ABSOLUTE GAP-SHEAVES AND EXTENSIONS OF
COHERENT ANALYTIC SHEAVES

BY

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Thimm introduced the concept of gap-sheaves for analytic subsheaves of finite
direct sums of structure-sheaves on domains of complex number spaces (Definition
9, [13]) and proved that these gap-sheaves are coherent if the subsheaves themselves
are coherent (Satz 3, [13]). This concept of gap-sheaves can be readily generalized
to analytic subsheaves of arbitrary analytic sheaves on general complex spaces
(Definition 1, [12]). All the gap-sheaves of coherent analytic subsheaves of arbitrary
coherent analytic sheaves on general complex spaces are coherent (Theorem 3, [12]).
The gap-sheaves of a given analytic subsheaf depend not only on the subsheaf
itself but also on the analytic sheaf in which the given subsheaf is embedded as a
subsheaf.

In this paper we introduce a new notion of gap-sheaves which we call absolute
gap-sheaves (Definition 3 below). These gap-sheaves arise naturally from the
problem of removing singularities of local sections of a coherent analytic sheaf.
They depend only on a given analytic sheaf and neither require nor depend upon an
embedding of the given sheaf as a subsheaf in another analytic sheaf. We give here a
necessary and sufficient condition for the coherence of absolute gap-sheaves of
coherent sheaves (Theorem 1 below). This yields some results concerning removing
singularities of local sections of coherent sheaves (see Remark following Corollary 2
to Theorem 1). Then we use absolute gap-sheaves to derive a theorem (Theorem 2
below) which generalizes Serre’s Theorem on the extension of torsion-free coherent
analytic sheaves (Theorem 1, [11]). Finally a result on extensions of global sections
of coherent analytic sheaves is derived (Theorem 4 below).

Unless specified otherwise, complex spaces are in the sense of Grauert (§1, [5]).
If \( \mathcal{F} \) is an analytic subsheaf of an analytic sheaf \( \mathcal{I} \) on a complex space \( (X, \mathcal{H}) \),
then \( \mathcal{F} : \mathcal{I} \) denotes the ideal-sheaf \( \mathcal{I} \) defined by \( \mathcal{I}_x = \{ s \in \mathcal{H}_x \mid s \mathcal{I}_x \subset \mathcal{I}_x \} \) for
\( x \in X \). \( E(\mathcal{F}, \mathcal{I}) \) denotes \( \{ x \in X \mid \mathcal{I}_x \neq \mathcal{I}_x \}. \) \( \text{Supp} \ \mathcal{F} \) denotes the support of \( \mathcal{F} \). If
\( t \in \Gamma(X, \mathcal{I}) \), then \( \text{Supp} \ t \) denotes the support of \( t \). For \( x \in X \), \( t_x \) denotes the germ
of \( t \) at \( x \). By the annihilator-ideal-sheaf \( \mathcal{A} \) of \( \mathcal{F} \) we mean the ideal-sheaf \( \mathcal{A} \) defined
by \( \mathcal{A}_x = \{ s \in \mathcal{H}_x \mid s \mathcal{F}_x = 0 \} \) for \( x \in X \). If \( \theta : (X, \mathcal{H}) \rightarrow (X', \mathcal{H}') \) is a holomorphic map
(i.e., a morphism of ringed spaces) from \( (X, \mathcal{H}) \) to another complex space \( (X', \mathcal{H}') \),
then \( R^0 \theta(\mathcal{F}) \) denotes the zeroth direct image of \( \mathcal{F} \) under \( \theta \). If \( f \in \Gamma(X, \mathcal{H}) \) and
\( x \in X \), we say that \( f \) vanished at \( x \) if \( f_x \) is not a unit in \( \mathcal{H}_x \).

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I. Absolute gap-sheaves.

Definition 1. Suppose \( \mathcal{F} \) is an analytic subsheaf of an analytic sheaf \( \mathcal{G} \) on a complex space \((X, \mathcal{O}_X)\) and \( \rho \) is a nonnegative integer. The \( \rho \)-th gap-sheaf of \( \mathcal{F} \) in \( \mathcal{G} \), denoted by \( \mathcal{G}^{(\rho)}\mathcal{F} \), is the analytic subsheaf of \( \mathcal{G} \) defined as follows: For \( x \in X \), \( s \in (\mathcal{G}^{(\rho)}\mathcal{F})_x \) if and only if there exist an open neighborhood \( U \) of \( x \) in \( X \), a subvariety \( A \) in \( U \) of dimension \( \leq \rho \), and \( t \in \Gamma(U, \mathcal{G}) \) such that \( t_x = s \) and \( t_y \in \mathcal{G}_y \) for \( y \in U - A \).

Denote the set \( \{x \in X \mid \mathcal{F}_x \neq (\mathcal{G}^{(\rho)}\mathcal{F})_x\} \) by \( E\rho^p(\mathcal{F}, \mathcal{G}) \).

Remark. When \( \mathcal{G} \) and \( \mathcal{F} \) are both coherent, then \( x \in E\rho^p(\mathcal{F}, \mathcal{G}) \) if and only if \( \mathcal{G}_x \) as an \( \mathcal{O}_x \)-submodule of \( \mathcal{F}_x \) has an associated prime ideal of dimension \( \leq \rho \) (Theorem 4, [12]). \( E\rho^p(\mathcal{F}, \mathcal{G}) = \emptyset \) means that for every \( x \in X \), \( \mathcal{G}_x \) as an \( \mathcal{O}_x \)-submodule of \( \mathcal{F}_x \) has no associated prime ideal of dimension \( \leq \rho \).

Definition 2. Suppose \( \mathcal{F} \) is an analytic subsheaf of an analytic sheaf \( \mathcal{G} \) on a complex space \((X, \mathcal{O}_X)\) and \( A \) is a subvariety of \( X \). Then the gap-sheaf of \( \mathcal{F} \) in \( \mathcal{G} \) with respect to \( A \), denoted by \( \mathcal{G}[A]\mathcal{F} \), is defined as follows: For \( x \in X \), \( s \in (\mathcal{G}[A]\mathcal{F})_x \) if and only if there exist an open neighborhood \( U \) of \( x \) in \( X \) and \( t \in \Gamma(U, \mathcal{G}) \) such that \( t_x = s \) and \( t_y \in \mathcal{G}_y \) for \( y \in U - A \).

Proposition 1. Suppose \( \mathcal{F} \) is a coherent analytic subsheaf of a coherent analytic sheaf \( \mathcal{G} \) on a complex space \((X, \mathcal{O}_X)\) and \( \rho \) is a nonnegative integer. Then \( \mathcal{G}^{(\rho)}\mathcal{F} \) is coherent and \( E\rho^p(\mathcal{F}, \mathcal{G}) \) is a subvariety of dimension \( \leq \rho \) in \( X \).

Proof. See Theorem 3 [12]. This can also be derived easily from Satz 3 [13]. Q.E.D.

Proposition 2. Suppose \( \mathcal{F} \) is a coherent analytic subsheaf of a coherent analytic sheaf \( \mathcal{G} \) on a complex space \((X, \mathcal{O}_X)\) and \( A \) is a subvariety of \( X \). Then \( \mathcal{G}[A]\mathcal{F} \) is coherent.

Proof. See Theorem 1 [12]. This can also be derived easily from [13, Satz 9]. Q.E.D.

Definition 3. Suppose \( \mathcal{G} \) is an analytic sheaf on a complex space \( X \) and \( \rho \) is a nonnegative integer. The \( \rho \)-th absolute gap-sheaf of \( \mathcal{G} \), denoted by \( \mathcal{G}^{(\rho)} \), is the analytic sheaf on \( X \) defined by the following presheaf: Suppose \( U \subset V \) are open subsets of \( X \). Then

\[
\mathcal{G}^{(\rho)}(U) = \lim\limits_{\text{ind}} \lim\limits_{\text{dir}} \Gamma(U - A, \mathcal{G}),
\]

where \( \mathcal{A}(U) \) is the directed set of all analytic subvarieties in \( U \) of dimension \( \leq \rho \) directed under inclusion. \( \mathcal{G}^{(\rho)}(V) \to \mathcal{G}^{(\rho)}(U) \) is induced by restriction.

Remarks. (i) \( \mathcal{G}^{(\rho)} = (\mathcal{G}^{(0)}\mathcal{G})^{(\rho)} \), where 0 is the zero-subsheaf of \( \mathcal{G} \).

(ii) There is a natural sheaf-homomorphism \( \mu: \mathcal{G} \to \mathcal{G}^{(\rho)} \). The kernel of \( \mu \) is \( \mathcal{G}^{(0)}\mathcal{G} \). When \( E\rho^p(0, \mathcal{G}) = \emptyset \), \( \mu \) is injective and we can regard \( \mathcal{G} \) as a subsheaf of \( \mathcal{G}^{(\rho)} \). In this case we denote the set \( \{x \in X \mid \mathcal{G}_x \neq (\mathcal{G}^{(\rho)})_x\} \) by \( E\rho^p(\mathcal{G}) \).
Lemma 1. Suppose \( F \) is a coherent analytic sheaf on a reduced complex space \((X, \mathcal{O})\) of pure dimension \( n \). Suppose \( 0 \leq \rho \leq n - 2 \). If \( E^{n-1}(0, F) = \mathcal{O} \), then \( \mathcal{F}^{[\rho]} \) is coherent and \( E^\rho(F) \) is a subvariety of dimension \( \leq \rho \).

Proof. Let \( \pi: (\bar{X}, \bar{\mathcal{O}}) \to (X, \mathcal{O}) \) be the normalization of \((X, \mathcal{O})\). Let \( \mathcal{F} \) be the inverse image of \( F \) under \( \pi \) (Definition 8, [6]). Let \( \mathcal{F}^e \) be the torsion-subsheaf of \( \mathcal{F} \) and \( \mathcal{G} = \mathcal{F}^e / \mathcal{F} \). Let \( Y = \text{Supp} \ F \). \( \mathcal{F} \) and \( \mathcal{G} \) are both coherent and \( \mathcal{G} \) is torsion-free (Proposition 6, [1]). \( \dim Y \leq n - 1 \) (Proposition 7, [1]). We claim that

\[ \mathcal{G}^{[\rho]} \text{ is coherent and } E^\rho(\mathcal{G}) \text{ is a subvariety of dimension } \leq \rho \text{ in } \bar{X}. \]

Take \( x \in \bar{X} \). On some open neighborhood \( U \) of \( x \) in \( \bar{X} \) \( \mathcal{G} \) can be regarded as a coherent subsheaf of \( \mathcal{O}^{[\rho]} \) for some \( \rho \) (Proposition 9, [1]). It is clear that \( \mathcal{G}^{[\rho]} \) is isomorphic to \( \mathcal{G}^{[\rho]}_{\eta, \mathcal{O}} \) on \( U \) and \( E^\rho(\mathcal{G}, \mathcal{O}^{[\rho]}) \cap U = E^\rho(\mathcal{G}) \cap U \). (1) follows from Proposition 1.

