SOME THEOREMS ON HOPFICITY

BY
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1. Introduction. Let $G$ be a group and let $\text{Aut } G$ be the group of automorphisms of $G$ and let $\text{End } G$ be the semigroup of endomorphisms of $G$ onto $G$. A group $G$ is called hopfian if $\text{End } G = \text{Aut } G$, that is, a group $G$ is hopfian if every endomorphism of $G$ onto itself is an automorphism. To put this in another way, $G$ is hopfian if $G$ is not isomorphic to a proper factor group of itself.

The question whether or not a group is hopfian was first studied by Hopf, who using topological methods, showed that the fundamental groups of closed two-dimensional orientable surfaces are hopfian [5].

Several problems concerning hopfian groups are still open. For instance, it is not known whether or not a group $H$ must be hopfian if $H \leq G$, $G$ abelian and hopfian and $G/H$ finitely generated. Also it is not known whether or not $G$ must be hopfian, if $G$ is abelian, $H \leq G$, $H$ hopfian, and $G/H$ finitely generated [2]. On the other hand, A. L. S. Corner [3], has shown the surprising result, that the direct product $A \times A$ of an abelian hopfian group $A$ with itself need not be hopfian.

Corner's result leads us to inquire: What conditions on the hopfian groups $A$ and $B$ will guarantee that $A \times B$ is hopfian? We shall prove, for example, in §3, that the direct product of a hopfian group and a finite abelian group is hopfian. Also we shall prove that the direct product of a hopfian abelian group and a group which obeys the ascending chain condition for normal subgroups (for short, an A.C.C. group) is hopfian (Theorems 3 and 5 respectively).

In §4 we examine various conditions on a hopfian group $A$ which guarantee $A \times B$ is hopfian for groups $B$ with a principal series. For example if the center of $A$, $Z(A)$, is trivial or if $A$ satisfies the descending chain condition for normal groups, (for short, $A$ is a D.C.C. group) then $A \times B$ is hopfian.

Theorem 3 is equivalent to: The direct product of a hopfian group and a cyclic group of prime power order is hopfian. In seeking to generalize this result we note that the normal subgroups of a cyclic group $C_{p^n+1}$ of prime power order $p^n+1$ form a chain and $C_{p^n+1}$ has exactly $n$-proper normal subgroups. We define an $n$-normal group as a group $G$ with exactly $n$-proper normal subgroups such that the normal subgroups of $G$ form a chain. Hence the simplest example of an $n$-normal group is $C_{p^n+1}$. (We only consider $n$ finite.) We then consider in §5 the direct product $G = A \times B$ of a hopfian group $A$ with an $n$-normal group $B$. In Theorem 16, we show that if $G$ is not hopfian, several anomalies arise with respect to $A$. For instance if $G$ is not hopfian we will show that there are infinitely many

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homomorphisms of $A$ onto $B$. We show that if $B$ is 0-normal or 1-normal, $A \times B$ is hopfian.

In §6 we explore briefly the concept of super-hopficity. If all homomorphic images of $A$ are hopfian, we say that $A$ is super-hopfian. We show for example that if $G$ is generated by a super-hopfian normal subgroup $A$ and a normal subgroup $B$ such that $B$ has finitely many normal subgroups, then $G$ is super-hopfian.

Unless otherwise stated, $A$ will always designate a hopfian normal subgroup of $G$ and $T$ will designate an element of $\text{End}$ on $G$. If $g \in G$, $O(g)$ will designate the order of $g$, $|G|$ will designate the cardinality of $G$. If $H \lhd G$ and $j$ is a positive integer, $HT^{-j}$ will designate the complete pre-image of $H$ under $T^j$.

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2. Some general theorems. We begin with a result that shows us that in some cases it suffices to consider infinitely generated hopfian groups $A$.

**Theorem 1.** If $G$ is a group containing a hopfian subgroup $N$ of index $[G : N] = r$, $r$ finite, such that $G$ contains only finitely many subgroups of index $r$, then $G$ is hopfian.

**Proof.** Suppose $G \cong G/K$, $K \neq 1$. If under an isomorphism of $G$ onto $G/K$, $K$ corresponds to $K_i/K_i$, we see $G \cong G/K \cong G/K_i$. Repeating the procedure, we see there exists subgroups $K_i$, where $K_i$ is a proper subgroup of $K_{i+1}$ such that

$$G \cong G/K_i, \quad i \geq 0, \quad K_0 = K.$$

Hence we may write $N \cong M_i/K_i$ so that $[G : N] = [G : M_i] = r$. Hence $M_i = M_j$ for some $i$ and $j$ with $i < j$. But then,

$$\frac{M_i/K_i}{K_j/K_i} \cong \frac{M_i}{K_i} \sim \frac{M_j}{K_j} \sim N \sim \frac{M_i}{K_j}$$

so that $N$ is not hopfian.

The following corollaries follow quite easily:

**Corollary 1.** Let $G$ be a group containing a hopfian normal subgroup $N$ of index $[G : N] = r$ ($r$ not necessarily finite) such that $G$ contains only finitely many normal subgroups of index $r$, then $G$ is hopfian.

**Corollary 2.** If $G$ is a finitely generated group containing a subgroup $N$ of finite index, $N$ hopfian, then $G$ is hopfian.

**Corollary 3.** If $A$ is finitely generated, and $|B| < \infty$, then $A \times B$ is hopfian.

**Lemma 1.** If $G/A$ is hopfian and if $AT \subset A$, then $T \in \text{Aut} G$. 

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Proof. $T$ induces an endomorphism of $G/A$ onto itself in the obvious way. Since $G/A$ is hopfian we conclude $AT^{-1} = A$ from which the conclusion easily follows.

**Theorem 2.** Let $A$ and $G/A$ be hopfian and suppose one of the following holds:
(a) $A \leq Z(G)$, $G/A$ centerless,
(b) $A$ a periodic group, $G/A$ torsion free,
(c) $A$ and $G/A = B$ both periodic groups such that if $a \in A$, $b \in B$, then $(O(a), O(b)) = 1$.

Then $G$ is hopfian.

**Proof.** Apply the previous lemma.

3. $G/A$ an A.C.C. group.

