

INFINITE PRIMES AND UNIQUE FACTORIZATION IN A PRINCIPAL RIGHT IDEAL DOMAIN⁽¹⁾

BY

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1. **Introduction.** Throughout this paper all rings are assumed to be integral domains with unity. If R is an integral domain R^* will denote the set of nonzero elements of R . An integral domain in which the sum and intersection of any two principal right ideals is principal whenever the intersection is nonzero is called a weak Bézout domain. In particular, if we do not require that the intersection be nonzero then R is called a right Bézout domain (see [2]). If R is a weak Bézout domain and $a \in R^*$ then the set $[aR, R]$ of all principal right ideals of R that contain aR is a sublattice of the lattice of all right ideals of R . In this case $\dim a$ is defined to be the dimension of the lattice $[aR, R]$, i.e., $\dim a$ is the length of the longest chain in $[aR, R]$. We let R' denote the set of finite dimensional elements of R . If each right ideal of an integral domain R is a principal right ideal then R is called a PRI (principal right ideal) domain. Thus each PRI domain is a weak Bézout domain.

The known types of unique factorization that occur in a PRI domain R deal with the members of R' . For example, it is well known [4, p. 34] that each nonunit z of R' can be factored into primes: $z = p_1 \cdots p_n$, and if $z = q_1 \cdots q_m$ is another such factorization, then $n = m$ and there is a permutation Π on $\{1, 2, \dots, n\}$ such that p_i and $q_{\Pi(i)}$ are similar, $i = 1, 2, \dots, n$. There is another type of unique factorization that occurs in R' which is described by R. E. Johnson in [6]. An element $a \in R'$ is called simple if $[aR, R]$ has the property that $[aR, R] = [aR, B] \cup [B, R]$ implies $B = aR$ or $B = R$. Johnson proves that each element z in R' can be factored into simple elements: $z = a_1 \cdots a_n$, and no subproduct $a_i \cdots a_j$, $i < j$, of z is simple. Any other factorization of z into simple elements of this type must have the form $z = (a_1 u_1)(u_1^{-1} a_2 u_2) \cdots (u_{n-1}^{-1} a_n)$ where u_1, \dots, u_{n-1} are units in R .

The type of unique factorization that we describe in the present paper concerns all of the nonzero elements of a PRI domain R . In §2 we develop some general results (in particular Theorem 2) for weak Bézout domains that satisfy the ascending chain condition for principal right ideals. In §3 we define $\text{inf}^{(\alpha)}$ primes for each non-limit ordinal α . $\text{Inf}^{(0)}$ primes are the usual primes, and $\text{inf}^{(\alpha)}$ primes have infinite dimension if $\alpha \neq 0$. The unique factorization theorem (Theorem 3) that follows

Received by the editors June 28, 1968 and, in revised form, November 12, 1968.

⁽¹⁾ This work is part of the author's Ph.D. dissertation which was done at the University of New Hampshire, Durham, New Hampshire.

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states that each nonzero element a of a PRI domain R can be factored as $a = z_{\alpha_1} \cdots z_{\alpha_n}$ where z_{α_i} is a uniquely determined product of $\text{inf}^{(\alpha_i)}$ primes. The factorization of a is unique in the sense that if $a = y_{\beta_1} \cdots y_{\beta_m}$ is another such factorization of a then $n = m$, $\alpha_i = \beta_i$ ($i = 1, 2, \dots, n$), and there are units u_1, \dots, u_{n-1} in R such that $z_{\alpha_1} = y_{\alpha_1} u_1$, $z_{\alpha_n} = u_{n-1}^{-1} y_{\alpha_n}$, and $z_{\alpha_i} = u_{i-1}^{-1} y_{\alpha_i} u_i$ ($i \neq 1, n$).

The important applications of Theorem 3 obviously occur in PRI domains R such that $R^* \neq R'$. Until recently few examples of such were available. Such examples however do occur, for instance, in P. M. Cohn [3]. In [5] A. V. Jategaonkar describes the method of skew polynomial extensions. Using these methods it is possible to construct PRI domains with $\text{inf}^{(\alpha)}$ primes, given an arbitrary ordinal α . We include this discussion in §4.

2. Right quotient monoids. Suppose R is an integral domain and $\emptyset \neq S \subset R^*$. We call S a *right quotient monoid* in R if S satisfies the following *right quotient conditions*:

- (1) $ab \in S$ iff a and $b \in S$, where $a, b \in R$.
- (2) $a \in R, b \in S$ implies there exists $\bar{a} \in R, \bar{b} \in S$ such that $a\bar{b} = b\bar{a}$.

In this case S contains the group of units of R . For, $a \in S$ for some element a of R . From $1a \in S$ we obtain $1 \in S$ by condition (1). Also, if u is a unit in R then $uu^{-1} = 1 \in S$ and therefore $u \in S$.