Let \( \mathcal{F}^* = R^0\pi(\mathcal{F}) \), \( \mathcal{G}^* = R^0\pi(\mathcal{G}) \), and \( (\mathcal{G}^{[\rho]})^* = R^0\pi(\mathcal{G}^{[\rho]}) \). Let \( \alpha: \mathcal{F}^* \to \mathcal{G}^* \) and \( \beta: \mathcal{F}^* \to (\mathcal{G}^{[\rho]})^* \) be induced respectively by the quotient map \( \mathcal{F} \to \mathcal{G} \) and the inclusion map \( \mathcal{G} \to \mathcal{G}^{[\rho]} \). We have a natural sheaf-homomorphism \( \lambda: \mathcal{F} \to \mathcal{F}^* \) (Satz 7(b), [6]). Let \( Z \) be the set of all singular points of \( X \). Let \( \mathcal{H} \) be the kernel of \( \alpha \lambda \). Then \( \text{Supp } \mathcal{H} \subseteq Z \cup \pi(Y) \). Since \( E^{n-1}(0, F) = \mathcal{O} \) and \( \dim \text{Supp } \mathcal{H} \leq n - 1 \), \( \mathcal{H} = 0 \). \( \gamma = \beta \alpha \lambda: \mathcal{F} \to (\mathcal{G}^{[\rho]})^* \) is injective. It is easily seen that \(((\mathcal{G}^{[\rho]})^*)_{[\rho]} = (\mathcal{G}^{[\rho]})^* \). \( \gamma \) induces a sheaf-monomorphism \( \gamma_1: \mathcal{F}^{[\rho]} \to (\mathcal{G}^{[\rho]})^* \). \( \mathcal{F}^{[\rho]} \approx \mathcal{F}^{[\rho]}_{[\rho]} = \gamma(\mathcal{F})_{[\rho]}(\mathcal{G}^{[\rho]})_1 \), and \( E^\rho(\mathcal{F}) = E^\rho(\gamma(\mathcal{F}), (\mathcal{G}^{[\rho]})^*) \). Since by Proposition 1 \( \gamma(\mathcal{F})_{[\rho]}(\mathcal{G}^{[\rho]})_1 \) is coherent and \( E^\rho(\gamma(\mathcal{F}), (\mathcal{G}^{[\rho]})^*) \) is a subvariety of dimension \( \leq \rho \) in \( X \), the Lemma follows. Q.E.D.

Lemma 2. Suppose \( F \) is a coherent analytic sheaf on a complex space \((X, \mathcal{H})\). Suppose \( x \in X \) and \( f \in \mathcal{H}_x \) such that for every nonnegative integer \( \rho \) either \( x \notin E^\rho(0, F) \) or \( f \) does not vanish identically on any branch-germ of \( E^\rho(0, F) \) at \( x \). Then \( f \) is not a zero-divisor for \( \mathcal{H}_x \).

Proof. Suppose the contrary. Then there exist \( s \in \Gamma(U, F) \) and \( g \in \Gamma(U, \mathcal{H}) \) for some open neighborhood \( U \) of \( x \) such that \( g_x = f \), \( g_s = 0 \), and \( s_x \neq 0 \). Let \( Z = \text{Supp } s \) and \( \dim Z_x = \rho \). By shrinking \( U \), we can assume that \( \dim Z = \rho \). Hence \( Z \subseteq E^\rho(0, F) \). Since \( \dim E^\rho(0, F) \leq \rho \), the union \( Z_0 \) of all \( \rho \)-dimensional branches of \( Z \) is equal to the union of some \( \rho \)-dimensional branches of \( E^\rho(0, F) \cap U \). By assumption \( g \) does not vanish identically on \( Z_0 \). For some \( y \in Z_0 \), \( g_y \) is a unit in \( \mathcal{H}_y \), \( s_y = 0 \), contradicting that \( Z = \text{Supp } s \). Q.E.D.

Lemma 3. Suppose \( F \) is a coherent analytic sheaf on a complex space \( X \) and \( \rho \) is a nonnegative integer. If \( E^\rho(0, F) = \mathcal{O} \), then for any nonnegative integer \( \sigma \) either \( E^\rho(0, F) = \mathcal{O} \) or every branch of \( E^\rho(0, F) \) has dimension \( > \rho \).

Proof. Suppose \( Y \) is a nonempty \( m \)-dimensional branch of \( E^\rho(0, F) \) for some nonnegative integer \( m \) such that \( m \leq \rho \). Take a Stein open subset \( U \) of \( X \) such that \( U \cap E^\rho(0, F) = U \cap Y \neq \emptyset \). Take \( x \in U \cap Y \). Since \( (0_{[\sigma]}\mathcal{F})_x \neq 0 \), there exists
Let \( s \in \Gamma(U, 0_{(3)}) \) such that \( s_x \neq 0 \). Supp \( s \subseteq E^0(0, \mathcal{F}) \cap U = U \cap Y \). dim Supp \( s \leq \rho \). Hence \( s \in \Gamma(U, 0_{(3)}) \). \( x \in E^0(0, \mathcal{F}) \), contradicting that \( E^0(0, \mathcal{F}) = \emptyset \). Q.E.D.

**Lemma 4.** Suppose \( \mathcal{F}_i, 1 \leq i \leq 3 \), are coherent analytic sheaves on a complex space \( (X, \mathcal{H}) \) and \( \rho \) is a nonnegative integer such that \( E^0(0, \mathcal{F}_i) = 0 \) for \( 1 \leq i \leq 3 \). Suppose \( 0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0 \) is an exact sequence of sheaf-homomorphisms. If \( (\mathcal{F}_1)^{[\rho]} \) is coherent and \( E^0(\mathcal{F}_1) \) is a subvariety of dimension \( \leq \rho \) for \( i = 1, 3 \), then \( (\mathcal{F}_2)^{[\rho]} \) is coherent and \( E^0(\mathcal{F}_2) \) is a subvariety of dimension \( \leq \rho \).

**Proof.** Let \( X_i = E^0(\mathcal{F}_i), i = 1, 3 \). The problem is local in nature. Take \( x_0 \in X \) and take an open Stein neighborhood \( U \) of \( x_0 \) in \( X \). \( \mathcal{F}_i \) is a coherent analytic subsheaf of \( (\mathcal{F}_i)^{[\rho]} \), \( i = 1, 3 \). Let \( \mathcal{A}_i = (\mathcal{F}_i)^{[\rho]} \), \( i = 1, 3 \). \( E(\mathcal{A}_i, \mathcal{H}) = X_i \), \( i = 1, 3 \). Let \( \mathcal{F}_i \) be the ideal-sheaf for \( X_i \), \( i = 1, 3 \). By Hilbert Nullstellensatz, after shrinking \( U \), we can find a natural number \( m \) such that \( \mathcal{A}_i^{[m]} \subseteq \mathcal{A}_i \) on \( U \). \( i = 1, 3 \). By Lemma 3 for any nonnegative integer \( \sigma \) every nonempty branch of \( E^0(0, \mathcal{F}_2) \) has dimension \( > \rho \). Since \( \dim X_i \leq \rho \), \( i = 1, 3 \), we can choose \( f \in \Gamma(U, \mathcal{F}_1^{[\sigma]} \cap \mathcal{F}_3^{[\sigma]}) \) such that \( f \neq 0 \) does not vanish identically on any nonempty branch-germ of \( E^0(0, \mathcal{F}_2) \) at \( x_0 \) for any nonnegative integer \( \sigma \). By Lemma 2 \( f_{x_0} \) is not a zero-divisor for \( (\mathcal{F}_2)^{[\rho]} \). Let \( \mathcal{H} \) be the kernel of the sheaf-homomorphism \( \alpha : \mathcal{F}_2 \rightarrow \mathcal{F}_2 \) on \( U \) defined by multiplication by \( f \). Then \( \mathcal{H}_{x_0} = 0 \). By shrinking \( U \), we can assume that \( \mathcal{H} = 0 \) on \( U \). \( \alpha \) induces a sheaf-monomorphism \( \beta : (\mathcal{F}_2)^{[\rho]} \rightarrow (\mathcal{F}_2)^{[\rho]} \). Let \( \gamma = \beta \circ \alpha \). We claim that \( \gamma((\mathcal{F}_2)^{[\rho]}) \subseteq \mathcal{F}_2 \) is \( U \). Take \( s \in ((\mathcal{F}_2)^{[\rho]})_x \) for some \( x \in U \). \( s \) is defined by some \( t \in \Gamma(W - A, \mathcal{F}_3) \), where \( W \) is an open neighborhood of \( x \) in \( U \) and \( A \) is a subvariety of dimension \( \leq \rho \) in \( W \). \( \gamma(t) \in \Gamma(W - A, \mathcal{F}_3) \) defines an element \( a \) of \( ((\mathcal{F}_3)^{[\sigma]})_x \). \( f_s a \in ((\mathcal{F}_3)^{[\sigma]})_x \). By shrinking \( W \) we can find \( u \in \Gamma(W, \mathcal{F}_3) \) such that \( u \) agrees with \( f_s \) on \( W - A \) and we can find \( v \in \Gamma(W, \mathcal{F}_2) \) such that \( v = u \). \( v - f_s \) defines an element \( b \) of \( ((\mathcal{F}_2)^{[\rho]})_x \). \( f_s b \in ((\mathcal{F}_2)^{[\rho]})_x \). By shrinking \( W \) we can find \( w \in \Gamma(W, \mathcal{F}_2) \) such that \( w \) agrees with \( f(v - f_s) \) on \( W - A \). \( f(v - f_s) - w \in ((\mathcal{F}_2)^{[\rho]})_x \). Hence \( \gamma((\mathcal{F}_2)^{[\rho]}) \subseteq \mathcal{F}_2 \). It is easily seen that \( \gamma((\mathcal{F}_2)^{[\rho]}) = \gamma((\mathcal{F}_2)^{[\rho]} \cap U) \). \( U = E^0(\mathcal{F}_2) \cap U = E^0((\gamma(\mathcal{F}_2)^{[\rho]} \cap U). \) Q.E.D.

**Lemma 5.** Suppose \( \mathcal{F} \) is a coherent analytic sheaf on a complex space \( (X, \mathcal{H}) \) of pure dimension \( n \) and \( 0 \leq \rho \leq n - 2 \). If \( E^{n-1}(0, \mathcal{F}) = \emptyset \), then \( (\mathcal{F})^{[\rho]} \) is coherent and \( E^0(\mathcal{F}) \) is a subvariety of dimension \( \leq \rho \).