**Theorem 3.** If $B$ is a finite abelian group then $G = A \times B$ is hopfian.

**Proof.** It suffices to assume that $B$ is cyclic of prime power order, say, $|B| = p^n$ $B = \langle b \rangle$. Throughout this discussion and the next one, we will use symbols $a, a_i$ to designate elements of $A$.

Suppose first for a given $T$, we have $bT = a$. Let $b'a_1$ be a pre-image of $b$ under $T$. Let $u = ba_1$ and let $v = ba_1^{-1}$. We may then verify,

$$
G = \langle b \rangle \times A = \langle u \rangle A = \langle v \rangle \times A
$$

and $uT = v$. Let $A^1 = \langle v \rangle T^{-1} \cap A$ so that $\langle v \rangle T^{-1} = \langle u \rangle A^1$. Hence,

$$
A \sim (G/\langle v \rangle) \sim (G/\langle v \rangle T^{-1}) = (\langle u \rangle A)/(\langle u \rangle A^1) \sim A/A^1.
$$

Hence $A^1 = 1$. Hence $T$ is an isomorphism on $A$ and without too much difficulty, one sees that $T \in \text{Aut } G$.

Now suppose $bT \notin A$, say $bT = b'a$. If $(q, p) = 1$, we can find an automorphism $S$ of $G$ such that $bTS = b$, so that by Lemma 1, $TS \in \text{Aut } G$ and a fortiori, $T \in \text{Aut } G$. Hence we may assume $(q, p) \neq 1$.

If $aT \in A$ and if $f_p$ designates the greatest power of $p$ dividing the integer $f$ then $bT^2 = b'a_2$ where $r_p > q_p$. If $aT = b'a_3$, and $aT \notin A$, and if $s_p \leq q_p$, then for a suitable integer $u$, if $z = ba^u$, $zT \in A$ and $G = \langle z \rangle \times A$. If $s_p > q_p$ then $bT^2 = b'a_4$ where $v_p > q_p$. Hence if $(q, p) \neq 1$, we see that we may find an element $w$ of $G$, such that $G = \langle w \rangle \times A$ and $wT^i \in A$ for some integer $i$, $1 \leq i < 2^n$. Hence $T^i$ and $T$ are automorphisms.

**Theorem 4.** If $A$ is abelian and $B$ is finitely generated and abelian then $G = A \times B$ is hopfian.

**Proof.** By the previous theorem, we may assume $B = \langle b \rangle \sim C_\infty$.

By Lemma 1, if $A \leq AT^{-1}$ then $T \in \text{Aut } G$. Hence we may assume $A/\left(A \cap AT^{-1}\right)$ is infinite cyclic, that is,

$$
A = \langle a \rangle \times A \cap AT^{-1}.
$$
But \( A/(A \cap AT) \) is contained isomorphically in \( G/AT \) which in turn is a homomorphic image of \( G/A \). Hence we may write, \( A = \langle a_i \rangle A \cap AT \). Hence there is an element \( S, S \in \text{End on } A \) which agrees with \( T \) on \( A \cap AT^{-1} \) such that \( aS = a_i \). It easily follows that \( T \in \text{Aut } G \).

**Corollary.** If \( A \) is abelian and if \( B \) is finitely generated and \( B' \) the commutator group of \( B \) then \( A \times B = G \) is hopfian.

**Proof.** \( B'T = B' \) so that \( B'T^{-1} = B' \) or else \( (A \times B)/B' \sim A \times (B/B') \) is not hopfian.

**Corollary.** If \( Z(A) \) and \( A/Z(A) \) are hopfian and if \( B \) is a finitely generated abelian group, then \( A \times B \) is hopfian.

**Proof.** \( [Z(A) \times B]T^{-1} = Z(A) \times B \). Now apply the theorem.

We present here some general observations concerning \( T \) in relation to \( G/A \), where \( G \) and \( T \) are arbitrary and \( G/A \) is an A.C.C. group. (\( A \) need not be hopfian in this discussion.)

We note \( T \) induces in a natural way, a homomorphism of \( G/AT^i \) onto \( G/AT^{i+1} \). Since \( G/A \) is an A.C.C. group we see that ultimately all these homomorphisms are isomorphisms that is, for \( s \geq r \)

\[
(\text{AT}^{i+j})T^{-j} = \text{AT}^i , \quad j \geq 1
\]

so that kernel \( T^i \subset \text{AT}^s \). Hence

\[
\text{kernel } T^i \subset \bigcap_{s \geq r} \text{AT}^s , \quad j \geq 1.
\]

It follows that a necessary and sufficient condition that \( T \in \text{Aut } G \) is that \( T^i \) be an isomorphism on \( A \) for all \( i \geq 1 \). Moreover in seeking to prove that \( T \in \text{Aut } G \) it is not restrictive to assume that, for \( i \geq 1 \) and \( j \geq 1 \),

\[
(1) \quad G/AT^i \sim G/AT^{i+1} , \quad (\text{AT}^{i+j})T^{-j} = \text{AT}^i , \quad \text{kernel } T^j \subset \text{AT}^i.
\]

For if \( T \) does not obey the above conditions some power \( T_1 \) of \( T \) does and we could work with \( T_1 \) instead of \( T \). We will assume (1) whenever it is convenient.

We now resume our convention that \( A \) is hopfian.

**Theorem 5.** If every proper homomorphic image of \( A \) is abelian and \( B \) is an A.C.C. group then \( G = A \times B \) is hopfian.

**Proof.** Deny. Then we may find \( T, T \) not an isomorphism on \( A \) such that the conditions (1) hold. Let,

\[
G_1 = \text{gp } (A, AT, AT^2, AT^3, \ldots).
\]

Then \( G_1 T \subset G_1 \), so that \( G_1 T^{-1} = G_1 \). However \( AT^i \subset Z(G) \), \( i \geq 1 \) because \( G = AT^i \cdot BT^i \) and \( AT^i \) is abelian. Hence \( G_1 = A \times B_1 \) where \( B_1 \subset Z(B) \). Hence \( B_1 \) is finitely generated so that \( A \times B_1 \) is hopfian which implies \( T \) is an isomorphism on \( G_1 \), a contradiction of our hypothesis.
Corollary. If $A$ is abelian and $B$ is an A.C.C. group, then $A \times B$ is hopfian.