If S is a right quotient monoid in R then the set $K = RS^{-1} = \{rs^{-1} \mid r \in R, s \in S\}$ can be made into a ring in the usual way (see Bourbaki [1, p. 162])⁽⁹⁾. It is easy to prove that K is an integral domain with the property that $s \in R$ is a unit in K iff $s \in S$. Further, if A is a right ideal of R then $AS^{-1} = \{as^{-1} \mid a \in A, s \in S\}$ is a right ideal of K , and if R is a PRI domain then so is K . The easy proofs of these facts are omitted.

LEMMA 1. *Let R be an integral domain and let S be a right quotient monoid in R . Let A, B be right ideals of R . Then*

- (1) $(A \cap B)S^{-1} = AS^{-1} \cap BS^{-1}$.
- (2) $(A + B)S^{-1} = AS^{-1} + BS^{-1}$.

Proof. Clearly $(A \cap B)S^{-1} \subset AS^{-1} \cap BS^{-1}$. Now suppose $as_1^{-1} = bs_2^{-1} \in AS^{-1} \cap BS^{-1}$ ($s_1, s_2 \in S$ and $a, b \in R$). Because S is a right quotient monoid we can choose $\bar{s}_1, \bar{s}_2 \in S$ such that $s_1\bar{s}_2 = s_2\bar{s}_1$. It follows that $a\bar{s}_2 = b\bar{s}_1 \in A \cap B$. Also, $as_1^{-1} = a\bar{s}_2(s_2\bar{s}_1)^{-1} \in (A \cap B)S^{-1}$. This proves $AS^{-1} \cap BS^{-1} \subset (A \cap B)S^{-1}$ and (1) is established.

Now it is obvious that $(A + B)S^{-1} \subset AS^{-1} + BS^{-1}$. To show the reverse inclusion let $as_1^{-1} + bs_2^{-1} \in AS^{-1} + BS^{-1}$ ($s_1, s_2 \in S$ and $a, b \in R$). Choose $\bar{s}_1, \bar{s}_2 \in S$ such that $s_1\bar{s}_2 = s_2\bar{s}_1$. Then $as_1^{-1} + bs_2^{-1} = (a\bar{s}_2 + b\bar{s}_1)(s_1\bar{s}_2)^{-1}$. Hence $as_1^{-1} + bs_2^{-1} \in (A + B)S^{-1}$. This proves (2). Q.E.D.

⁽⁹⁾ For this purpose condition (1) in the definition of right quotient monoid is usually replaced by the weaker condition that S be multiplicatively closed.

COROLLARY. *If R is a weak Bézout or a right Bézout domain, then so is $K=RS^{-1}$*

We recall from [2] that if $a, \bar{a} \in R$, then a and \bar{a} are similar ($a \sim \bar{a}$ or $a \sim_R \bar{a}$) if $R/aR \cong R/\bar{a}R$ as right R -modules, and this is true iff there exists $b \in R$ such that $aR+bR=R$ and $aR \cap bR=b\bar{a}R$. It is also shown in [2] that the definition of similarity is left-right symmetric.

LEMMA 2. *Let S be a right quotient monoid in an integral domain R and let $K=RS^{-1}$. If $a, \bar{a} \in R$ and $a \sim_R \bar{a}$ then $a \sim_K \bar{a}$. Further, if $a \sim_R \bar{a}$ and $a \in S$ then $\bar{a} \in S$.*

Proof. Let $a, \bar{a} \in R$ with $a \sim_R \bar{a}$. Then $aR+bR=R$ and $aR \cap bR=b\bar{a}R$ for some $b \in R$. Therefore $aK+bK=K$ and $aK \cap bK=b\bar{a}K$ by Lemma 1. Hence $a \sim_K \bar{a}$. If in addition $a \in S$ then a is a unit in K . Since $a \sim_K \bar{a}$ it follows that \bar{a} is a unit in K . Hence $\bar{a} \in S$. Q.E.D.

LEMMA 3. *Suppose R is a right Bézout domain and $\emptyset \neq S \subset R^*$ such that $ab \in S$ iff a and $b \in S$. Then S is a right quotient monoid in R iff elements similar to members of S belong to S , that is, iff $a \in S, \bar{a} \in R$, and $a \sim \bar{a}$ implies $\bar{a} \in S$.*

Proof. Assume the hypotheses. If S is a right quotient monoid in R then elements similar to members of S belong to S by Lemma 2. Conversely assume that S has the property that $a \in S, \bar{a} \in R$, and $a \sim \bar{a}$ implies $\bar{a} \in S$. To show that S is a right quotient monoid in R we need only establish condition (2) of the definition. Accordingly let $a \in S$ and $b \in R$. Because R is a right Bézout domain we can choose $d, m \in R$ such that $aR+bR=dR$ and $aR \cap bR=mR$. Then $a=da', b=db'$, and $m=a\bar{b}=b\bar{a}$ for some $a', b', \bar{a}, \bar{b} \in R$. It follows that $a'R+b'R=R$ and $a'R \cap b'R=b'\bar{a}R$. Therefore $a' \sim \bar{a}$. Now $a \in S$ implies $a' \in S$. Hence $\bar{a} \in S$ by our assumption. This shows that S is a right quotient monoid in R . Q.E.D.