**Proof.** Let \( \mathcal{N} \) be the subsheaf of all nilpotent elements of \( \mathcal{H} \) and \( \mathcal{O} = \mathcal{H} \mid \mathcal{N} \). Since the lemma is local in nature, we can suppose that for some nonnegative integer \( k \) \( \mathcal{N}^k = 0 \). For \( 0 \leq l \leq k \) define \( (\mathcal{F})^{(l)} \) inductively as follows: \( (\mathcal{F})^{(0)} = \mathcal{F} \) and, for \( 1 \leq l \leq k \), \( (\mathcal{F})^{(l)} = (\mathcal{N} \mathcal{F}^{(l-1)})^{(l-1)} \). Let \( Y = \bigcup_{l=1}^k E^{n-1}(\mathcal{N} \mathcal{F}^{(l-1)}, \mathcal{F}^{(l-1)}) \). \( Y \) is a subvariety of dimension \( \leq n - 1 \). On \( X - Y \) \( (\mathcal{F})^{(l)} = (\mathcal{F})^{(l)} \) for \( 1 \leq l \leq k \). Hence \( (\mathcal{F})^{(0)} = 0 \) on \( X - Y \). Since \( (\mathcal{F})^{(k)} \subseteq \mathcal{F} \) and \( E^{n-1}(0, \mathcal{F}) = \emptyset \). \( (\mathcal{F})^{(k)} = 0 \). From the definition of \( (\mathcal{F})^{(l)} \) we see that \( E^{n-1}(\mathcal{F})^{(l)}, \mathcal{F}^{(l)} = \emptyset \) for \( 1 \leq l \leq k \). Hence \( E^{n-1}(0, \mathcal{F}^{(l)}) = \emptyset \). \( (\mathcal{F})^{(l)} = \emptyset \) for \( 1 \leq l \leq k \). \( E^{n-1}(0, \mathcal{F}) = \emptyset \) implies that \( E^{n-1}(0, \mathcal{F}^{(l)}) = \emptyset \) for \( 0 \leq l \leq k \). Since \( (\mathcal{F}^{(l)}) \subseteq (\mathcal{F})^{(l)} \), \( (\mathcal{F})^{(l)} \) can be regarded as a coherent analytic sheaf on \( (X, \mathcal{O}) \).
1 \leq l \leq k. By Lemma 1 \( (\mathcal{F}^{(l-1)}/\mathcal{F}^{(l)})^{(l)} \) is coherent and \( E^l(\mathcal{F}^{(l-1)}/\mathcal{F}^{(l)}) \) is a subvariety of dimension \( \leq \rho \). Since \( \mathcal{F}^{(k)} = 0 \), from Lemma 4 and the exact sequences

\[ 0 \rightarrow \mathcal{F}^{(l)} \rightarrow \mathcal{F}^{(l-1)} \rightarrow \mathcal{F}^{(l-1)}/\mathcal{F}^{(l)} \rightarrow 0, \ 1 \leq l \leq k \]

we conclude by backward induction on \( l \) that \( (\mathcal{F}^{(l)})^{(l)} \) is coherent and \( E^l(\mathcal{F}^{(l)}) \) is a subvariety of dimension \( \leq \rho \) for \( 0 \leq l \leq k \). The Lemma follows from \( \mathcal{F} = \mathcal{F}^{(l)} \). Q.E.D.

Lemma 6. Suppose \( \mathcal{F} \) is a coherent analytic sheaf on a complex space \((X, \mathcal{H})\) and \( \rho \) is a nonnegative integer. Let \( Y \) be the union of \((\rho + 1)\)-dimensional branches of \( E^{\rho+1}(0, \mathcal{F}) \). Then for \( x \in Y \) \( (\mathcal{F}^{(l)})_x \) is not finitely generated over \( \mathcal{H}_x \).

Proof. We can assume that \( Y \neq \emptyset \). Let \( \mathcal{I} = \mathcal{F}/0_{(l)}\mathcal{F} \). Since \( E^0(0, \mathcal{I}) = \emptyset \), by Lemma 3 and Proposition 1 every branch of \( E^{\rho+1}(0, \mathcal{I}) \) is \((\rho+1)\)-dimensional. Since \( \mathcal{I} \) agrees with \( \mathcal{F} \) on \( X - E^0(0, \mathcal{F}) \), \( E^{\rho+1}(0, \mathcal{I}) - E^0(0, \mathcal{F}) = E^{\rho+1}(0, \mathcal{F}) - E^0(0, \mathcal{F}) \). Dim \( E^0(0, \mathcal{F}) \leq \rho \) implies that \( E^{\rho+1}(0, \mathcal{F}) = Y \).

Fix \( x \in Y \). Suppose \( (\mathcal{F}^{(l)})_x \) is finitely generated over \( \mathcal{H}_x \). Let \( \mathcal{I} = 0_{(l+1)}\mathcal{F} \). Since \( E^0(0, \mathcal{I}) \subset E^0(0, \mathcal{F}) = \emptyset \), \( \mathcal{I} \subset \mathcal{I}^{(l)} \subset \mathcal{I}^{(l)} = \mathcal{F}^{(l)} \). Since \( \text{Supp } \mathcal{I} = E^{\rho+1}(0, \mathcal{F}) \), \( (\mathcal{F}^{(l)})_x \) is a nonzero finitely generated \( \mathcal{H}_x \)-module. Let \( (\mathcal{F}^{(l)})_x \) be generated by \( s_1, \ldots, s_n \in (\mathcal{F}^{(l)})_x \). For some open neighborhood \( U \) of \( x \) in \( X \) and for some subvariety \( A \) of dimension \( \leq \rho \) in \( U \), \( s_i \) is induced by \( t_i \in \Gamma(U - A, \mathcal{F}) \), \( 1 \leq i \leq m \). By shrinking \( U \), we can choose \( f \in \Gamma(U, \mathcal{F}) \) such that \( W = Z(f) \cap Y \) is a subvariety of dimension \( \rho \) in \( U \) and \( x \in Z(f) \), where \( Z(f) = \{ y \in U | f_y \neq 0 \} \) is not a unit in \( \mathcal{H}_y \). There exists a unique \( g \in \Gamma(U - Z(f), \mathcal{H}) \) such that \( gf = 1 \) on \( U - Z(f) \). For \( 1 \leq i \leq m \) define \( u_i \in \Gamma(U - (A \cup W), \mathcal{F}) \) by \( (u_i)_y = 0 \) for \( y \in U - Y \) and \( (u_i)_y = (g)_y \) for \( y \in Y \cap (U - (A \cup W)) \). \( u_i \) induces \( v_i \in (\mathcal{F}^{(l)})_x \), \( 1 \leq i \leq m \). \( f_i v_i = s_i \), \( 1 \leq i \leq m \). For some \( a_{ij} \in \mathcal{H}_x \), \( v_i = \sum_{j=1}^m a_{ij} s_j \), \( 1 \leq i \leq m \). \( s_i = f_i v_i = \sum_{j=1}^m a_{ij} s_j \), \( 1 \leq i \leq m \). \( (\mathcal{F}^{(l)})_x \) is a nonzero \( \mathcal{H}_x \)-module. Since \( f_x \) is not a unit in \( \mathcal{H}_x \), by [8, (4.1)] we have \( (\mathcal{F}^{(l)})_x = 0 \) (contradiction). Q.E.D.

Theorem 1. Suppose \( \mathcal{F} \) is a coherent analytic sheaf on a complex space \((X, \mathcal{H})\) and \( \rho \) is a nonnegative integer. Then \( \mathcal{F}^{(l)} \) is coherent if and only if \( \text{dim } E^{\rho+1}(0, \mathcal{F}) < \rho + 1 \). In that case \( E^0(\mathcal{F}/0_{(l)}\mathcal{F}) \) is a subvariety of dimension \( \leq \rho \).

Proof. It follows from Lemma 6 that, if \( \mathcal{F}^{(l)} \) is coherent, then \( \text{dim } E^{\rho+1}(0, \mathcal{F}) < \rho + 1 \). We are going to prove that \( \mathcal{F}^{(l)} \) is coherent and \( E^0(\mathcal{F}/0_{(l)}\mathcal{F}) \) is a subvariety of dimension \( \leq \rho \) in \( X \). Since \( \mathcal{F} \) agrees with \( \mathcal{F}/0_{(l)}\mathcal{F} \) on \( X - E^0(0, \mathcal{F}) \), \( E^{\rho+1}(0, \mathcal{F}/0_{(l)}\mathcal{F}) \) is contained in the subvariety \( E^0(0, \mathcal{F}) \cup E^{\rho+1}(0, \mathcal{F}) \) of dimension \( \leq \rho \). \( E^0(0, \mathcal{F}/0_{(l)}\mathcal{F}) = \emptyset \) implies \( E^{\rho+1}(0, \mathcal{F}/0_{(l)}\mathcal{F}) = \emptyset \) by Lemma 3. Since \( \mathcal{F}^{(l)} = (\mathcal{F}/0_{(l)}\mathcal{F})^{(l)} \), by replacing \( \mathcal{F} \) by \( \mathcal{F}/0_{(l)}\mathcal{F} \), we can assume that \( E^{\rho+1}(0, \mathcal{F}) = \emptyset \). Since the problem is local in nature, we can suppose that \( X \) is of finite dimension \( n \). If \( n < \rho + 2 \), \( E^{\rho+1}(0, \mathcal{F}) = \emptyset \) implies that \( \mathcal{F} = 0 \), \( \mathcal{F}^{(l)} = 0 \) is coherent and \( E^l(\mathcal{F}) = \emptyset \). So we can assume that \( n \geq \rho + 2 \). For \( \rho + 1 \leq m \leq n \) let \( \mathcal{G}^{(m)} = 0_{(m)}\mathcal{F} \). \( \mathcal{G}^{(m+1)} = 0 \), because \( E^{\rho+1}(0, \mathcal{F}) = \emptyset \). For \( \rho + 2 \leq m \leq n \) let \( X_m = \text{Supp } (\mathcal{G}^{(m)})/\mathcal{G}^{(m-1)} \). \( X_m \) is the union of all \( m \)-dimensional branches of \( E^m(0, \mathcal{F}) \),
\( \rho + 2 \leq m \leq n \). Let \( E^{m-1}(0, \mathcal{O}(m)/\mathcal{O}(m-1)) = \mathcal{O} \) for \( \rho + 2 \leq m \leq n \). For \( \rho + 2 \leq m \leq n \) let \( \mathcal{A}(m) \) be the annihilator-ideal-sheaf for \( \mathcal{O}(m)/\mathcal{O}(m-1) \). Then \( (\mathcal{O}(m)/\mathcal{O}(m-1)) | X_m \) can be regarded as a coherent analytic sheaf on the complex space \( (X_m, (\mathcal{A}/\mathcal{A}(m)) | X_m) \) which is either empty or of pure dimension \( m, \rho + 2 \leq m \leq n \). By Lemma 5

\[
(\mathcal{O}(m)/\mathcal{O}(m-1))^{[\rho]} \simeq ((\mathcal{O}(m)/\mathcal{O}(m-1)) | X_m)^{[\rho]} 
\]

is coherent and \( E^\rho(\mathcal{O}(m)/\mathcal{O}(m-1)) = E^\rho((\mathcal{O}(m)/\mathcal{O}(m-1)) | X_m) \) is a subvariety of dimension \( \leq \rho, \rho + 2 \leq m \leq n \). Since \( \mathcal{G}^{(\rho + 2)} = \mathcal{G}^{(\rho + 1)} \), from Lemma 4 and the exact sequences \( 0 \to \mathcal{G}(m-1) \to \mathcal{G}(m) \to \mathcal{G}(m)/\mathcal{G}(m-1) \to 0, \rho + 3 \leq m \leq n \), we conclude by induction on \( m \) that \( \mathcal{G}(m)^{[\rho]} \) is coherent and \( E^\rho(\mathcal{G}(m)) \) is a subvariety of dimension \( \leq \rho, \rho + 2 \leq m \leq n \). Theorem follows from \( \mathcal{F} = \mathcal{G}(n) \). Q.E.D.