In view of the last theorem, it might be of some interest to give an example of a hopfian group $A$, which is not an A.C.C. group and which is not abelian, but yet every proper homomorphic image of $A$ is abelian. We proceed to do this.

Definition. Let $H$ be a group and $F$ a group of automorphisms. We will say $G$ is an extension of $H$ by $F$, if $G$ consists of elements $f_A, f_F, h_H$, where multiplication in $G$ is defined by

$$(f_A h_A)(f_F h_F) = (f_A f_F)(h_H h_F)$$

for $f_A \in F$ and $h_A \in H$, where $h_F$ is the image of $h_A$ under $f_A$.

Theorem 6. Let $H$ be a simple group and let $L$ be a hopfian group of automorphism of $H$. Furthermore, suppose

(2) $L \cap$ inner-automorphism $H=1$.

Then if $G$ is an extension of $H$ by $L$ then $G$ is hopfian. In fact if $L$ is super-hopfian, then $G$ is super-hopfian.

Proof. If $N \triangleleft G$ and $N \neq 1$ then $H \subset N$, for if $H \cap N=1$ the elements of $H$ and $N$ commute element-wise, which leads to a contradiction of (2). Hence, if $f \in \text{End}$ on $G$, by Lemma 1, $HT \neq 1$. Hence $H \subset HT$. But $HT \sim H$ since $H$ is simple. Hence $H=HT$. By Lemma 1 again, $f \in \text{Aut} G$. If $L$ is super-hopfian, every proper homomorphic image of $G$ is a homomorphic image of $L$ so that $G$ is super-hopfian.

As an application, let $H$ be the alternating group on an infinite countable set. Let $p_i, i=1, 2, 3, \ldots$, be a sequence of distinct primes. Then $H$ has a group of automorphisms $L$ which is the restricted direct product of cyclic groups of order $p_i, i=1, 2, \ldots$, and such that (2) holds. $L$ is super-hopfian. Hence $G$ is not an A.C.C. group, every proper homomorphic image of $G$ is abelian and $G$ is super-hopfian.

Somewhat along the lines of the previous theorem, we have

Theorem 7. Let every proper normal subgroup of $A$ be an A.C.C. group. Let every normal subgroup of $B$ be an A.C.C. group. Then if $G\subseteq A\times B$, then $G$ is hopfian.

Proof. Deny. Suppose $T$ is not an isomorphism on $A$ and kernel $T\subseteq AT$. Hence

$$B_1 = (A \cdot AT)/A \sim AT/A \cap AT \sim A/A_1.$$

Now $B_1$ is contained isomorphically in $B$ as a normal subgroup. Hence $A/A_1$ is an A.C.C. group. But $A_1 \neq A$ or else $AT \subset A$ contradicting Lemma 1. Hence $A_1$ is an A.C.C. group. But then so is $A$ and certainly then so is $G$ implying that $G$ is hopfian after all.

We now present some observations concerning the group $G$ where $G/A$ has finitely many normal subgroups.
Suppose $G$ is not hopfian. Then we may choose $T$ satisfying the conditions (1), $T$ not an isomorphism on $A$. Moreover, we may choose positive integer $r$ and $k$, $r<k$ such that

$$A \cdot AT^{-k} = A \cdot AT^{-r} = L,$$

$$AT^k \cdot A = AT^r \cdot A = M.$$

Hence, $MT^{-r-k} = M$, so that $M$ is not hopfian. If $G/A$ is finite, but $G$ is not hopfian, we might begin by choosing $[G : A]$ as small as possible so that if $M$ is constructed as above, $M = G$. But then $G/A$ is a homomorphic image of $A$. We may summarize part of the previous remarks as

**Theorem 8.** The statement,

If $A$ is hopfian and $G/A$ is finite then $G$ is hopfian is universally true if and only if the statement,

If $A$ is hopfian and $G/A$ is a finite homomorphic image of $A$, then $G$ is hopfian,

is universally true.

4. $A \times B$, where $B$ has a principal series.

**Definition.** We say that a group $B$ may be cancelled in direct products if whenever

$$C \times B \sim C^1 \times B^1$$

and

$$B \sim B^1$$

then $C \sim C^1$ (for any $C$).

**Lemma 2.** If $B$ has a principal series, $B$ may be cancelled in direct products.

**Proof.** See [4].

**Theorem 9.** If $B$ has a principal series, a necessary and sufficient condition for $A \times B$ to be hopfian is that $AT \cap BT = 1$ for arbitrary $T$ of End on $(A \times B)$.

**Proof.** The necessity part of the theorem is clear. Now suppose that $AT \cap BT = 1$ for any $T \in$ End on $(A \times B)$. By the remarks preceding (1), we can choose $r > 0$ such that

$$\text{kernel } T' \subseteq AT^s \quad \text{for } j \geq 1 \text{ and } s \geq r,$$

where $r$ depends on $T$. By hypothesis, $AT^r \cap BT^r = 1$ so,

$$A \times B = AT^r \times BT^r.$$

Hence if $K = \text{kernel } T' \cap B$, then

$$B/K \sim BT^r$$

and

$$A \times (B/K) \sim (AT^r/K) \times BT^r.$$

Hence by Lemma 2 we see $A \sim AT^r/K$. It follows without difficulty that $T'$ and $T$ are automorphisms.
COROLLARY 1. If $B$ has a principal series, then a sufficient condition for $T$ to be an automorphism, for $T$ in $\text{End} (A \times B)$, is

$$AT^i \cap BT^i = 1, \quad i \geq 1.$$  

COROLLARY 2. A sufficient condition for $T \in \text{Aut} (A \times B)$ is kernel $T^i \subseteq A$, $i \geq 1$ (where $B$ has a principal series).

COROLLARY 3. If $G$ has a principal series and if $T \in \text{End} (A \times B)$, and if $AT \cap BT = 1$, and if kernel $T \cap B \subseteq AT$ then $T$ is an automorphism.

THEOREM 10. If $B$ has a principal series and if there are only finitely many possible kernels for homomorphisms of $A$ into normal subgroups of $B$, then $A \times B$ is hopfian.