COROLLARY. *If R is a right Bézout domain then R^* is a right quotient monoid in R .*

THEOREM 1. *Let R be a weak Bézout domain satisfying the ascending chain condition for principal right ideals, and let S be a right quotient monoid in R . Each $z \in R^*$ can be written as $z=xs$ where $s \in S$ and x has no nonunit right factor in S . The factorization is unique in the sense that if $z=x_1s_1=x_2s_2$ are two such factorizations of z then there is a unit $u \in R$ such that $x_1=x_2u$ (and $us_1=s_2$).*

Proof. Let $z \in R^*$. Let C_S be the collection of submodules R/sR of R/zR such that $s \in S$. Note that $zR/zR \cong R/R$ is a member of C_S . We claim that C_S is closed under sums. For if $aR/zR, bR/zR \in C_S$, then $aR/zR \cong R/xR, bR/zR \cong R/yR$ where $z=ax=by$ and $x, y \in S$. Since R is a weak Bézout domain, $aR+bR=dR, aR \cap bR=mR$ for some $d, m \in R$. Choose $a', b', \bar{a}, \bar{b} \in R$ with $a=da', b=db', m=a\bar{b}=b\bar{a}$. Then $a'R+b'R=R, a'R \cap b'R=b'\bar{a}R$. Hence $a' \sim \bar{a}$. Now $ax=by \in mR$ implies

$by = mr$ for some $r \in R$. Hence $y = \bar{a}r$. This shows that $\bar{a} \in S$ because $y \in S$. Consequently $a' \in S$. Then $aR/zR + bR/zR = dR/zR = dR/da'xR \cong R/a'xR$ and $a'x \in S$. We conclude $aR/zR + bR/zR \in C_S$.

To prove the theorem observe that C_S has the ascending chain condition by hypothesis. Thus we may select a (not necessarily proper) maximal member xR/zR of C_S . Thus $xR/zR \cong R/sR$ where $z = xs$ and $s \in S$. Since xR/zR is maximal x has no nonunit right factor in S . Also, xR/zR is the unique maximal member of C_S because C_S is closed under sums. Therefore the factorization is unique. Q.E.D.

Let R be an integral domain and let $I = \{\alpha \mid 0 \leq \alpha \leq \alpha_0\}$ be an initial segment of ordinals. A collection $\{S_\alpha \mid \alpha \in I\}$ of right quotient monoids in R is called a *right quotient chain* in R if the following conditions hold:

- (1) $S_\alpha \subsetneq S_{\alpha+1}$ for each $\alpha \in I$, $\alpha \neq \alpha_0$.
- (2) $S_\alpha = \bigcup_{\beta < \alpha} S_\beta$ if α is a limit ordinal.

For convenience we let S_{-1} denote the group of units of R . Then S_{-1} is contained in each S_α . Let $K_\alpha = R(S_\alpha)^{-1}$ if $\alpha = -1$ or if $\alpha \in I$. Then because of condition (1) $K_{\alpha-1} \subset K_\alpha$ for each $\alpha \in I$.

THEOREM 2. *Let R be a weak Bézout domain satisfying the ascending chain condition for principal right ideals. Let $I = \{\alpha \mid 0 \leq \alpha \leq \alpha_0\}$ be an initial segment of ordinals and let $\{S_\alpha \mid \alpha \in I\}$ be a right quotient chain in R . Each $z \in R^*$ can be factored as $z = ra_{\alpha_1} \cdots a_{\alpha_n}$ where α_i are nonlimit ordinals such that $\alpha_0 \geq \alpha_1 > \cdots > \alpha_n$, $a_{\alpha_i} \in S_{\alpha_i}$, a_{α_i} has no nonunit right factor in S_{α_i-1} , $r \in R$ and r has no nonunit right factor in S_{α_0} . The factorization is unique in the sense that if $z = sb_{\beta_1} \cdots b_{\beta_m}$ is another such factorization of z then $n = m$, $\alpha_i = \beta_i$ ($i = 1, 2, \dots, n$), and there are units u_0, u_1, \dots, u_{n-1} in R such that $r = su_0$, $a_{\alpha_n} = u_n^{-1}b_{\alpha_n}$, and $a_{\alpha_i} = u_i^{-1}b_{\alpha_i}u_i$ ($i \neq 0, n$).*