**Corollary 1.** Suppose \( \mathcal{F} \) is a coherent analytic sheaf on a complex space \( X \), \( \rho \) is a nonnegative integer, and \( x \in X \). \( \mathcal{F}^{[\rho]} \) is coherent at \( x \) if and only if \( x \) does not belong to a \( (\rho + 1) \)-dimensional branch of \( E^{\rho + 1}(0, \mathcal{F}) \). Hence the set of points where \( \mathcal{F}^{[\rho]} \) is not coherent is either empty or it is a subvariety of pure dimension \( \rho + 1 \).

**Remark.** Under the assumption of Corollary 1 to Theorem 2 \( x \) does not belong to a \( (\rho + 1) \)-dimensional branch of \( E^{\rho + 1}(0, \mathcal{F}) \) if and only if the zero submodule of \( \ mathcal{F} x \) has no associated prime ideal of dimension \( \rho + 1 \) [12, Theorem 4]. This gives us an algebraic criterion for the coherence of \( \mathcal{F}^{[\rho]} \) at \( x \).

**Corollary 2.** Suppose \( \mathcal{F} \) is a coherent analytic sheaf on a complex space \( X \) and \( \rho \) is a nonnegative integer. Let \( \mu: \mathcal{F} \to \mathcal{F}^{[\rho]} \) be the natural sheaf-homomorphism. Then \( Z = \{ x \in X | \mu_x \) is not surjective} \) is a subvariety of dimension \( \leq \rho + 1 \).

**Proof.** Let \( Y \) be the union of all \( (\rho + 1) \)-dimensional branches of \( E^{\rho + 1}(0, \mathcal{F}) \). By Lemma 6 \( Y \subseteq Z \). Since \( \mathcal{F}^{[\rho]} \) agrees with \( \mathcal{F}^{[0]} \) on \( X - Y \), \( Z - Y = E^\rho(\mathcal{F})^{[0]}(X - Y) \) is a subvariety of dimension \( \leq \rho + 1 \). Q.E.D.

**Remark.** Corollary 2 to Theorem 1 can be stated alternatively in the following way: The set of points where we cannot always remove closed singularities contained in subvarieties of dimension \( \rho \) for local sections of a coherent analytic sheaf \( \mathcal{F} \) satisfying \( E^\rho(0, \mathcal{F}) = \mathcal{O} \) is a subvariety of dimension \( \leq \rho + 1 \).

The weaker statement that this set of points is contained in a subvariety of dimension \( \leq \rho + 1 \) is an easy consequence of Satz III, [9] and Satz 5, [10].

**II. Extension of coherent sheaves.** Suppose \( S \) is a subvariety of a complex space \( X \) and \( \mathcal{F} \) is a coherent analytic sheaf on \( X - S \). \( \mathcal{F} \) is said to satisfy \( (*)_x, S \) if for every \( x \in S \) there exists some open neighborhood \( U \) of \( x \) in \( X \) such that \( \Gamma(U - S, \mathcal{F}) \) generates \( \mathcal{F} \) on \( U - S \).

**Lemma 7.** Suppose \( S \) is a subvariety of codimension \( \geq 2 \) in a reduced complex space \( (X, \emptyset) \) of pure dimension \( n \). Let \( \theta: X - S \to X \) be the inclusion map. Suppose \( \mathcal{F} \)
is a coherent analytic sheaf on $X-S$ such that $E^{s-1}(0, \mathcal{F}) = \emptyset$. If $\mathcal{F}$ satisfies $(*)_{x, s}$, then $R^0\theta(\mathcal{F})$ is coherent.

**Proof.** Let $\pi: (\overline{X}, \overline{\mathcal{O}}) \to (X, \mathcal{O})$ be the normalization of $(X, \mathcal{O})$. Let $\overline{S} = \pi^{-1}(S)$ and $\pi' = \pi|_{(\overline{X} - \overline{S})}$. Let $\vartheta: \overline{X} - \overline{S} \to \overline{X}$ be the inclusion map. Let $\mathcal{F}$ be the inverse image of $\mathcal{F}$ under $\pi'$. Let $\mathcal{G}$ be the torsion-subsheaf of $\vartheta_*(\mathcal{F})$, $\mathcal{G} = \mathcal{F}/\mathcal{G}$, and $Y = \text{Supp} \mathcal{F}$. Since $\mathcal{F}$ satisfies $(*)_{x, S}$, $\mathcal{G}$ satisfies $(*)_{x, S}$. This implies that $\mathcal{G}$ satisfies $(*)_{x, S}$. By Theorem 1, $R^0\theta(\mathcal{G})$ is coherent on $\overline{X}$. Let $\mathcal{G}^* = R^0\pi'\mathcal{F}$ and $\mathcal{G}_* = R^0\pi(R^0\theta(\mathcal{G}))$. $\mathcal{G}^*$ is coherent on $X$. Let the sheaf-homomorphism $\alpha: \mathcal{G}^* \to \mathcal{F}^*$ on $X-S$ be induced by the quotient map $\mathcal{F} \to \mathcal{G}$. We have a natural sheaf-homomorphism $\lambda: \mathcal{F} \to \mathcal{F}^*$. Let $Z$ be the set of all singular points on $X$. Let $\mathcal{K}$ be the kernel of $\alpha$. Then $\text{Supp} \mathcal{K} \subseteq Z \cup \pi(Y)$. Since $E^{s-1}(0, \mathcal{F}) = \emptyset$ and $\dim \text{Supp} \mathcal{K} \leq n-1$, $\mathcal{K} = 0$. $\alpha\lambda$ is injective. Since $R^0\theta(\mathcal{G}^* | X-S) = \mathcal{G}^*$, $\alpha\lambda$ induces a sheaf-homomorphism $\beta: R^0\theta(\mathcal{F}) \to \mathcal{G}^*$. Take $x \in S$. There exists an open neighborhood $U$ of $x$ in $X$ such that $\Gamma(U-S, \mathcal{F})$ generates $\mathcal{F}$ on $U-S$. Let $\eta$ be the torsion-subsheaf of $\vartheta_*(\mathcal{F})$, $\eta = \mathcal{F}/\eta$. By Proposition 2 $\eta$ is coherent. Hence $R^0\theta(\mathcal{F})$ is coherent. Q.E.D.

**Lemma 8.** Suppose $S$ is a subvariety in a complex space $(X, \mathcal{H})$. Let $\theta: X-S \to X$ be the inclusion map. Suppose $\mathcal{F}_i$, $1 \leq i \leq 3$, are coherent analytic sheaves on $X-S$ such that $R^0\theta(\mathcal{F}_3)$ is coherent. Suppose $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is an exact sequence of sheaf-homomorphisms on $X-S$. If $\mathcal{F}_2$ satisfies $(*)_{x, s}$ and $E^{s+1}(0, \mathcal{F}_2) = \emptyset$, then $R^0\theta(\mathcal{F}_2)$ is coherent.

**Proof.** Take $x \in S$. There is an open neighborhood $U$ of $x$ in $X$ such that $\Gamma(U-S, \mathcal{F}_2)$ generates $\mathcal{F}_2$ on $U-S$. Let $W$ be a Stein open neighborhood of $x$ in $U$. We claim that $\Gamma(W-S, \mathcal{F}_2)$ generates $\mathcal{F}_2$ on $W-S$. There exist $s_i \in \Gamma(U-S, \mathcal{F}_2)$, $1 \leq i \leq m$, generating $(\mathcal{F}_2)_y$. Define a sheaf-homomorphism $\varphi: \mathcal{H}^m \to \mathcal{F}_2$ on $U-S$ by $\varphi(a_1, \ldots, a_m) = \sum_{i=1}^m \alpha_i(s_i)z$ for $a_1, \ldots, a_m \in \mathcal{H}_2$ and $z \in U-S$. $\eta(s_i)$ can be extended uniquely to an element of $\Gamma(U, R^0\theta(\mathcal{F}_3))$, $1 \leq i \leq m$. There is a unique sheaf-homomorphism $\psi: \mathcal{H}^n \to R^0\theta(\mathcal{F}_3)$ on $U$ which agrees with $\eta \varphi$ on $U-S$. Let $\mathcal{K}$ be the kernel of $\psi$. $\mathcal{K}$ is coherent. There exist $u_i \in \Gamma(W, \mathcal{H})$, $1 \leq i \leq n$, generating $\mathcal{K}_y$. Let $v_i = \psi(u_i | (W-S))$, $1 \leq i \leq n$. Then $v_i \in \Gamma(W-S, \mathcal{F}_2)$, $1 \leq i \leq n$, and $(\mathcal{F}_2)_y$ is generated by $v_1, \ldots, v_n$. Q.E.D.

**Lemma 9.** Suppose $S$ is a subvariety of dimension $p$ in a complex space $X$. Let $\theta: X-S \to X$ be the inclusion map. Suppose $\mathcal{F}_i$, $1 \leq i \leq 3$, are coherent analytic sheaves on $X-S$ such that $R^0\theta(\mathcal{F}_j)$ is coherent for $j=1, 3$. Suppose $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is an exact sequence of sheaf-homomorphisms on $X-S$. If $\mathcal{F}_2$ satisfies $(*)_{x, s}$ and $E^{s+1}(0, \mathcal{F}_2) = \emptyset$, then $R^0\theta(\mathcal{F}_2)$ is coherent.

**Proof.** Take $x \in S$. We need only prove that $R^0\theta(\mathcal{F}_2)$ is coherent at $x$. There is a Stein open neighborhood $U$ of $x$ in $X$ such that $\Gamma(U-S, \mathcal{F}_2)$ generates $\mathcal{F}_2$ on $U-S$.
The exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ induces the exact sequence $0 \to R^0\theta(\mathcal{F}_1) \to R^0\theta(\mathcal{F}_2) \to R^0\theta(\mathcal{F}_3)$. For $s \in \Gamma(U - S, \mathcal{F}_2)$ let $\tilde{s} \in \Gamma(U, R^0\theta(\mathcal{F}_2))$ be the unique extension of $s$ and let $\eta(\tilde{s})$. Let $\mathcal{I}$ be the subsheaf of $R^0\theta(\mathcal{F}_2)$ on $U$ generated by $\{ \tilde{s} \mid s \in \Gamma(U - S, \mathcal{F}_2) \}$ and $\mathcal{I}$ be the subsheaf of $R^0\theta(\mathcal{F}_3)$ on $U$ generated by $\{ s \mid s \in \Gamma(U - S, \mathcal{F}_3) \}$.