Proof. Choose $T$ obeying the conditions (1). Then write,

$$A \cdot AT^k = A \times B_k, \quad k \geq 1, \quad B_k \triangle B.$$  

The above gives rise to a homomorphism of $A$ onto $B_k$, whose kernel is $A \cap AT^{-k}$. Hence we have for say $0 < r < s$,

$$A \cap AT^{-r} = A \cap AT^{-s}.$$  

Hence, $AT^r \cap AT^{s-r} = AT^s \cap A$, so that kernel $T^i \subseteq A$, $i \geq 1$ and we may apply Corollary 2, of the previous theorem.

COROLLARY 1. If $B$ is finite and $A$ has only finitely many normal subgroups $A_i$ such that $[A : A_i] = 1$, then $A \times B$ is hopfian.

COROLLARY 2. If $B$ has a principal series and if there are only finitely many homomorphisms of $A$ into $B$, then $A \times B$ is hopfian.

LEMMA 3. Let $B$ be a group with a principal series. Let $P$ be a property of groups such that:

(a) $A$ has a nontrivial normal group $A_*$ such that $A/A_*$ has property $P$.

(b) If $A$ has property $P$, then $A \times B$ is hopfian.

(c) If $A_* \triangle A$ and $A/A_*^j$ has $P$ and if $T \in \text{End} (A \times B)$, then $A/(A_* \cap A^*T^j)$ has property $P$ for all integers $j$.

(d) $A$ satisfies the descending chain condition for normal subgroups $A^*$ such that $A/A^*$ has property $P$.

Then $A \times B$ is hopfian.

Proof. Choose a minimal normal group $A^*$ such that $A^* \neq 1$ and $A/A^*$ has property $P$. Then we may assume $A^*T^j \cap A^* = A^*$ for any $j$ so that $A^*T^{-j} = A^*$ for $j \geq 0$. Now apply Corollary 2 of Theorem 9.

THEOREM 11. Suppose either

(a) $B$ is finite, and $A$ satisfies the descending chain condition for normal subgroups of finite index, or
(b) \( B \) has a composition series and \( A \) satisfies the descending chain condition for normal subgroups \( A^* \) such that \( A/A^* \) has a composition series, or

(c) \( B \) has a principal series and \( A \) satisfies the descending chain condition for normal groups \( A^* \) such that \( A/A^* \) has a principal series.

Then \( G = A \times B \) is hopfian.

**Proof.** For instance, for (c) take \( P \) the property of having a principal series. Let \( A/A^* \) have property \( P \). Let

\[
H = A/A_1, \quad E = A^*/A_1, \quad F = A/A^*
\]

where \( A_1 = A^*T^i \cap A^* \). One can show \( E \) obeys the ascending and descending chain conditions for normal subgroups of \( H \), that is any ascending or descending chain of subgroups of \( E \) which are normal in \( H \) terminates. It follows that \( H \) has a principal series.

**Corollary.** If \( A \) is a D.C.C. group, and if \( B \) has a principal series, then \( A \times B \) is hopfian.

**Theorem 12.** If \( A \) satisfies the ascending chain condition for normal nonhopfian subgroups, and if \( B \) has a principal series, then \( G = A \times B \) is hopfian.

**Proof.** Deny. Choose \( T \) satisfying the conditions (1), but \( T \) not an isomorphism on \( A \). Let,

\[
A_i = \bigcap A T^q A_i, \quad i = 0, 1, 2, \ldots
\]

where \( q \) ranges over all integers. Then \( A_i T^{q_i} = A_i \) so that the \( A_i \) are nonhopfian. Hence we may find \( j \) so that \( A_i = A_{i+1} \). Hence \( A_{i+1} T^{q_i} = A_j \). It follows that kernel \( T^i \subseteq A, i \geq 1 \). Now apply Corollary 2 of Theorem 9 to obtain a contradiction.

In view of the former result, it might be interesting to give an example of a hopfian group \( G \) such that \( G \) contains a normal nonhopfian subgroup and such that \( G \) obeys the ascending chain condition for normal nonhopfian subgroups. (The example we give will be of special interest in Theorem 18.)

Let \( p \) be a prime and \( K \) be the field with \( p \) elements. Let \( m \) be an integer, \( m \geq 3 \). Let \( SL (m, K) \) be the group of nonsingular, unimodular, linear transformations of a vector space \( V \) of dimension \( m \) over \( K \). Let

\[
PSL (m, K) = SL (m, K)/Z
\]

where \( Z = \text{center of } SL (m, K) \).

**Lemma 4.** \( Z \) is the subgroup of diagonal linear transformations of \( SL (m, K) \) i.e., \( Z \) consists of those transformations \( T \) which have the form

\[
xT = \lambda x, \quad \lambda^m = 1 \quad \text{for all } x \in V.
\]

Also, \( PSL (m, K) \) is simple.
Proof. This is a special case of a more general result. See [6].

Now let \( \langle a \rangle \) be a cyclic group of order \( p, i = 1, 2, 3, \ldots \). Let \( G \) be the restricted direct sum of the \( \langle a \rangle \). Let \( G_r \) be the direct sum of the groups \( \langle a \rangle \) for \( 1 \leq i \leq r \) and let \( G' \) be the restricted direct sum of the groups \( \langle a \rangle \) for \( i > r \). Hence \( G \) is the direct sum of \( G_r \) and \( G' \). Now let \( F_* \) be the set of automorphisms \( T \) of \( G \) such that there exists an \( r \) such that \( T \) fixes the group \( G_r \), that is \( G_rT = G_r \), and such that \( T \) is the identity map on \( G' \), i.e., if \( x \in G' \), \( xT = x \). One can see that \( F_* \) is a group of automorphisms of \( G \). Now if \( T \in F_* \) we may choose \( r \) such that \( G_rT = G_r \) and \( T \) is the identity on \( G' \). Now on \( G_r \), \( T \) acts as a linear transformation and we define \( |T| \) as the determinant of the matrix representing \( T \) on \( G_r \). It may be verified that \( |T| \) is well defined, and independent of \( r \). Now let \( F \) be the subgroup of \( F_* \) of those transformations \( T \), with \( |T| = 1 \). We claim that \( F \) is simple. To see this let \( F_n \) be those elements \( T \) of \( F \) such that \( G_nT = G_{n-1} \), and \( T \) the identity on \( G_n \). We see \( F \) is the union of the \( F_n \). Since the union of an ascending sequence of simple groups is simple, we need only show that the groups \( F_n \) are simple. However one can see that \( F_n \approx \text{SL}(p^n, K) \) and since \( \lambda p^n = \lambda \) in \( K \), \( \text{SL}(p^n, K) \) has no center and so is simple by the previous lemma.