Proof. To prove existence of the factorization let $z \in R^*$. If z has no nonunit right factor in S_{α_0} then we are finished. Otherwise by Theorem 1 $z = rs_0$ for some nonunit $s_0 \in S_{\alpha_0}$ and r has no nonunit right factor in S_{α_0} . Let α_1 be the least ordinal such that $s_0 \in S_{\alpha_1}$. Clearly α_1 is not a limit ordinal and $\alpha_0 \geq \alpha_1$. It follows by Theorem 1 that $s_0 = a_{\alpha_1}s_1$ for some element $s_1 \in S_{\alpha_1-1}$ and a_{α_1} has no nonunit right factor in S_{α_1-1} . Clearly $a_{\alpha_1} \in S_{\alpha_1}$ because $s_0 \in S_{\alpha_1}$. If s_1 is not a unit let α_2 be the least ordinal such that $s_1 \in S_{\alpha_2}$. Then $\alpha_1 > \alpha_2$ and α_2 is not a limit ordinal. Another application of Theorem 1 yields $s_1 = a_{\alpha_2}s_2$ where $s_2 \in S_{\alpha_2-1}$ and a_{α_2} has no nonunit right factor in S_{α_2-1} . Clearly $a_{\alpha_2} \in S_{\alpha_2}$ because $s_1 \in S_{\alpha_2}$. If s_2 is not a unit we may repeat the argument. Now this process cannot continue indefinitely since we would obtain an infinite sequence $\alpha_1 > \alpha_2 > \cdots$ contradicting the well ordering of the ordinals. Thus the process stops, say, with the integer n . That is, a_{α_n} has no nonunit right factor in S_{α_n-1} and s_n is a unit. This establishes the existence of the factorization.

To prove uniqueness suppose $z = ra_{\alpha_1} \cdots a_{\alpha_n} = sb_{\beta_1} \cdots b_{\beta_m}$ are two factorizations of z of the type stated in the theorem. Then Theorem 1 applies and yields $r = su_0$ for some unit $u_0 \in R$. Therefore $a_{\alpha_1} \cdots a_{\alpha_n} = u_0^{-1}b_{\beta_1} \cdots b_{\beta_m}$. Evidently $\alpha_1 = \beta_1$. Again Theorem 1 applies and yields $a_{\alpha_1} = u_0^{-1}b_{\beta_1}u_1$ for some unit $u_1 \in R$. Cancelling this

factor we obtain $a_{\alpha_2} \cdots a_{\alpha_n} = u_1^{-1} b_{\beta_2} \cdots b_{\beta_m}$. Uniqueness now follows by induction. Q.E.D.

3. Unique factorization and infinite primes. In this section we shall construct a natural set $\{R^{(\alpha)} \mid \alpha \in I\}$ which is a right quotient chain in a right Bézout domain R . We shall then apply Theorem 2 to this right quotient chain. We begin by characterizing the peculiar factors that appear in Theorem 2.

Let $I = \{\alpha \mid 0 \leq \alpha \leq \alpha_0\}$ be an initial segment of ordinals and let $\{S_\alpha \mid \alpha \in I\}$ be a right quotient chain in an integral domain R . If α is a nonlimit ordinal in I then $x \in S_\alpha$ is called an α -prime if xR is maximal in $\{xR \mid x \in S_\alpha \setminus S_{\alpha-1}\}$.

LEMMA 4. *Let $I = \{\alpha \mid 0 \leq \alpha \leq \alpha_0\}$ be an initial segment of ordinals and let $\{S_\alpha \mid \alpha \in I\}$ be a right quotient chain in a PRI domain R . If α is a nonlimit ordinal in I and x is an α -prime then x is prime in $K_{\alpha-1}$.*

Proof. Assume the hypotheses and let x be an α -prime. Suppose $xK_{\alpha-1} \subsetneq yK_{\alpha-1} \subsetneq K_{\alpha-1}$. Then $xR \subsetneq xK_{\alpha-1} \cap R \subsetneq yK_{\alpha-1} \cap R \subsetneq R$. Let $\bar{y} \in R$ be such that $\bar{y}R = yK_{\alpha-1} \cap R$. Then $xR \subsetneq \bar{y}R$. The definition of α -prime implies that $\bar{y} \in S_{\alpha-1}$ and therefore \bar{y} is a unit in $K_{\alpha-1}$ and $\bar{y}K_{\alpha-1} = K_{\alpha-1}$. Now $\bar{y}K_{\alpha-1} = yK_{\alpha-1}$ and so $yK_{\alpha-1} = K_{\alpha-1}$. This shows that x is prime in $K_{\alpha-1}$. Q.E.D.