$\eta^* (\mathcal{I}) = \mathcal{I}$. Since $R^0\theta(\mathcal{F}_2)$ is coherent, $\mathcal{I}$ being generated by global sections is coherent. Since $R^0\theta(\mathcal{F}_1)$ is coherent and $U$ is Stein, on $U$ $R^0\theta(\mathcal{F}_1)$ is generated by $\Gamma(U, R^0\theta(\mathcal{F}_1)) \approx \Gamma(U - S, \mathcal{F}_1) \subset \Gamma(U - S, \mathcal{F}_2)$. $R^0\theta(\mathcal{F}_1) \subset \mathcal{I}$. We have an exact sequence $0 \to R^0\theta(\mathcal{F}_1) \to \mathcal{I} \to \mathcal{I} \to 0$, where $\eta$ is induced by $\eta$ and $\tilde{s}$ is the inclusion map. Since $R^0\theta(\mathcal{F}_1)$ and $\mathcal{I}$ are both coherent, $\mathcal{I}$ is coherent. $E^{p+1}(0, \mathcal{I}) \subset E^{p+1}(0, \mathcal{F}_2) = \emptyset$. By Theorem $1\beta(\mathcal{I})$ is coherent. Since $\dim S = \rho$, $R^0\theta(\mathcal{F}_2) = \mathcal{I}$. The inclusion map $\mathcal{F}_2 \to \mathcal{I}$ on $U - S$ induces on $U$ a sheaf-monomorphism $\beta: R^0\theta(\mathcal{F}_2) \to \mathcal{I}$. $\beta(R^0\theta(\mathcal{F}_2)) = \mathcal{I}[\mathcal{S}]$. Since $\mathcal{I}[\mathcal{S}]$ is coherent by Proposition 2, $R^0\theta(\mathcal{F}_3)$ is coherent on $U$. Q.E.D.

**Lemma 10.** Suppose $S$ is a subvariety of codimension $\geq 2$ in a complex space $(X, \mathcal{K})$ of pure dimension $n$. Let $\theta: X - S \to X$ be the inclusion map. Suppose $\mathcal{F}$ is a coherent analytic sheaf on $X - S$. If $\mathcal{F}$ satisfies $(*), E^{n-1}(0, \mathcal{F}) = \emptyset$, then $R^0\theta(\mathcal{F})$ is coherent on $X$.

**Proof.** Let $\mathcal{K}$ be the subsheaf of all nilpotent elements of $\mathcal{K}$ and $\mathcal{F} = \mathcal{K}$. Since the Lemma is local in nature, we can suppose that for some nonnegative integer $k \mathcal{K}^k = 0$. For $0 \leq l \leq k$ define coherent analytic sheaves $\mathcal{F}^l(\mathcal{K})$ on $X - S$ inductively as follows: $\mathcal{F}^0(\mathcal{K}) = \mathcal{F}$ and, for $1 \leq l \leq k$, $\mathcal{F}^l(\mathcal{K}) = (\mathcal{K} \mathcal{F}^l(\mathcal{K}))_{(n-1)\mathcal{F}^{l-1}}$. Let

$$Y = \bigcup_{l=1}^{k} E^{n-1}(X \mathcal{F}^l(\mathcal{K}), \mathcal{F}^{l-1})$$

$Y$ is a subvariety in $X - S$ of dimension $\leq n - 1$. On $X - (S \cup Y)$, $\mathcal{F}^l = \mathcal{K} \mathcal{F}^l(\mathcal{K})$ for $1 \leq l \leq k$. Hence $\mathcal{F}^k = 0$ on $X - (S \cup Y)$. Since $\mathcal{F}^k(\mathcal{K}) \subset \mathcal{F}$ and $E^{n-1}(0, \mathcal{F}) = \emptyset$, $\mathcal{F}^k = 0$ on $X - S$. From the definition of $\mathcal{F}^l$ we see that $E^{n-1}(\mathcal{F}^l, \mathcal{F}^{l-1}) = \emptyset$ for $1 \leq l \leq k$. Hence $E^{n-1}(0, \mathcal{F}^{l-1} \mathcal{F}^l(\mathcal{K}) = 0$ for $1 \leq l \leq k$. $E^{n-1}(0, \mathcal{F}^l) = \emptyset$ implies that $E^{n-1}(0, \mathcal{F}^l) = \emptyset$ for $1 \leq l \leq k$. Since $\mathcal{F}^l \subset \mathcal{F}^l(\mathcal{K})$, $\mathcal{F}^{l-1} \mathcal{F}^l(\mathcal{K})$ can be regarded as a coherent analytic sheaf on $(X - S, \mathcal{O} | (X - S))$, $1 \leq l \leq k$.

Set $\mathcal{F}^{k+1} = 0$. We are going to prove $(2)$, for $0 \leq l \leq k$ by induction on $l$:

$(2)_l$ $\mathcal{F}^l$ satisfies $(*)_{X,S}$ and $R^0\theta(\mathcal{F}^l(\mathcal{K}), \mathcal{F}^{l+1})$ is coherent.

Since $\mathcal{F}^0 = \mathcal{F}$, $\mathcal{F}^l$ satisfies $(*)_{X,S}$. $\mathcal{F}^l(\mathcal{K}) / \mathcal{F}^{l+1}$ satisfies $(*)_{X,S}$. By Lemma 7

$$R^0\theta(\mathcal{F}^l(\mathcal{K}), \mathcal{F}^{l+1})$$

is coherent. $(2)_0$ is true. Suppose for some $0 \leq m < k$ $(2)_m$ is true. By Lemma 8 and
the exact sequence $0 \rightarrow \mathcal{F}(m + 1) \rightarrow \mathcal{F}(m) \rightarrow \mathcal{F}(m)/\mathcal{F}(m + 1) \rightarrow 0$, we conclude that $\mathcal{F}(m + 1)$ satisfies $(*)_{X,S}$. Hence $\mathcal{F}(m + 1)/\mathcal{F}(m + 2)$ satisfies $(*)_{X,S}$. By Lemma 7

$$R^0\theta(\mathcal{F}(m + 1)/\mathcal{F}(m + 2))$$

is coherent. $(2)_{m+1}$ is true. Hence $(2)_l$ holds for $0 \leq l \leq k$.

Now we are going to prove $(3)_l$ for $0 \leq l \leq k$ by backward induction on $l$:

$$(3)_l$$

Since $\mathcal{F}^{(k)}=0$, $(3)_k$ is true. Suppose $(3)_m$ is true for some $0 < m \leq k$. From $(2)_{m-1}$, $(3)_m$, Lemma 10 and the exact sequence $0 \rightarrow \mathcal{F}(m) \rightarrow \mathcal{F}(m - 1) \rightarrow \mathcal{F}(m - 1)/\mathcal{F}(m) \rightarrow 0$, we conclude that $(3)_{m-1}$ is true. Hence $(3)_l$ holds for $0 \leq l \leq k$. The Lemma follows from $(3)_0$. Q.E.D.

**Lemma 11.** Suppose $S$ is a subvariety of dimension $p$ in a complex space $(X, \mathcal{H})$. Suppose $\mathcal{F}$ is a coherent analytic sheaf on $X-S$ such that $\operatorname{Supp} \mathcal{F}$ is a subvariety of pure dimension $n > p$ and $E^{n-1}(0, \mathcal{F})=0$. Then there exists a complex subspace $(Y, \mathcal{K})$ of pure dimension $n$ in $(X, \mathcal{H})$ such that $Y-S=\operatorname{Supp} \mathcal{F}$ and $\mathcal{F}|(Y-S)$ can be regarded as a coherent analytic sheaf on $(Y-S, \mathcal{K}|(Y-S))$.

**Proof.** By [7, V.D.5] the topological closure $Y$ of $\operatorname{Supp} \mathcal{F}$ in $X$ is a subvariety of pure dimension $n$. Let $Y=\bigcup_{\alpha \in A} Y_{\alpha}$ be the decomposition into irreducible branches. Let $\mathcal{J}_{\alpha}$ be the ideal-sheaf for $Y_{\alpha}, \alpha \in A$. Choose $x_\alpha \in Y_{\alpha}-(S \cup (\bigcup_{\beta \in B, \beta \neq \alpha} Y_{\beta}))$. Let $\mathcal{J}$ be the annihilator-ideal-sheaf for $\mathcal{F}$. Then $E(\mathcal{J}, \mathcal{H}|(X-S))=Y-S$. By Hilbert Nullstellensatz, there exists a natural number $m_\alpha$ such that $(\mathcal{J}^{m_\alpha})_{\alpha} \subset \mathcal{J}_{\alpha}, \alpha \in A$. Let $\mathcal{J}=\prod_{\alpha \in A} \mathcal{J}^{m_\alpha}$. Then $\mathcal{J}$ is coherent and $(\mathcal{J}^{m})_{\alpha}=0$ for $\alpha \in A$. $\operatorname{Supp} \mathcal{F}$ is a subvariety of dimension $<n$ in $X-S$. $E^{n-1}(0, \mathcal{F})=0$ implies that $\mathcal{J}^{n}=0$. Set $\mathcal{K}=(\mathcal{J}^{n})|Y$. Then $(Y, \mathcal{K})$ satisfies the requirements. Q.E.D.

**Theorem 2.** Suppose $S$ is a subvariety of dimension $p$ in a complex space $(X, \mathcal{H})$. Let $\theta: X-S \rightarrow X$ be the inclusion map. Suppose $\mathcal{F}$ is a coherent analytic sheaf on $X-S$ such that $E^{p+1}(0, \mathcal{F})=0$ or equivalently for every $x \in X-S$ the zero $\mathcal{H}_x$-submodule of $\mathcal{F}_x$ has no associated prime ideal of dimension $\leq p+1$. Then the following conditions are equivalent:

(i) $R^0\theta(\mathcal{F})$ is coherent.

(ii) There exists a coherent analytic sheaf on $X$ which extends $\mathcal{F}$.

(iii) $\mathcal{F}$ satisfies $(*)_{X,S}$.