Now let \( M \) be the extension of the group \( G \) by \( F \). One sees that if \( g_1 \) and \( g_2 \) are elements in \( G \), \( g_1 \neq 1, g_2 \neq 1 \), there exists \( T \in F \) such that \( g_1T = g_2 \). One can now see that \( G \) is the only normal subgroup of \( M \) so that certainly \( M \) is hopfian and has a nonhopfian normal subgroup, namely \( G \), and \( M \) obeys the ascending chain condition for normal nonhopfian groups.

Lemma 5. Let \( C \triangleleft G \) and suppose that \( C \) has finitely many normal subgroups. If \( T \in \text{End} \) on \( G \), then either \( C \cap CT^i = 1 \) for all positive \( i \) sufficiently large, or we can find \( C^* \), \( C^* \triangleleft C \), \( C^* \triangleleft G \), \( C^* \neq 1 \), and a positive integer \( j \) such that \( C^*T^i = C^* \).

Proof. If \( C \cap CT^i \neq 1 \), for all \( i \) sufficiently large, we may find positive integers \( r \) and \( s, r < s \), and normal groups \( C_r \) and \( C^* \) of \( C \) such that if \( u \) is either \( r \) or \( s \),

\[ CT^u \cap C = C^*T^u = C^* \neq 1. \]

Hence if \( j = s - r \), \( C^*T^j = C^* \).

We note at this point that if \( A \) is hyper-hopfian, that is if every normal subgroup of \( A \) is hopfian, then certainly Theorem 12 guarantees \( A \times B \) is hopfian if \( B \) has a principal series. For instance if the groups \( M_i \) are torsion-hyper-hopfian groups such that elements \( m_i, m_i \) of \( M_i \) and \( M_i \) respectively, \( i \neq j \), have relatively prime orders, then the restricted direct product of the \( M_i \) is hyper-hopfian. In particular one may choose the \( M_i \) to be finite groups.

Theorem 13. If \( A \times B \) is not hopfian and \( B \) has a principal series, then there exists a homomorphic image \( C \) of \( B \) such that \( A \times C \) is not hopfian, and \( \mathbb{Z}(C) \neq 1 \), and if \( T \) is an arbitrary element of \( \text{End} \) on \( (A \times C) \), then \( T \) is an isomorphism on \( C \). Also if \( C_1 \triangleleft C \), \( C_1 \neq 1 \), then \( A \times (C/C_1) \) is hopfian. Furthermore, if \( B \) has finitely many normal subgroups, and \( A \times B \) is not hopfian, we can find \( C \) with the former
properties, and in addition with the property that if \( T \in \text{End on } (A \times C) \), \( T \notin \text{Aut}(A \times C) \), then \( CT^i \cap C = 1 \), for all positive \( i \) sufficiently large.

**Proof.** Choose a group \( C \), \( C \) a homomorphic image of \( B \), with the number of terms in a principal series for \( C \) minimal with respect to \( A \times C \) being nonhopfian. This guarantees that for all \( T \in \text{End on } (A \times C) \), \( T \) is an isomorphism on \( C \) and \( A \times C/C_1 \) is hopfian if \( C_1 \neq 1 \). Furthermore, since \( A \times C \) is not hopfian, we may choose \( T \in \text{End on } (A \times C) \) so that \( AT \cap CT \neq 1 \). Hence \( Z(CT) \sim Z(C) \neq 1 \). Furthermore if \( B \) has finitely many normal subgroups, so does \( C \) so that if \( T \) is any element of \( \text{End on } (A \times C) \), \( T \notin \text{Aut}(A \times C) \), then \( CT^i \cap C = 1 \) for all \( i \) sufficiently large or else we could choose \( C^* \) as in the previous lemma and \( A \times (C/C^*) \) would not be hopfian.

**Corollary.** Suppose \( A \) cannot be written in the form

\[
A = A_1 \circ A_2, A_i \triangle A, A_i \neq A, i = 1, 2,
\]

(3)

Then if \( B \) has finitely many normal subgroups, \( A \times B \) is hopfian. Moreover, if the homomorphic images of \( A \) are indecomposable as a direct product, then \( A \times B \) is hopfian. Finally if \( B \) is fixed, and \( A \) cannot be written in the form (3) with the additional stipulation that \( A_2 \) be a homomorphic image of \( B \), then \( A \times B \) is hopfian.

**Proof.** If \( A \times B \) is not hopfian, choose \( C \) as in the previous theorem and \( T \in \text{End on } (A \times C) \), \( C \cap CT = 1 \), and \( T \) not an isomorphism on \( A \). Let \( N = CT^{-1} \) so that \( (A \times C)/N \sim A \) so that we may take \( A_1 = (AN)/N \), and \( A_2 = (CN)/N \sim C \). If \( A \) is written in the form (3), then \( A/A_1 \cap A_2 = A_1/A_1 \cap A_2 \times A_2/A_1 \cap A_2 \).

**Theorem 14.** If \( Z_0 = 1 \) and \( Z_{n+1}/Z_n = Z(A/Z_n) \), \( n \geq 0 \), and \( A/Z_n \) and its center are hopfian for all \( n \geq 0 \), then if \( B \) has a principal series, \( A \times B \) is hopfian.

**Proof.** Deny. Choose a group \( E \) with a principal series and an integer \( r \geq 0 \), such that \( H = A/Z_r \times E \) is not hopfian and \( \triangle(E) = \text{length of a principal series for } E \) is minimal. That is, if \( A/Z_r \times D \) is not hopfian, and if \( D \) has a principal series, then \( \triangle(E) \leq \triangle(D) \). Consequently, the group \( C \) we may associate with \( E \), by the previous theorem, is \( E \) itself, so \( Z(E) \neq 1 \). If \( T \in \text{End on } H \), but \( T \notin \text{Aut } H \), we see from the minimality of \( \Delta(E) \) that

\[
Z(H)T^{-1} = Z(H) = Z_{r+1}/Z_r \times Z(E).
\]

However, \( Z(E) \) is finite and this contradicts Theorem 3.

**Corollary.** If \( Z(A) = 1 \), and if \( A \) is hopfian and \( B \) has a principal series, then \( A \times B \) is hopfian.

