibliography


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