**Proof.** It is clear that (i) implies (ii) and (ii) implies (iii). We need only prove that (iii) implies (i). Suppose $\mathcal{F}$ satisfies $(*)_{X,S}$. We are going to prove that $R^0\theta(\mathcal{F})$ is coherent. Since the problem is local in nature, we can suppose that $X$ is of finite dimension $n$. If $n < p+2$, then $E^{p+1}(0, \mathcal{F})=0$ implies that $\mathcal{F}=0$. $R^0\theta(\mathcal{F})=0$ is coherent. So we can assume that $n \geq p+2$. For $p+1 \leq m \leq n$ let $\mathcal{F}^{(m)}=0_{m|\mathcal{F}}$. $\mathcal{F}^{(p+1)}=0$, because $E^{p+1}(0, \mathcal{F})=0$. For $p+2 \leq m \leq n$ let $X_m=\operatorname{Supp} \mathcal{F}^{(m)}/\mathcal{F}^{(m-1)}$. Then $X_m$ is the union of all $m$-dimensional branches of $E^m(0, \mathcal{F}), p+2 \leq m \leq n$. Coherence of $\mathcal{F}^{(m)}$ implies coherence of $\mathcal{F}^{(p+1)}$. Coherence of $\mathcal{F}^{(p+1)}$ implies coherence of $\mathcal{F}^{(p+2)}$.
Let $\theta_m: Y_m - S \to Y_m$ be the inclusion map $p+2 \leq m \leq n$. By Lemma 11 there exists a complex subspace $(Y_m, \mathcal{H}_m)$ of pure dimension $m$ in $(X, \mathcal{H})$ such that $Y_m - S = X_m$ and $(\mathcal{G}^{(m)}/\mathcal{G}^{(m-1)})(Y_m - S)$ can be regarded as a coherent analytic sheaf on $(Y_m - S, \mathcal{H}_m|_{(Y_m - S)})$, $p+2 \leq m \leq n$.

We are going to prove $(4)_m$ for $p+2 \leq m \leq n$ by backward induction on $m$:

$(4)_m$ $\mathcal{G}^{(m)}$ satisfies (*)$_{X,S}$ and $R^0\theta(\mathcal{G}^{(m)}/\mathcal{G}^{(m-1)})$ is coherent.

Since $\mathcal{G}^{(n)} = \mathcal{F}$, $\mathcal{G}^{(n)}$ satisfies (*)$_{X,S}$. $(\mathcal{G}^{(n)}/\mathcal{G}^{(n-1)})(Y_n - S)$ satisfies (*)$_{Y_n,Y_n,S}$. By Lemma 10 $R^0\theta(\mathcal{G}^{(n)}/\mathcal{G}^{(n-1)})(Y_n - S)$ is coherent. $(4)_n$ is true. Suppose for some $p+2 < q \leq n$, $(4)_q$ is true. From Lemma 8, $(4)_q$, and the exact sequence $0 \to \mathcal{G}^{(q-1)} \to \mathcal{G}^{(q)} \to \mathcal{G}^{(q)}/\mathcal{G}^{(q-1)} \to 0$ we conclude that $R^0\theta(\mathcal{G}^{(q-1)})$ satisfies (*)$_{X,S}$. $(\mathcal{G}^{(q-1)}/\mathcal{G}^{(q-2)})(Y_{q-1} - S)$ satisfies (*)$_{Y_{q-1},Y_{q-1},S}$. By Lemma 10 $R^0\theta(\mathcal{G}^{(q-1)}/\mathcal{G}^{(q-2)})(Y_{q-1} - S)$ is coherent. $(4)_{q-1}$ is true. Hence $(4)_m$ holds for $p+2 \leq m \leq n$.

Now we are going to prove $(5)_m$ for $p+1 \leq m \leq n$ by induction on $m$:

$(5)_m$ $R^0(\mathcal{G}^{(m)})$ is coherent.

Since $\mathcal{G}^{(p+1)} = 0$, $(5)_{p+1}$ is true. Suppose $(5)_q$ is true for some $p+1 \leq q < n$. From $(4)_{q+1}$, $(5)_q$, Lemma 9, and the exact sequence $0 \to \mathcal{G}^{(q)} \to \mathcal{G}^{(q+1)} \to \mathcal{G}^{(q+1)}/\mathcal{G}^{(q)} \to 0$ we conclude that $R^0\theta(\mathcal{G}^{(q+1)})$ is coherent. $(5)_{q+1}$ is true. Hence $(5)_m$ holds for $p+1 \leq m \leq n$. Since $\mathcal{G}^{(m)} = \mathcal{F}$, $(5)_m$ implies that $R^0\theta(\mathcal{F})$ is coherent. Q.E.D.

**Corollary.** Suppose $S$ is a subvariety of dimension $p$ in a complex space $(X, \mathcal{H})$ and $\theta: X - S \to X$ is the inclusion map. Suppose $\mathcal{F}$ is a coherent analytic sheaf on $X - S$ such that the homological codimension (p. 358, [9]) of the $\mathcal{H}_x$-module $\mathcal{I}_x \geq p+2$ for $x \in X$. Then the following conditions are equivalent:

(i) $R^0\theta(\mathcal{F})$ is coherent.

(ii) There exists a coherent analytic sheaf on $X$ which extends $\mathcal{F}$.

(iii) $\mathcal{F}$ satisfies (*)$_{X,S}$.

**Proof.** Follows from Theorem 2 and Satz I [9]. Q.E.D.

**Remark.** [14, (4.1)] is a special case of the Corollary to Theorem 2.

**III. Extensions of global sections of coherent sheaves.**

**Definition 4.** Suppose $\rho$ is a natural number. A real-valued function $v$ on a complex space $X$ is said to be *-strongly $\rho$-convex at $x \in X$ if there exist a nowhere degenerate holomorphic map $\varphi$ from some open neighborhood $U$ of $x$ in $X$ to an open subset $D$ of $\mathbb{C}^n$ and a real-valued $C^2$ function $\delta$ on $D$ such that $v = \overline{\varphi} \delta$ on $U$ and at every point in $D$ the Hermitian matrix $(\partial^2 v/\partial z_i \partial \overline{z}_j)_{1 \leq i,j \leq n}$ has at least $n - \rho + 1$ positive eigenvalues.

**Definition 5.** Suppose $\rho$ is a natural number. An open subset $D$ of a complex space $X$ is said to be *-strongly $\rho$-concave at $x \in X$ if there is a *-strongly $\rho$-convex
function $v$ on some open neighborhood $U$ of $x$ in $X$ such that $D \cap U = \{ y \in U \mid v(y) > v(x) \}$.

**Lemma 12.** Suppose $\mathcal{F}$ is a coherent analytic sheaf on a reduced complex space $(X, \mathcal{O})$ of pure dimension $n$ such that $E^{n-1}(0, \mathcal{F}) = \emptyset$. Suppose $1 \leq p < n$, $x \in X$, and $D$ is an open subset of $X$ which is $*$-strongly $p$-concave at $x$. Then there exist an open neighborhood $U$ of $x$ in $X$, a subvariety $V$ of dimension $< p$ in $U$, and a natural number $m$ satisfying the following: If for some open neighborhood $W$ of $x$ in $U$ $f \in \Gamma(W, \mathcal{O})$ vanishes identically on $V \cap W$ and $s \in \Gamma(W \cap D, \mathcal{F})$, then $f^m s \mid W' \cap D$ can be extended to an element of $\Gamma(W', \mathcal{F})$ for some open neighborhood $W'$ of $x$ in $W$.

**Proof.** Let $\pi: (X, \mathcal{O}) \to (X, \mathcal{O})$ be the normalization of $(X, \mathcal{O})$. Let $\mathcal{F}$ be the inverse image of $\mathcal{F}$ under $\pi$, $\mathcal{F}^\prime$ is the torsion subsheaf of $\mathcal{F}$, and $\mathcal{S} = \mathcal{F}^\prime / \mathcal{F}$. Let $\pi^{-1}(x) = \{ y_1, \ldots, y_k \}$. For every $1 \leq i \leq k$ there exists a sheaf-monomorphism $\alpha_i: \mathcal{G} \to \mathcal{F}$ on some open neighborhood $U_i$ of $y_i$ in $X$. By shrinking $U_i$, $1 \leq i \leq k$, we can suppose that $U_i \cap U_j = \emptyset$ for $i \neq j$. There is an open neighborhood $U^\ast$ of $x$ in $X$ such that $\pi^{-1}(U^\ast) = \bigsqcup_{i=1}^k U_i$. Define a coherent analytic sheaf $\mathcal{F}$ on $\pi^{-1}(U^\ast)$ by setting $\mathcal{F} = \mathcal{G}$ on some open neighborhood $U_i$ of $y_i$ in $X$. By shrinking $U_i$, $1 \leq i \leq k$, we can suppose that $U_i \cap U_j = \emptyset$ for $i \neq j$. There is an open neighborhood $U^\ast$ of $x$ in $X$ such that $\pi^{-1}(U^\ast) = \bigsqcup_{i=1}^k U_i$. Define a coherent analytic sheaf $\mathcal{F}$ on $\pi^{-1}(U^\ast)$ by setting $\mathcal{F} = \mathcal{G}$ on some open neighborhood $U_i$ of $y_i$ in $X$. By shrinking $U_i$, $1 \leq i \leq k$, we can suppose that $U_i \cap U_j = \emptyset$ for $i \neq j$. There is an open neighborhood $U^\ast$ of $x$ in $X$ such that $\pi^{-1}(U^\ast) = \bigsqcup_{i=1}^k U_i$. Define a coherent analytic sheaf $\mathcal{F}$ on $\pi^{-1}(U^\ast)$ by setting $\mathcal{F} = \mathcal{G}$ on some open neighborhood $U_i$ of $y_i$ in $X$. By shrinking $U_i$, $1 \leq i \leq k$, we can suppose that $U_i \cap U_j = \emptyset$ for $i \neq j$. There is an open neighborhood $U^\ast$ of $x$ in $X$ such that $\pi^{-1}(U^\ast) = \bigsqcup_{i=1}^k U_i$. Define a coherent analytic sheaf $\mathcal{F}$ on $\pi^{-1}(U^\ast)$ by setting $\mathcal{F} = \mathcal{G}$ on some open neighborhood $U_i$ of $y_i$ in $X$. By shrinking $U_i$, $1 \leq i \leq k$, we can suppose that $U_i \cap U_j = \emptyset$ for $i \neq j$. 

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vanishes identically on $V \cap W$ and $s \in \Gamma(W \cap D, \mathcal{F})$, then $f^m s|W' \cap D$ can be extended to an element of $\Gamma(W', \mathcal{F})$ for some open neighborhood $W'$ of $x$ in $W$.