5. \( A \times B \), \( B \) \( n \)-normal. We begin by giving some examples of \( n \)-normal groups. As we have mentioned, we have the groups \( C_t, t = p^{n+1}, p \) a prime. Or if \( F \) is a
simple group and $B$ is an $n$-normal group of automorphisms of $F$, such that $B$ does not contain any inner-automorphism (different from 1), then the extension of $F$ by $B$ is $n+1$ normal. In particular, if $B \sim C_q^n$, $p$ a prime, we can find a prime $q$, $q = 1 \mod p^n$ so that $C_q$ has a group of automorphisms, $B$, and extending $C_q$ by $B$ gives us a nonabelian $n$-normal group. Similarly if $H$ is the alternating group of arbitrary infinite cardinality, and if $R \in \text{Aut} \ H$, $O(R) = p^n$, and if $\langle R \rangle$ contains no inner-automorphism except 1, if we extend $H$ by $\langle R \rangle$ we obtain an infinite $n$-normal group so that nonabelian $n$-normal groups of arbitrary infinite cardinality exist.

Until further notice, $B$ shall represent an $n$-normal group, with normal subgroups,

$$1 = B_0, B_1, \ldots, B_n, B_{n+1} = B, B_i \subset B_{i+1}.$$  

**Lemma 6.** If $T \in \text{End on } G$, $G = A \times B$, $T \notin \text{Aut } G$, then $G = A \cdot AT$ and $B \cdot AT$ is a proper subgroup of $G$.

**Proof.** Either $A \cdot AT \subset B \cdot AT$ or $B \cdot AT \subset A \cdot AT$. Hence all we need show is that $A \cdot AT$ is not a subgroup of $B \cdot AT$. But if $A \cdot AT \subset B \cdot AT$, then $G = B \cdot AT$ and hence $A \sim AT/B \cap AT$, from which we could easily deduce that $T \in \text{Aut } G$.

**Lemma 7.** Suppose $T \in \text{Aut } G$, and $AT \cap BT = B_i T$ where kernel $T \cap B = B_i \subset B_i$. Then if $B \cap AT = B_j$, then $j > i$.

**Proof.** Use the previous lemma to see that $B/B_i$ is contained isomorphically as a proper normal subgroup of $G/AT \sim B/B_i$.

**Lemma 8.** If $BT \sim B$, then $AT \cap BT \neq 1$.

**Proof.** Deny. Then $G = A \times B = AT \times BT$. By the previous lemma, $B_i \subset AT$. But then Corollary 3 of Theorem 9 implies that $T \in \text{Aut } G$.

**Theorem 15.** A necessary and sufficient condition that $T \in \text{Aut } G$ is $AT \cap BT = 1$.

**Proof.** The previous lemma and Lemma 2.

In our next theorem, we show that if $A \times B$ is not hopfian $A$ must enjoy several anomalous properties.

**Theorem 16.** Suppose $G = A \times B$ is not hopfian. Then,

1. There exists infinitely many homomorphisms of $A$ onto $B$.
2. Also there exist normal subgroups of $A \times B$, $R^i, R, R_i, R_i^i, i \geq 0$ such that
3. $R^i \subset R^{i+1} \subset R^i, R^0 = R_0, R_i \cap R_{i+1} \subset R$ for all $i$.
4. $R^i = \cap R^i, R = \cup R_i$, where the intersection and union are taken over all $i \geq 0$. Also the containments in 2) are proper.
5. The $R_i$ are subgroups of $A$.
6. $R^* = \cup R_i, R = \cap R_i$ for all $i$ and $j$.
7. $R_i^j \sim R_i \sim R_{i+1}/R_i \sim R_{i+1}/R_i$ for all $i$ and $j$.
8. $R_i^j \sim R_i \sim R_{i+1}/R_i \sim R_{i+1}/R_i$ for all $i$ and $j$. 

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(8) There exist normal subgroups $A_i \subseteq A_i$, $i=1, 2, \ldots$, $A_i \subseteq A_{i+1}$ properly, such that $A_{i+1}/A_i \sim R_2/R_1$ for all $i$.

(9) There exist normal subgroups $K_i$, $K_i \subseteq K_{i+1}$, $i \geq 0$, such that if $L = \bigcup K_i$, then $L$ is nonhopfian, and

$$R_i/K_j \sim R_{i+j}, \quad R_i/L \sim R_{j+i}, \quad K_{i+j}/K_i \sim K_i \quad \text{and} \quad L/K_j \sim L.$$

**Proof.** Let $T \in \text{End on } G$, $T \in \text{Aut } G$. Then by Lemma 6, $A \cdot AT^j = G$ for all $j > 0$, which implies $A \cdot AT^{-j} = G$ for $j > 0$. Hence $A/A \cap AT^{-j} \sim B$ and one may show (as in Theorem 10) that if the groups $A \cap AT^{-j}$, $j = 1, 2, 3, \ldots$ are not distinct, then $T$ is an automorphism.

Now let us assume, without loss of generality, that $T$ satisfies the condition (1), and that $AT^r \cap BT^r = B_iT^r$ for all $r \geq 1$ (for some fixed $i$, $i \geq 1$) and that $i$ is maximal in the sense that if

$$B_iT^q \subset AT^q \cap BT^q \quad \text{for some } q \geq 1, \quad \text{then } u \leq i.$$

(For if $T$ does not obey these conditions, some power of $T$ does, and we could then work with this power of $T$.)

Now we define,

$$R_i = \bigcap_{j \geq 0} AT^j, \quad j \geq 0, \quad R = \bigcup_{j \geq 0} R_j.$$

With the aid of (1), we see $R_iT = R_{i+1}$ so that $RT = R$, and $R \neq 1$, since kernel $T \subseteq R$. Moreover, the groups $R_j$ are all distinct, for if say $R_m = R_{m+1}$, then $R_j = R_m$ for $j > m$ and hence $R = R_m$. But then with the aid of Lemma 7, we see $B_{i+m} \subseteq R_i \subseteq R$.

Hence,

$$A \cdot R = A \times B_s, \quad s > i.$$

Hence, $(A \cdot R)^s = AT^m \cap BT^m = AT^m \cap R = AT^m = AT^m \cdot B_s T^m$ so that $B_s T^m \subseteq BT^m \cap AT^m$, a contradiction of the maximality of $i$.