LEMMA 5. *Let R be a weak Bézout domain, let I be an initial segment of ordinals, and let $\{S_\alpha \mid \alpha \in I\}$ be a right quotient chain in R . If x_1, \dots, x_k are α -primes, then $x_1 \cdots x_k$ has no nonunit right factor that belongs to $S_{\alpha-1}$.*

Proof. The proof is by induction on k . The lemma is true if $k=1$ by the definition of α -prime. Assume k is an integer greater than 1 and the lemma holds for positive integers less than k . Suppose $x_1 \cdots x_k = ab$ with $b \in S_{\alpha-1}$, $a \in R$ and x_i are α -primes. We shall show that b must be a unit. If $aR \subsetneq x_1R$ then $a = x_1s$, $s \in R$. Hence $x_2 \cdots x_k = sb$. It follows by induction that b must be a unit. Suppose on the other hand that $aR \not\subsetneq x_1R$. Then since $x_1R \cap aR \neq 0$ and R is a weak Bézout domain it follows that $x_1R + aR = dR$ and $x_1R \cap aR = mR$ for some $d, m \in R$. Choose $x', a', \bar{a}, \bar{x}_1 \in R$ such that $x_1 = dx'$, $a = da'$, and $m = x_1\bar{a} = a\bar{x}_1$. Then $x'R + a'R = R$ and $x'R \cap a'R = a'\bar{x}_1R$. Consequently $x' \sim \bar{x}_1$. Now $x_1 \cdots x_k = x_1\bar{a}z$ for some $z \in R$. Therefore $ab = x_1 \cdots x_k = x_1\bar{a}z$ and so $b = \bar{x}_1z$. Hence $z \in S_{\alpha-1}$ since $b \in S_{\alpha-1}$. It follows from $x_2 \cdots x_k = \bar{a}z$ and by induction that z is a unit. Consequently b is a right associate of \bar{x}_1 . Therefore $x' \sim \bar{x}_1$ yields $x' \sim b$ and hence $x' \in S_{\alpha-1}$ by Lemma 2. Now $d \in S_{\alpha-1}$ because $x_1R \subsetneq dR$ and x_1 is an α -prime. Therefore $x_1 = dx' \in S_{\alpha-1}$. However this contradicts the fact that x_1 is an α -prime. Q.E.D.

Whenever R is a weak Bézout domain satisfying the ascending chain condition for principal right ideals the converse to Lemma 5 holds as follows.

LEMMA 6. *Let R be a weak Bézout domain satisfying the ascending chain condition for principal right ideals. Let I be an initial segment of ordinals and let $\{S_\alpha \mid \alpha \in I\}$ be a right quotient chain in R . Let α be a nonlimit ordinal in I , and let $a \in S_\alpha \cap (K_{\alpha-1})'$. If a has no nonunit right factor that belongs to $S_{\alpha-1}$ then a is a product of α -primes.*

Proof. Assume the hypotheses. Since $a \in S_\alpha \setminus S_{\alpha-1}$ we may choose x_1 (by the ascending chain condition for principal right ideals) such that x_1R is maximal in $\{xR \mid aR \subset xR \text{ and } x \in S_\alpha \setminus S_{\alpha-1}\}$. Then x_1 is an α -prime and $a = x_1s_1$ for some $s_1 \in R$. Clearly $s_1 \in S_\alpha$ and if s_1 is not a unit then $s_1 \in S_\alpha \setminus S_{\alpha-1}$ because of the assumption on a . We repeat the argument and obtain $s_1 = x_2s_2$ where x_2 is an α -prime and $s_2 \in S_\alpha$. If this process does not terminate we obtain, for each positive integer i , $s_i = x_{i+1}s_{i+1}$ where x_{i+1} is an α -prime and hence a nonunit in $K_{\alpha-1}$. Let $r_i = x_1 \cdots x_i$. Then $a = r_i s_i$ and $r_{i+1} = r_i x_{i+1}$ for each positive integer i . Therefore $aK_{\alpha-1} \subsetneq \cdots \subsetneq r_{i+1}K_{\alpha-1} \subsetneq r_iK_{\alpha-1} \subsetneq \cdots \subsetneq r_1K_{\alpha-1}$. This contradicts the hypothesis that $a \in (K_{\alpha-1})'$. Therefore the process terminates, say, with the integer k . Thus $a = x_1 \cdots x_k s_k$ and s_k is a unit in R . Q.E.D.

Each right Bézout domain contains a natural right quotient monoid as follows.

LEMMA 7. *Let R be a right Bézout domain. Then R' is a right quotient monoid in R .*

Proof. Clearly $\emptyset \neq R' \subset R^*$. Suppose $a, b \in R^*$. Then $aR/abR \cong R/bR$ and therefore $\dim ab = \dim a + \dim b$. Hence $ab \in R'$ iff $a, b \in R'$. Also if $a \in R'$ and $\bar{a} \in R$ with $a \sim \bar{a}$ then $aR + bR = R$ and $aR \cap bR = b\bar{a}R$ for some $b \in R$. Therefore $[aR, R] = [aR, aR + bR] \cong [aR \cap bR, bR] = [b\bar{a}R, bR] \cong [\bar{a}R, R]$ as lattices. Thus $\dim a = \dim \bar{a}$ and so $\bar{a} \in R'$. It follows by Lemma 3 that R' is a right quotient monoid in R . Q.E.D.