**Proof.** Let $\mathcal{H}$ be the subsheaf of all nilpotent elements of $\mathcal{H}$ and $\mathcal{O} = \mathcal{H}/\mathcal{H}$. Since the Lemma is local in nature, we can suppose that $\mathcal{H}^k = 0$ for some natural number $k$. For $0 \leq l \leq k$ define $\mathcal{F}^{(l)}$ inductively as follows:

$$\mathcal{F}^{(0)} = \mathcal{F}, \quad \text{and, for } 1 \leq l \leq k, \quad \mathcal{F}^{(l)} = (\mathcal{H}^{(l-1)}|_{\mathcal{O}^{(l-1)}}).$$

As in the Proof of Lemma 5, we have the following:

$$\mathcal{F}^{(k)} = 0; \quad E^{n-1}(0, \mathcal{F}^{(l-1)}|\mathcal{F}^{(l)}) = \emptyset \quad \text{for } 1 \leq l \leq k;$$

and $\mathcal{F}^{(0)} = \mathcal{F}^{(l)}/\mathcal{F}^{(l+1)}$, $0 \leq l \leq k-1$, can be regarded as a coherent analytic sheaf on the reduced complex space $(X, \mathcal{O})$. By Lemma 12 for $0 \leq l \leq k-1$ we have a subvariety $V_l$ of dimension $< \rho$ in some open neighborhood $U_l$ of $x$ in $X$ and a natural number $p_l$ satisfying the following: If for some open neighborhood $W$ of $x$ in $U_l$ $f \in \Gamma(W, \mathcal{O})$ vanishes identically on $V_l \cap W$ and $s \in \Gamma(W \cap D, \mathcal{F}^{(l)})$, then $f^m s|W' \cap D$ can be extended to an element of $\Gamma(W', \mathcal{F}^{(l)})$ for some open neighborhood $W'$ of $x$ in $W$.

Let $U = \bigcap_{l=0}^{k-1} U_l$ and $V = \bigcup_{l=0}^{k-1} (V_l \cap U)$. Let $m_l = \sum_{i=1}^{l-1} p_i$, $0 \leq l \leq k-1$. By considering the exact sequences $0 \to \mathcal{F}^{(l+1)} \to \mathcal{F}^{(l)} \to \mathcal{F}^{(l)} \to 0$, $0 \leq l \leq k-1$, and by backward induction on $l$, we conclude the following for $0 \leq l \leq k-1$: If $f \in \Gamma(W, \mathcal{H})$ vanishes identically on $W \cap V$ and $s \in \Gamma(W \cap D, \mathcal{F}^{(l)})$ for some open neighborhood $W$ of $x$ in $U$, then $f^m s|W' \cap D$ can be extended to an element of $\Gamma(W', \mathcal{F}^{(l)})$ for some open neighborhood $W'$ of $x$ in $W$. Hence $U$, $V$, and $m = m_0$ satisfy the requirements. Q.E.D.

**Lemma 14.** Suppose $\mathcal{F}$ is a coherent analytic sheaf on a complex space $(X, \mathcal{O})$ and $\rho$ is a natural number such that $E^p(0, \mathcal{F}) = \emptyset$. Suppose $x \in X$ and $D$ is an open subset of $X$ which is *-strongly $\rho$-concave at $x$. Then there exist an open neighborhood $U$ of $x$ in $X$, a subvariety $V$ of dimension $< \rho$ in $U$, and a natural number $m$ satisfying the following: For some open neighborhood $W$ of $x$ in $U$, $f \in \Gamma(W, \mathcal{H})$ vanishes identically on $W \cap V$ and $s \in \Gamma(W \cap D, \mathcal{F})$, then $f^m s|W' \cap D$ can be extended to an element of $\Gamma(W', \mathcal{F})$ for some neighborhood $W'$ of $x$ in $W$.

**Proof.** Since the problem is local in nature, we can suppose that $X$ is of finite dimension $n$. If $n \leq \rho$, $E^p(0, \mathcal{F}) = \emptyset$ implies that $\mathcal{F} = 0$ and what is to be proved is trivial. So we can suppose that $n > \rho$. Define $\mathcal{G}^{(l)} = 0_{k \leq n}$ for $\rho \leq k \leq n$. $\mathcal{G}^{(0)} = 0$. For $\rho < k \leq n$ let $X_k = \text{Supp} \mathcal{G}^{(l)}/\mathcal{G}^{(k-1)}$ and let $\mathcal{A}^{(k)}$ be the annihilator-ideal-sheaf for $\mathcal{G}^{(k)}/\mathcal{G}^{(k-1)}$. For $\rho < k \leq n$ $X_k$ is empty or of pure dimension $k$, $E^{k-1}(0, \mathcal{G}^{(k)}/\mathcal{G}^{(k-1)}) = \emptyset$, and $(\mathcal{G}^{(k)}/\mathcal{G}^{(k-1)})|X_k$ can be regarded as a coherent analytic sheaf on the complex space $(X_k, (\mathcal{H}^{(k)}/\mathcal{A}^{(k)})|X_k)$. By Lemma 13, for $\rho < k \leq n$, if $x \in X_k$, there exist a subvariety $V_k$ of dimension $< \rho$ in some open neighborhood $U_k$ of $x$ in $X_k$ and a
natural number $p_k$ satisfying the following: If for some open neighborhood $W$ of $x$ in $U_k f \in \Gamma(W, (\mathcal{H}^k/\mathcal{G}^{(k)}|X_k)$ vanishes identically on $W \cap V_k$ and

$$s \in \Gamma(W \cap D, \mathcal{G}^{(k)}/\mathcal{G}^{(k-1)}),$$

then $f^{p_k}s|W' \cap D$ can be extended to an element of $\Gamma(W', \mathcal{G}^{(k')}/\mathcal{G}^{(k-1)})$ for some open neighborhood $W'$ of $x$ in $W$. For $p < k \leq n$, if $x \in X_k$, choose an open neighborhood $U_k$ of $x$ in $X$ such that $U_k \cap X_k = U_k$; and, if $x \notin X_k$, let $U_k = X$, $V_k = \emptyset$, and $p_k = 1$.

Let $U = \bigcap_{k=0}^n U_k$ and $V = \bigcup_{k=0}^n (U \cap V_k)$. Set $m_k = \sum_{i=0}^k p_i$. By considering the exact sequences $0 \to \mathcal{G}^{(k)} \to \mathcal{G}^{(k+1)} \to \mathcal{G}^{(k+1)}/\mathcal{G}^{(k)} \to 0$, $\rho \leq k \leq n-1$, and by induction on $k$, we conclude the following for $p < k \leq n$: If for some open neighborhood $W$ of $x$ in $U f \in \Gamma(W, \mathcal{H})$ vanishes on $V \cap W$ and $s \in \Gamma(W \cap D, \mathcal{G}^{(k)})$, then $f^{p_k}s|W' \cap D$ can be extended to an element of $\Gamma(W', \mathcal{G}^{(k)})$ for some open neighborhood $W'$ of $x$ in $W$. The Lemma follows from $F = \mathcal{G}^{(n)}$ and $m = m_n$.

Q.E.D.

**Theorem 3 (Local Extension).** Suppose $\mathcal{F}$ is a coherent analytic sheaf on a complex space $(X, \mathcal{H})$ and $\rho$ is a natural number such that $\mathcal{F} = \mathcal{F}^{(\rho-1)}$. Suppose $x \in X$ and $D$ is an open subset of $X$ which is *-strongly $\rho$-concave at $x$. Then the following is satisfied: If $s \in \Gamma(W \cap D, \mathcal{F})$ for some open neighborhood $W$ of $x$ in $X$, then $s|W' \cap D$ can be extended to an element $t$ of $\Gamma(W', \mathcal{F})$ for some open neighborhood $W'$ of $x$ in $W$ and $t_x$ is uniquely determined.

**Proof.** Since $\mathcal{F} = \mathcal{F}^{(\rho-1)}$, by Theorem 1, and the definition of $\mathcal{F}^{(\rho-1)}$, $E^\sigma(0, \mathcal{F}) = \emptyset$. There exist an open neighborhood $U$ of $x$ in $X$, a subvariety $V$ of dimension $< \rho$ in $U$, and a natural number $m$ satisfying the requirements of Lemma 14. By Lemma 3 every branch of $E^\sigma(0, \mathcal{F})$ has dimension $> \rho$ for every nonnegative integer $\sigma$. By shrinking $U$ we can assume that there is $f \in \Gamma(U, \mathcal{H})$ such that $f$ vanishes identically on $V$ and $f$ does not vanish identically on any branch of $E^\sigma(0, \mathcal{F}) \cap U$ for any nonnegative integer $\sigma$. By Lemma 2 the sheaf-homomorphism $\alpha: \mathcal{F} \to \mathcal{F}$ on $U$ defined by multiplication by $f^m$ is injective.

Suppose $s \in \Gamma(W \cap D, \mathcal{F})$. For some open neighborhood $W'$ of $x$ in $W$ $a(s)|W' \cap D$ can be extended to an element $t \in \Gamma(W', \mathcal{F})$. $Z = \{y \in W' | ~ i_y \notin a(\mathcal{F})_y\}$ is a subvariety in $W'$. Since $D$ is *-strongly $\rho$-concave at $x$ and $Z \cap D = \emptyset$, either $x \notin Z$ or $\dim Z_x < \rho$. By shrinking $W'$, we can assume that either $Z \cap W' = \emptyset$ or $\dim Z < \rho$. $i \in \Gamma(W', a(\mathcal{F})_{(\rho-1)})$. $\mathcal{F} = \mathcal{F}^{(\rho-1)}$ implies that $a(\mathcal{F})_{(\rho-1)} = a(\mathcal{F})$. Hence $i \in \Gamma(W', a(\mathcal{F}))$. $t = a^{-1}(i) \in \Gamma(W', \mathcal{F})$ extends $s|W' \cap D$.

Suppose for some other open neighborhood $W''$ of $x$ in $W$ there is $t' \in \Gamma(W'', \mathcal{F})$ extending $s|W'' \cap D$. We are going to prove that $t'_x = t_x$. By shrinking both $W'$ and $W''$, we can assume that $W'' = W''$. $Y = \{y \in W' | ~ t'_y \neq t_y\}$ is a subvariety in $W'$. Since $D$ is *-strongly $\rho$-concave at $x$ and $Y \cap D = \emptyset$, either $x \notin Y$ or $t'_x - t_x \in (0_{(\rho-1)})_x = 0$. Q.E.D.
**Theorem 4 (Global Extension).** Suppose \( \rho \) is a natural number and \( v \) is a \(*\)-strongly \( \rho \)-convex function on a complex space \( X \) such that \( \{ x \in X \mid \lambda < v(x) < \mu \} \) is relatively compact in \( X \) for any two real numbers \( \lambda < \mu \). Suppose \( \mathcal{F} \) is a coherent analytic sheaf on \( X \) satisfying \( \mathcal{F} = \mathcal{F}^{(0,-1)} \). Then for \( \lambda \in \mathbb{R} \) every section of \( \mathcal{F} \) on \( X_\lambda = \{ x \in X \mid v(x) > \lambda \} \) is uniquely extendible to a section of \( \mathcal{F} \) on \( X \).