We now define

$$R^0 = R_0 \quad \text{and} \quad R^{n+1} = R^n T^{-1} R_0, \quad n \geq 0.$$

By induction and the previous lemma, we see that $R^{n+1}$ is a proper subgroup of $R^n$ and $R^n = \bigcap AT^j$ where $j$ ranges over all integers $\geq -n$ for each $n \geq 0$. Moreover if we consider the homomorphism of $R^n$ onto $R^{n-1}$, induced by $T$ for $n \geq 1$, we see that the preimage of $R^n$ is exactly $R^{n+1}$. So that

$$R^n/R^{n+1} \sim R^{n-1}/R^n, \quad n \geq 1.$$

Furthermore, if we consider the homomorphism of $R^0 = R_0$ onto $R_1$ induced by $T$, we see that the preimage of $R_0$ is exactly $R^1$ so that $R^0/R^1 \sim R_1/R_0$.

Now one may see that $R_1/R_0$ is isomorphic to a normal subgroup of $AT/A \cap AT \sim B$. Also with the aid of (1) we see,

$$R_{j+1}/R_j \sim R_{j+2}/R_{j+1}, \quad j > 1.$$

Furthermore, $R_2/R_1$ is isomorphic to a normal subgroup of $AT^2/AT \cap AT^2 \sim B/B_1$.  

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If \( A_k = R_k \cap A \), ultimately the \( A_k \) are distinct and by a suitable reindexing, the \( A_k \) may be seen to have the properties asserted in the Theorem. One may verify the remaining assertions by taking \( K_j = \text{kernel } T', j \geq 1 \), and \( L = \bigcup_{j \geq 1} K_j \), and by noting that \( R^*T^{-1} = R^* \).

**COROLLARY.** If \(|B| > |A|\), then \( A \times B \) is hopfian.

**Proof.** \( B \) cannot be a homorphic image of \( A \).

We now find some particular values of \( n \) for which \( A \times B \) is hopfian.

**LEMMA 9.** If \(|B| = p^{n+1}, p \text{ a prime}, \) then \( B \sim C_p^{n+1} \).

**Proof.** Use induction on \( n \), and the fact that \( Z(B) \neq 1 \).

**LEMMA 10.** If \( T \in \text{End on } (A \times B) \) and \( BT \subseteq A \) and \( B \subseteq AT \), then \( T \) is an isomorphism on \( A \).

**Proof.** \( AT = B \times A \cap AT \) and \( A = BT(A \cap AT) \). These two decompositions give rise to a homomorphism \( S \) of \( AT \) onto \( A \) such that \( S \) agrees with \( T \) on \( B \) and \( S \) is the identity on \( A \cap AT \).

**LEMMA 11.** Let \( k \) be the least integer, \( k \geq 0 \) (if one exists), such that \( A \times B \) is not hopfian for some \( A \) and for some \( k \) normal group \( B \). Then if \( T \in \text{End on } (A \times B) \), \( T \notin \text{Aut } (A \times B) \), then \( B \cap BT = 1 \) and \( T \) is an isomorphism on \( B \).

**Proof.** Deny. Then \( B_1T \subseteq B_1 \) and \( A \times B/B_1 \) is not hopfian, which contradicts the minimality of \( k \) if \( B_1 \neq B \), or the hopficity of \( A \) if \( B_1 = B \).

**THEOREM 17.** If \( B \) is \( n \)-normal, \( 0 \leq n \leq 1 \), then \( A \times B \) is hopfian.

**Proof.** Let \( k \) be as in the last lemma, \( A \times B \) not hopfian, \( B \) \( k \)-normal. We will show \( k \geq 2 \). Let \( T \in \text{End on } (A \times B) \), \( T \) not an isomorphism on \( A \). Let \( A \cdot BT = A \times B_r, B \cdot AT = (B_iT)(AT), AT \cap BT = B_iT, B \cap AT = B_i \) where \( 1 \leq i < j \). Using Lemma 11 we see \( B_r, B_i, B_i \) are central groups of \( B \) and hence are cyclic \( p \) groups for some prime \( p \). Furthermore, we see \( A \cap BT = (B_{k-r+1})T \) and \( B/B_1 \sim B_q/B_r, B/B_{k-r+1} \sim B_r, q = k + i - j + 1 \). Hence we must have,

\[
j > r, \quad j > q, \quad k-r+1 > q, \quad k-r+1 > r
\]

or otherwise \( B \) would be a finite \( p \) group and hence \( B \) would be cyclic, a contradiction of Theorem 3. In summary we have,

\[
0 \leq r < \frac{k+1}{2} \leq \frac{k+i}{2} < \frac{k+i+1}{2} < j \leq k+1.
\]

And with the aid of Lemma 10, we see \( 1 \leq i < j-r \leq k \). Hence we see \( k=0 \) or \( 1 \) is impossible.

**COROLLARY 1.** If \( C = D \cdot E, D \triangle C, D \cap E = 1 \) where \( D \) and \( E \) are simple, then \( A \times C \) is hopfian.
Proof. Either $C$ is 1-normal or $C \sim D \times E$.

Corollary 2. If $B$ is 2-normal and if $T \in \text{End}(A \times B)$, $T$ not an automorphism, then $B \subseteq AT$, $A \cap BT = B_2T$, $B_2B_2 \sim B_3 \sim C_9$ for some prime $p$ and $B_1 = Z(B)$.

Corollary 3. If $r$ is a positive integer, then $A \times \text{symmetric} (r)$ is hopfian.

Proof. Symmetric 4 is 2-normal and centerless. If $r \neq 4$, symmetric $r$ is 1-normal.

Corollary 4. If $B$ is a group such that $B$ has exactly one normal group in a principal series, i.e., $B$ has a principal series of the form $1, B_*, B$, then $A \times B$ is hopfian.

Proof. Either $B$ is 1-normal or $B$ is the direct product of simple groups.

Corollary 5. If $G = A \times B$, $B$ n-normal and if $BT$ is i-normal, $i = 0$ or 1, then $T \in \text{Aut} G$.

Theorem 18. Let $E$ be a class of hopfian groups such that any hopfian group is isomorphic to a unique group of $E$. Then there exists a class $E_*$ of hopfian groups such that:

(a) $E$ and $E_*$ have the same cardinality.