Let R be a right Bézout domain. We construct, by transfinite induction, a natural chain $\{R^{(\alpha)} \mid \alpha \text{ is an ordinal}\}$ of right quotient monoids in R as follows.

Let $R^{(0)} = R'$. Let α be an ordinal greater than zero and assume $R^{(\beta)}$ has been defined and is a right quotient monoid in R whenever $\beta < \alpha$, and let $K_\beta = R(R^{(\beta)})^{-1}$. Then K_β is a right Bézout domain (Corollary to Lemma 1) and hence K'_β is a right quotient monoid in K_β by Lemma 7. We define $R^{(\alpha)}$ by

$$R^{(\alpha)} = \bigcup_{\beta < \alpha} R^{(\beta)} \text{ if } \alpha \text{ is a limit ordinal,}$$

$$R^{(\alpha)} = (K_{\alpha-1})' \cap R \text{ if } \alpha \text{ is not a limit ordinal.}$$

To show that the induction is valid we must show that $R^{(\alpha)}$ is a right quotient monoid in R . If α is a limit ordinal the proof is obvious. Assume that α is not a limit ordinal. Clearly $\emptyset \neq R^{(\alpha)} \subset R^*$. Also $ab \in R^{(\alpha)}$ iff $a, b \in R^{(\alpha)}$ because $(K_{\alpha-1})'$ has this property. Now if $a \in R^{(\alpha)}$, $\bar{a} \in R$ and $a \sim_R \bar{a}$ then $a \sim_{K_{\alpha-1}} \bar{a}$ by Lemma 2. It follows (as in the proof of Lemma 7) that $\dim_{K_{\alpha-1}} a = \dim_{K_{\alpha-1}} \bar{a}$. Hence $\bar{a} \in (K_{\alpha-1})'$ since $a \in (K_{\alpha-1})'$. It follows that $\bar{a} \in R^{(\alpha)}$. The hypotheses of Lemma 3 are satisfied and therefore $R^{(\alpha)}$ is a right quotient monoid in R . Q.E.D.

If α, β are ordinals such that $\alpha \leq \beta$, then $R^{(\alpha)} \subset R^{(\beta)} \subset R$. Also $R^{(\alpha)} = R^{(\alpha+1)}$ for some ordinal α . For if $R^{(\alpha)} \neq R^{(\alpha+1)}$ for each ordinal α then $\text{card}(R^{(\alpha)}) \geq \text{card}(\alpha)$ for each ordinal α . Choosing β such that $\text{card}(\beta) > \text{card}(R)$ we obtain $\text{card}(\beta) > \text{card}(R) \geq \text{card}(R^{(\beta)})$, a contradiction. We let α_0 denote the least ordinal such that $R^{(\alpha_0)} = R^{(\alpha_0+1)}$, and we call $\{R^{(\alpha)} \mid 0 \leq \alpha \leq \alpha_0\}$ the *right D-chain* (Dimension chain) in R . In this situation $R^{(-1)}$ will denote the group of units of R .

Evidently the right D -chain in a right Bézout domain R is a right quotient chain in R . If R is a PRI domain then the right D -chain has the following additional property.

LEMMA 8. *Let R be a PRI domain and let $\{R^{(\alpha)} \mid 0 \leq \alpha \leq \alpha_0\}$ be the right D -chain in R . Then $R^{(\alpha_0)} = R^*$.*

Proof. Suppose $R^* \neq R^{(\alpha_0)}$. Then by the maximum condition in the PRI domain R we may choose x such that xR is maximal in $\{xR \mid x \in R^* \setminus R^{(\alpha_0)}\}$. The proof of Lemma 4 can be used to show that x is prime in K_{α_0} . In particular $x \in (K_{\alpha_0})' \cap R = R^{(\alpha_0+1)}$. This contradicts $R^{(\alpha_0+1)} = R^{(\alpha_0)}$. Therefore $R^* = R^{(\alpha_0)}$. Q.E.D.

Let R be a PRI domain and let $\{R^{(\alpha)} \mid 0 \leq \alpha \leq \alpha_0\}$ be the right D -chain in R . We shall call the α -primes in R $\text{inf}^{(\alpha)}$ primes. If α is a nonlimit ordinal such that $\alpha \leq \alpha_0$ we let $Z^{(\alpha)}$ be the set of (finite) products of $\text{inf}^{(\alpha)}$ primes.