**Proof.** We can assume that \( X \) as a topological space is connected. Since \( E^0(0, \mathcal{F}) = \emptyset \), we can assume that every branch of \( X \) has dimension \( > \rho \). Fix \( \lambda_0 \in \mathbb{R} \) and \( s \in \Gamma(X_{\lambda_0}, \mathcal{F}) \). We can assume that \( X_{\lambda_0} \neq \emptyset \). Let \( \Lambda = \{ \lambda \in \mathbb{R} \mid \lambda \leq \lambda_0 \} \) and \( s \) can be extended to \( s_\lambda \in \Gamma(X_\lambda, \mathcal{F}) \). Clearly, if \( \lambda \in \Lambda \) and \( \lambda < \mu \), then \( \mu \in \Lambda \). We are going to prove:

\[
(6) \quad \text{If } \lambda \in \Lambda \text{ and } s_\lambda, s'_\lambda \in \Gamma(X_\lambda, \mathcal{F}) \text{ both extend } s, \text{ then } s_\lambda = s'_\lambda.
\]

Suppose the contrary. Then \( Z = \{ x \in X_\lambda \mid (s_\lambda)_x \neq (s'_\lambda)_x \} \) is a nonempty subvariety in \( X_\lambda \). Let \( Z_0 \) be a branch of \( Z \). Take \( x^* \in Z_0 \) and let \( \lambda^* = v(x^*) \). Let \( \xi = \sup \{ v(x) \mid x \in Z_0 \} \). Since \( Z \cap X_{\lambda_0} = \emptyset \), \( \xi \) is the supremum of \( v \) on the compact set \( Z_0 \cap \{ x \in X \mid \lambda^* \leq v(x) \leq \lambda_0 \} \). \( \xi = v(y) \) for some \( y \in Z_0 \). Since \( X_\xi \) is \(*\)-strongly \( \rho \)-concave at \( y \) and \( Z_0 \cap X_\xi = \emptyset \), we have \( \text{dim}(Z_0)_y < \rho \). Since \( Z_0 \) is irreducible, \( \text{dim} Z_0 < \rho \). Hence \( \text{dim} Z < \rho \). \( s_\lambda - s'_\lambda \in \Gamma(X_\lambda, 0^{(\rho-1)}) \). (6) follows from \( 0^{(\rho-1)} = 0 \).

For \( \lambda \in \Lambda \) denote the unique element of \( \Gamma(X_\lambda, \mathcal{F}) \) which extends \( s \) by \( s_\lambda \). To finish the proof, we need only prove that \( \Lambda \) has no lower bound, because in that case \( \Lambda = \{ \lambda \in \mathbb{R} \mid \lambda \leq \lambda_0 \} \) and by (6) \( s^* \in \Gamma(X, \mathcal{F}) \) defined by \( s^*|_{X_\lambda} = s_\lambda \) for \( \lambda \in \Lambda \) extends \( s \). Suppose the contrary. Then \( \eta = \inf \Lambda \) exists and is finite. Since \( X \) is connected, this implies that \( X_\eta \) is not closed in \( X \). By Theorem 3 for every \( x \) in the boundary \( \partial X_\eta \) of \( X_\eta \) there exists an open neighborhood \( U_x \) of \( x \) in \( X \) such that \( s_n \) can be extended to \( t_n \in \Gamma(U_x \cup X_\eta, \mathcal{F}) \). For \( x, x' \in \partial X_\eta \) let \( Y_{(x,x')} = \{ z \in U_x \cap U_{x'} \mid (t_n)_z \neq (t_n')_z \} \). Since \( 0^{(\rho-1)} = \emptyset \), \( Y_{(x,x')} \) is either empty or every branch of \( Y_{(x,x')} \) has dimension \( \geq \rho \). Since \( X_\eta \) is \(*\)-strongly \( \rho \)-concave at every one of its boundary points,

\[
(7) \quad Y_{(x,x')} \cap \partial X_\eta = \emptyset \quad \text{for } x, x' \in \partial X_\eta.
\]

Since \( \partial X_\eta \) is compact we can choose \( x_1, \ldots, x_k \in \partial X_\eta \) such that \( \partial X_\eta \subset \bigcup_{i=1}^k U_{x_i} \). For \( 1 \leq i \leq k \) choose a relatively compact open neighborhood \( W_i \) of \( x_i \) in \( U_{x_i} \) such that \( \partial X_\eta \subset \bigcup_{i=1}^k W_i \). Let \( W_i^- \) be the closure of \( W_i \) in \( X \). (7) implies that we can choose an open neighborhood \( W \) of \( \partial X_\eta \) in \( \bigcup_{i=1}^k W_i \) such that \( W \) does not intersect the closed set \( \bigcup_{1 \leq i,j \leq k, i \neq j} Y_{(x_i,x_j)} \cap W_i^- \cap W_j^- \). For some \( \lambda < \eta \), \( X_\lambda \subset W \cup X_\eta \) because of Proposition 2.7 of [3]. Define \( t \in \Gamma(X_\lambda, \mathcal{F}) \) by setting \( t = s_{(x_1)} \) on \( (U_{x_1} \cup X_\eta) \cap X_\lambda \). \( t \) extends \( s \), contradicting \( \lambda \notin \Lambda \).

Uniqueness follows from (6). Q.E.D.

**Remarks.** (i) Theorem 3 generalizes the Theorem on p. 279 of [4] and Theorem 4 generalizes Corollary 5.2 of [4] because of Theorem 4.3 of [4]. Theorems 3 and 4...
here have the advantage that, if $\mathcal{F}$ does not satisfy $\mathcal{F} = \mathcal{F}^{|0,-1}$, we can always construct the coherent analytic sheaf $\mathcal{G} = (\mathcal{F}^{|0,-1})^{|0,-1}$ which satisfies $\mathcal{G} = \mathcal{G}^{|0,-1}$.

(ii) Suppose $\mathcal{F}$ is a coherent analytic sheaf on a complex space $(X, \mathcal{H})$ and $x \in X$. The condition $\mathcal{F}_x = (\mathcal{F}^{|0})_x$ is equivalent to the condition $\text{codh } \mathcal{F}_x \geq 2$. It can be proved in the following way: If $\mathcal{F}_x = (\mathcal{F}^{|0})_x$, then $E^0(0, \mathcal{F}) = \emptyset$ and by Lemmas 2 and 3 we can find $f \in \Gamma(U, \mathcal{F})$ for some open neighborhood $U$ of $x$ in $X$ such that $f_x$ is not a unit of $\mathcal{H}_x$ and $f_x$ is not a zero-divisor for $\mathcal{F}_x$. By shrinking $U$, we can assume that $f_y$ is not a zero-divisor for $\mathcal{F}_y$ for $y \in U$. Suppose $x \in E^0(f\mathcal{F}, \mathcal{F}|U)$. By shrinking $U$, we can find $g \in \Gamma(U, \mathcal{F})$ such that $g_y \in (f\mathcal{F})_y$ for $y \in U - \{x\}$ and $g_x \notin (f\mathcal{F})_x$. Then $h \in \Gamma(U, \mathcal{F}^{|0})$ defined by $g_y = f_y h_y$ for $y \in U - \{x\}$ does not satisfy $h_x \in \mathcal{F}_x$. This is a contradiction. Hence $x \notin E^0(f\mathcal{F}, \mathcal{F}|U)$. By Lemmas 2 and 3 we can find $s \in \mathcal{H}_x$ which vanishes at $x$ and is not a zero-divisor for $(f\mathcal{F})_x$. $\text{codh } \mathcal{F}_x \geq 2$. On the other hand $\text{codh } \mathcal{F}_x \geq 2$ implies $\mathcal{F}_x = (\mathcal{F}^{|0})_x$ by Korollar zu Satz III, [9].

The equivalence of $\mathcal{F}_x = (\mathcal{F}^{|0})_x$ and $\text{codh } \mathcal{F}_x \geq 2$ is also a consequence of [14, (1.1)]. However, the proof presented here is more conceptual than the proof in [14].

(iii) In the case of Stein spaces we have the following stronger version of Theorem 4 which generalizes Theorem 5.4 of [4]:

Suppose $\mathcal{F}$ is a coherent analytic sheaf on a Stein space $X$ such that $\mathcal{F} = \mathcal{F}^{|0}$. Suppose $K$ is a compact subset of $X$ such that, if $A$ is a branch of $E^\sigma(0, \mathcal{F})$ for any $\sigma \geq 2$, then $A - K$ is irreducible. Then for every open neighborhood $U$ of $K$ in $X$ every element of $\Gamma(U - K, \mathcal{F})$ can be extended uniquely to an element of $\Gamma(U, \mathcal{F})$.

(8) It can be proved in the following way: Suppose $s \in \Gamma(U - K, \mathcal{F})$. Since $H^1(X, \mathcal{F}) = 0$, from the Mayer-Vietoris sequence of $\mathcal{F}$ on $X = (X - K) \cup U$ (p. 236, [2]) we conclude that for some $f \in \Gamma(X - K, \mathcal{F})$ and $g \in \Gamma(U, \mathcal{F})$ we have $f - g = s$ on $U - K$. From Theorem 4 we can find $f \in (X, \mathcal{F})$ which agrees with $f$ outside some compact subset of $X$. Since $E^\sigma(0, \mathcal{F}) = \emptyset$ for $\sigma \leq 1$ and $A - K$ is irreducible for any branch $A$ of $E^\sigma(0, \mathcal{F})$ with $\sigma \geq 2$, $f$ agrees with $f^\beta$ on $X - K$. $(f^\beta|U) - g$ extends $s$. The extension is clearly unique, because $E^0(0, \mathcal{F}) = \emptyset$.

In view of the equivalence of $\mathcal{F}_x = (\mathcal{F}^{|0})_x$ and $\text{codh } \mathcal{F}_x \geq 2$, in the above proof we can use Theorem 15 of [2] instead of Theorem 4. So (8) can be proved also by the finiteness theorems of pseudoconvex spaces in [2].

(8) generalizes Theorem 5.4 of [4] because of the following:

Suppose $K$ is a closed subset of an irreducible complex space $X$ and $U$ is an open neighborhood of $K$ in $X$ such that for every branch $A$ of $U - A$ is irreducible. Then $X - K$ is irreducible.
Let $R$ be the set of all regular points of $X$. To prove (9), we need only show that $R - K$ is connected. Suppose $R \cap U = \bigcup_{i \in I} R_i$ is the decomposition into topological components. Then $R_i - K$ is connected for $i \in I$. The restriction map $\Gamma(R \cap U, C) \rightarrow \Gamma(R \cap (U - K), C)$ is an isomorphism. From the following portion of the Mayer-Vietoris sequence of the constant sheaf $C$ on $R = (R \cap U) \cup (R - K)$:
$$0 \rightarrow \Gamma(R, C) \rightarrow \Gamma(R - K, C) \oplus \Gamma(R \cap U, C) \rightarrow \Gamma(R \cap (U - K), C),$$
we conclude that the restriction map $\Gamma(R, C) \rightarrow \Gamma(R - K, C)$ is an isomorphism. $R - K$ is connected.

REFERENCES


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