(b) No two distinct groups of $E_*$ are isomorphic.

(c) Any hopfian group is contained isomorphically as a normal subgroup of some group in $E_*$.

(d) Every group in $E_*$ has a nonhopfian normal subgroup.

Proof. Let $E_*$ be the set of groups which is formed by taking the direct product of groups in $E$ with the group $M$ of the example following Theorem 12, i.e. $E_* = \{(A \times M) | A \in E\}$.

Our assertions follow from the previous theorem, the definition of $M$ and Lemma 2.

6. Super-hopficity. We terminate this paper with an investigation of the concept of super-hopficity. For an illustration of super-hopficity, we note that the restricted direct product of periodic super-hopfian groups $M_i$, such that $(O(m_i), O(m_j)) = 1$ for $m_i \in M_i$, $m_j \in M_j$, $i \neq j$, is super-hopfian. In particular, the $M_i$ might be chosen as finite groups.

We no longer assume that $B$ designates an $n$-normal group.

Lemma 12. Let $A$ be super-hopfian and let $H = A \cdot B$, $A \triangle H$, $B \triangle H$. Suppose $T \in \text{End}(H)$ and $B \triangle R$, $R \triangle H$ and $RT^{-1} \subseteq R$. Then $RT^{-1} = R$.

Proof. If $RT^{-1} \neq R$, $H/RT^{-1}$ is a homomorphic image of $A$, but $H/RT^{-1}$ is not hopfian.

Corollary. If $H$ and $T$ are as in the lemma, and if $r > 0$ and if $L_r$ is the subgroup of $H$ generated by the groups $BT^n$, $i \geq 0$, then $L_rT^{-r} = L_r$. 

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Lemma 13. If $H$ and $L_r$ and $T$ are as in the preceding corollary and if $B \cap BT^{i'} = 1$ for fixed $r$ and for all $i \geq 1$, then $B$ abelian.

Proof. Since $L_rT' = L_r$, $L_r$ is generated by the groups $BT^{i'}$, $i \geq 1$, and $B$ commutes element-wise with each $BT^{i'}$, $i \geq 1$. Hence $B \subseteq Z(L_r)$.

Theorem 19. Let $H = AB$ where $A \triangle H$ and $B \triangle H$ and where $A$ is super-hopfian. Suppose $B$ satisfies any one of the following conditions:

(a) $B$ is a finitely generated A.C.C. group.
(b) $B$ has finitely many normal subgroups or,
(c) $B$ is an A.C.C. group and if $B_*$ is any homomorphic image of $B$ and if $B_1 \triangle B_*$ and $B_2 \triangle B_*$ and if $B_1 \sim B_2$ then $B_1 = B_2$.

Then $H$ is super-hopfian.

Proof. It suffices to prove $H$ is hopfian since any homomorphic image of $H$ satisfies the same hypothesis as $H$ in any of the three situations. Let us assume that (a) holds. Let $T \in \text{End}$ on $H$. In the notation of the corollary to Lemma 12, we have $B \subseteq L_1 = L_1 T$ and $L_1T$ is generated by the groups $BT^i$, $i \geq 1$. Hence, since $B$ is finitely generated, we can find $r$ such that

$$B \subseteq BT \cdot BT^2 \cdot BT^3 \cdot \ldots \cdot BT^{r-1} \cdot BT^r = E.$$ 

Hence,

$$BT \subseteq BT^2 \cdot BT^3 \cdot BT^r \cdot BT^{r+1} = ET.$$ 

Consequently, $E \subseteq ET$ and hence,

$$ET^i \subseteq ET^{i+1}, \quad i \geq 0.$$  

Now since $B$ is an A.C.C. group, so is $BT^i$, $i \geq 0$, and hence so is $E$. Consequently, $T$ is an isomorphism on $ET^i$ for all $i$ sufficiently large and positive. However, $L_1$ is the union of the groups $ET^i$, $i \geq 1$. Hence in view of (4), we see $T$ is an isomorphism on $L_1$. But from the corollary to Lemma 12, $L_1 = L_1 T^{-1}$ and so $T$ is an automorphism.

Now suppose the assertion of (b) is false and choose a counterexample $A \cdot B = H$ so that $B$ has the fewest number of normal subgroups among all possible counterexamples. Let $T \in \text{End}$ on $H$, $T$ not an isomorphism on $A$. Then we can find $r > 0$ such that $B \cap BT^i = 1$ for all $i \geq r$ or else by Lemma 5, we can find $j > 0$ such that $B_* T^j = B_*$ for some normal subgroup, $B_*$ of $B$, $B_* \neq 1$. Furthermore, $T^j$ is an isomorphism on $B$ because of the “minimality” of $B$. Hence,

$$H / B_* = [(AB_*) / (B_*)] / (B / B_*)$$

is not hopfian, which contradicts the “minimality” of $B$. Hence $r$ exists as asserted, and so we see from Lemma 13 that $B$ is abelian. Hence $B$ is finite. This contradicts part (a) of our theorem.

Finally for (c) we may proceed by denying that $G$ is hopfian. Hence we may choose $B^*$ and $A^*$ such that $H^* = A^* \cdot B^*$, $A^* \triangle H^*$, $B^* \triangle H^*$, $A^*$ super-hopfian, $B^*$ a
homomorphic image of $B$, $H^*$ not hopfian, and such that if $H_1 = A_1 \cdot B_1$, $A_1$ super-hopfian, $B_1$ a proper homomorphic image of $B^*$, then $H_1$ is hopfian.

Choose $T \in \text{End on } H^*$, $T$ not an isomorphism on $A^*$. Note $T^i$ must be an isomorphism on $B^*$ for $i \geq 1$. Now if $B^* \cap B^* T^j \neq 1$ for some $j, j \geq 1$, we may write $B_\bullet = B_2 T^j$, $B_\bullet \subseteq B^*$, $B_\bullet \neq 1$, $B_\bullet \neq B_2$ (or else $G/B_\bullet$ is not hopfian, etc.) but $B_\bullet \sim B_2$, a contradiction of our hypothesis. Hence, $B^* \cap B^* T^j = 1$ for $j \geq 1$, so that $B^*$ is abelian and finitely generated, a contradiction of part (a) of our theorem.

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