If R is a PRI domain and α is a nonlimit ordinal such that $\alpha \leq \alpha_0$ then $x \in R$ is an $\text{inf}^{(\alpha)}$ prime iff xR is maximal in $\{xR \mid x \in R \setminus R^{(\alpha-1)}\}$. For if xR is maximal in $\{xR \mid x \in R \setminus R^{(\alpha-1)}\}$ then the proof of Lemma 4 can be used to show that x is prime in $K_{\alpha-1}$. In particular $x \in (K_{\alpha-1})'$ and so $x \in R^{(\alpha)}$. Hence x is an $\text{inf}^{(\alpha)}$ prime. As a consequence we note that $\text{inf}^{(\alpha)}$ primes exist for each nonlimit ordinal $\alpha \leq \alpha_0$ by the maximum condition in R . In fact for each $z \notin R^{(\alpha-1)}$, $zR \subset xR$ for some $\text{inf}^{(\alpha)}$ prime x .

If R is a PRI domain we can combine Lemmas 5 and 6 into the following.

LEMMA 9. *Let R be a PRI domain and let $\{R^{(\alpha)} \mid 0 \leq \alpha \leq \alpha_0\}$ be the right D -chain in R . Let α be a nonlimit ordinal such that $\alpha \leq \alpha_0$, and let $z \in R^{(\alpha)}$. Then z has no nonunit right factor in $R^{(\alpha-1)}$ iff $z \in Z^{(\alpha)}$, i.e., iff z is a product of $\text{inf}^{(\alpha)}$ primes.*

Using Lemmas 8 and 9 we can state Theorem 2 for the present case as follows.

THEOREM 3. *Let R be a PRI domain and let $\{R^{(\alpha)} \mid 0 \leq \alpha \leq \alpha_0\}$ be the right D -chain in R . Each $a \in R^*$ can be written as $a = z_{\alpha_1} \cdots z_{\alpha_k} u$ where α_i are nonlimit ordinals such that $\alpha_0 \geq \alpha_1 > \cdots > \alpha_k$ and $z_{\alpha_i} \in Z^{(\alpha_i)}$ and u is a unit in R . This factorization is unique in the sense that if $a = y_{\beta_1} \cdots y_{\beta_n} v$ is another such factorization of a then $h = k$, $\alpha_i = \beta_i$ ($i = 1, 2, \dots, k$), and there are units u_1, \dots, u_{k-1} in R such that $z_{\alpha_1} = y_{\alpha_1} u_1$, $z_{\alpha_k} = u_{k-1}^{-1} y_{\alpha_k}$, and $z_{\alpha_i} = u_{i-1}^{-1} y_{\alpha_i} u_i$ ($i \neq 1, k$).*

We note that the factors that appear in Theorem 3 are themselves uniquely determined as follows. Let $\alpha \in [0, \alpha_0]$ be a nonlimit ordinal and let $z \in Z^{(\alpha)}$, z a nonunit. Then z is a product of $\text{inf}^{(\alpha)}$ primes. Since an $\text{inf}^{(\alpha)}$ prime is a prime in $K_{\alpha-1}$ (Lemma 4) the decomposition of z is unique up to similarity in $K_{\alpha-1}$. That is, if $z = x_1 \cdots x_n = y_1 \cdots y_m$ where x_i and y_i are $\text{inf}^{(\alpha)}$ primes in R , then $n = m$ and there is a permutation Π on $\{1, 2, \dots, n\}$ such that $x_i \sim_{K_{\alpha-1}} y_{\Pi(i)}$ ($i = 1, 2, \dots, n$).

4. **Example.** If L is a ring and σ is a monomorphism from L into L we shall denote by $H = L[x, \sigma]$ the ring of skew polynomials in an indeterminate x with coefficients in L (written on the right of x). Addition in H is the usual pointwise

addition and multiplication is determined by the associative and distributive laws and by the commutation rule $ax = xa^\sigma$ ($a \in L$). It is easy to prove that L is an integral domain iff H is an integral domain (see Ore [7]). Further, it is shown in [5] that H is a PRI domain iff L is a PRI domain and σ maps L^* into the group of units of L .

We turn our attention to skew polynomial extensions which are defined by A. V. Jategaonkar in [5]. Let $\bar{\alpha}$ be an ordinal, let $I = [0, \bar{\alpha}]$, and let L be an integral domain. Let H be a right twisted polynomial extension of L with $\{H_{(\alpha)} \mid \alpha \in I\}$ a chain of twisted subdomains from L to H . Thus for each nonlimit ordinal $\alpha \in I$ there exists a monomorphism $\rho_\alpha: H_{(\alpha-1)} \rightarrow H_{(\alpha-1)}$ and an indeterminate x_α such that

$$\begin{aligned}
 H &= H_{(\bar{\alpha})}, L = H_{(-1)}, \\
 H_{(\alpha)} &= H_{(\alpha-1)}[x_\alpha, \rho_\alpha] \text{ if } \alpha \in I \text{ and } \alpha \text{ is not a limit ordinal,} \\
 H_{(\alpha)} &= \bigcup_{\beta < \alpha} H_{(\beta)} \text{ if } \alpha \in I \text{ and } \alpha \text{ is a limit ordinal.}
 \end{aligned}$$

In addition we assume that L is a skew field and $\rho_\alpha: H_{(\alpha-1)} \rightarrow L$ for each nonlimit ordinal $\alpha \in I$. Then H is a PRI domain [5]. Now each member of H is a polynomial in a finite number of indeterminates. Let S be the set of polynomials of H with nonzero constant terms in L . Then it can be shown [5] that S is a right quotient monoid in H , and $R = HS^{-1}$ is a local PRI domain with unique maximal ideal x_0R such that

$$\begin{aligned}
 (*) \quad x_{\alpha+1}R &= \bigcap_{n=0}^{\infty} (x_\alpha)^n R \text{ if } \alpha \neq \bar{\alpha} \text{ and } \alpha \text{ is not a limit ordinal,} \\
 x_{\alpha+1}R &= \bigcap_{\beta < \alpha} x_\beta R \text{ if } \alpha \neq \bar{\alpha} \text{ and } \alpha \text{ is a limit ordinal.}
 \end{aligned}$$

That $\alpha_0 = \bar{\alpha}$ is a consequence of the next lemma.

LEMMA 10. *If $\bar{\alpha} \neq \alpha \in I$ or $\alpha = -1$ then $R \setminus R^{(\alpha)} = x_{\alpha+1}R$.*

Proof. The proof is by transfinite induction. Clearly $R \setminus R^{(-1)} = x_0R$ since x_0R is the unique maximal ideal of R . Let $\alpha \in I$, $\alpha \neq \bar{\alpha}$ and assume that $R \setminus R^{(\beta)} = x_{\beta+1}R$ if $\beta < \alpha$. If α is a limit ordinal then we obtain

$$R \setminus R^{(\alpha)} = R \setminus \left(\bigcup_{\beta < \alpha} R^{(\beta)} \right) = \bigcap_{\beta < \alpha} (R \setminus R^{(\beta)}) = \bigcap_{\beta < \alpha} x_{\beta+1}R = x_{\alpha+1}R$$

by (*) and the induction hypothesis. Now assume that α is not a limit ordinal. Then from $R \setminus R^{(\alpha-1)} = x_\alpha R$ it follows that x_α is a prime in $K_{\alpha-1}$. Also from (*) we obtain $x_{\alpha+1}K_{\alpha-1} = \bigcap_{n=0}^{\infty} (x_\alpha)^n K_{\alpha-1}$ and therefore $\dim_{K_{\alpha-1}}(x_{\alpha+1}) = \infty$. It follows that $x_{\alpha+1} \notin (K_{\alpha-1})'$. Therefore $x_{\alpha+1} \in R \setminus R^{(\alpha)}$ and $x_{\alpha+1}R \subset R \setminus R^{(\alpha)}$. To show the reverse inclusion let $f \in R \setminus R^{(\alpha)}$. Then $f \in R \setminus R^{(\alpha-1)} = x_\alpha R$. In fact $f \in \bigcap_{n=0}^{\infty} (x_\alpha)^n R$. Otherwise $f = (x_\alpha)^m g$ where m is the largest such integer. This implies $g \in R \setminus x_\alpha R = R^{(\alpha-1)}$ and so g is a unit in $K_{\alpha-1}$. Since x_α is prime in $K_{\alpha-1}$ it follows that $\dim_{K_{\alpha-1}}(f) < \infty$. Hence $f \in (K_{\alpha-1})' \cap R = R^{(\alpha)}$, a contradiction. Thus $f \in \bigcap_{n=0}^{\infty} (x_\alpha)^n R$, and so $f \in x_{\alpha+1}R$. Q.E.D.

COROLLARY. For each nonlimit ordinal $\alpha \leq \alpha_0$, x_α is the unique (up to right unit factor) $\text{inf}^{(\alpha)}$ prime in R .

Using the last corollary we may state Theorem 3 for the present case as follows.

THEOREM 4. Let R be the ring of polynomials constructed in this section. Each nonzero element $f \in R$ can be written in the form $f = x_{\alpha_1}^{n_1} \cdots x_{\alpha_k}^{n_k} u$ where n_i are positive integers, $\alpha_1 > \cdots > \alpha_k$ and u is a unit in R . This expression is unique in the sense that if $f = x_{\beta_1}^{m_1} \cdots x_{\beta_h}^{m_h} v$ is another such factorization of f , then $h = k$, $\alpha_i = \beta_i$ and $n_i = m_i$ ($i = 1, 2, \dots, k$).

Acknowledgement. The author is deeply indebted to Professor R. E. Johnson for encouragement and guidance in the development of the present work.

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