# ON THE EXTENSION OF PARTIAL ORDERS ON SEMIGROUPS OF RIGHT QUOTIENTS

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Let $S$ be a semigroup and let $T=Q(S, \Sigma)$ be a semigroup of quotients of $S$ with a subsemigroup $\Sigma$ of denominators contained in the centre of $S$. It was shown by Puttaswamaiah [9] that the condition

$$(1) \qquad a\xi \leqq b\xi \quad \text{with } a, b \in S, \xi \in \Sigma \quad \Rightarrow \quad a \leqq b$$

for a partial order $O_S$ of $S$ is sufficient in order to extend $O_S$ to a partial order $O_T$ of $T$(1). Beyond this we have proved in [13] that each partial order of $S$ can be extended to a partial order of $T$, and that one has the following situation: On the one hand, there is a 1-1 correspondence between all partial orders $O_T$ of $T$ and those partial orders $O_S$ of $S$ which obey (1). This correspondence is given by mapping each $O_T$ on its restriction $O_T|S=O_S$ on $S$ and conversely $O_T$ is the smallest extension of $O_S$ on $T$. On the other hand, each partial order $O_S$ of $S$ can be extended to such a partial order of $S$ which obeys (1), the smallest of them is unique.

In this paper we are going to generalize these results to a semigroup $T=Q_r(S, \Sigma)$ of right quotients of $S$. For this purpose we give in §1 a summary about semigroups of right quotients including an outline of a proof of their existence which is essentially given in [10]. It turns out that the first part of the result above is also true for semigroups of right quotients, replacing (1) by a similar two-sided condition (cf. Theorem 2). But the second only holds under supplementary assumptions (cf. Theorem 5), of course including the result above, and we give an example of a partial order on a certain semigroup $S$, which can not be extended to any semigroup of right quotients of $S$. Moreover, we obtain some results about the question of extending partial orders of $S$ to different semigroups of right quotients $T_i=Q_r(S, \Sigma_i)$ (cf. Theorem 3). For concepts and notations not defined in the text we refer to [2] and [5].

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## 1. Semigroups of right quotients.

DEFINITION 1. *Let $S=\{a, b, \ldots\}$ be a semigroup and let $\Sigma=\{\alpha, \beta, \ldots\}$ be a subsemigroup of $S$, such that each $\alpha \in \Sigma$ is cancellable in $S$. Then a semigroup $T$ with identity*, 1, *containing $S$ as a subsemigroup is called a semigroup of right quotients*

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(1) Concerning the other statements in [9] see [12].

*(briefly s.o.r.q.) of $S$ with respect to $\Sigma$, if each $\alpha \in \Sigma$ has an inverse $\alpha^{-1} \in T$ and the set of right quotients $\{a\alpha^{-1} \mid a \in S, \alpha \in \Sigma\} \subseteq T$ coincides with $T$.*

We shall outline a short elementary proof (cf. [10]) of the following theorem, the first part of which is due to Murata (cf. [7]).

THEOREM 1. *Let $S$ and $\Sigma$ be as in Definition 1. Then s.o.r.q. $T$ of $S$ with respect to $\Sigma$ exists if and only if the following Ore-Asano-Condition*(2) *$q_r(S, \Sigma)$ holds:*

*For each $a \in S$ and $\alpha \in \Sigma$ there are $l \in S$ and $\lambda \in \Sigma$ with $a\lambda = \alpha l$, i.e. $a\Sigma \cap \alpha S \neq \varnothing$. Moreover, we have the following rules for equality and multiplication in $T$:*

$$\text{(i)} \quad \begin{aligned} a\alpha^{-1} = b\beta^{-1} &\Leftrightarrow \text{there exist } l \in S \text{ and } \lambda \in \Sigma \text{ with } \alpha\lambda = \beta l \text{ and } a\lambda = bl \\ &\Leftrightarrow \text{for all } u \in S \text{ and } v \in S\ \alpha u = \beta v \text{ implies } au = bv \end{aligned}$$

$$\text{(ii)} \qquad a\alpha^{-1} \cdot b\beta^{-1} = at(\beta\tau)^{-1} \text{ with } b\tau = \alpha t,\ t \in S,\ \tau \in \Sigma.$$

*Therefore $T$ is, up to isomorphisms, uniquely determined by $S$ and $\Sigma$ and we may write $T = Q_r(S, \Sigma)$ for the s.o.r.q. of $S$ with respect to $\Sigma$.*

**Proof.** *The existence of one $T = Q_r(S, \Sigma)$ implies $q_r(S, \Sigma)$.* From $a \in S$ and $\alpha \in \Sigma$ we have $\alpha^{-1}a \in T$, hence there exist $l \in S$ and $\lambda \in \Sigma$ with $\alpha^{-1}a = l\lambda^{-1}$ and therefore $a\lambda = \alpha l$.

*From $q_r(S, \Sigma)$ follows the rule $z_r(S, \Sigma)$*, which is a generalization of a rule given by Malcev (cf. [6]) and is defined by:

*Let $a, b, c, y, u, v$ be elements of $S$ and let $\delta, \xi$ be elements of $\Sigma$, then in the following the starred equation is always a conclusion of the other three:*

$$\begin{aligned} & a\xi = by, \qquad c\xi = \delta y, \\ (*) \quad & au = bv, \qquad cu = \delta v. \end{aligned}$$

Indeed, by $q_r(S, \Sigma)$ there are $l \in S$ and $\lambda \in \Sigma$ with $u\lambda = \xi l$. From $\delta y l = c\xi l = cu\lambda = \delta v\lambda$ we have $yl = v\lambda$, hence $au\lambda = a\xi l = byl = bv\lambda$, which implies $au = bv$.

*In any $T = Q_r(S, \Sigma)$* (i) *and* (ii) *are true.* The existence of $l \in S$ and $\lambda \in \Sigma$ with $\alpha\lambda = \beta l$ comes from $q_r(S, \Sigma)$, which gives, when multiplied by $a\alpha^{-1} = b\beta^{-1}$, also $a\lambda = bl$, and conversely. Both right sides of (i) are equivalent alone by virtue of $q_r(S, \Sigma)$, using $z_r(S, \Sigma)$. The rule (ii) is obvious for each $t\tau^{-1} = \alpha^{-1}b$.

*Using $q_r(S, \Sigma)$ we construct $T = Q_r(S, \Sigma)$.* In $S \times \Sigma$ we define a relation $(a, \alpha) \sim (b, \beta)$ by the right sides of (i). By the second version reflexivity and symmetry are clear. In order to prove transitivity we assume also $(b, \beta) \sim (c, \gamma)$, take $k \in S$ and $\kappa \in \Sigma$ with $\alpha\lambda\kappa = \beta l\kappa = \gamma k$, from which we have $a\lambda\kappa = bl\kappa = ck$ and hence $(a, \alpha) \sim (c, \gamma)$, always using the first or the second version of (i) as it is convenient.

In the set $\overline{T}$ of equivalence classes of $S \times \Sigma$ by this relation we define a multiplication according to (ii).

$$(a, \alpha) \cdot (b, \beta) \sim (at, \beta\tau) \quad \text{with } t \in S,\ \tau \in \Sigma \text{ and } b\tau = \alpha t.$$

(2) This condition was used by Ore (cf. [8], with $\Sigma = S \backslash \{0\}$) and Asano (cf. [1]) in the case, where $S$ is a ring.

This definition does not depend on the choice of $t$ and $\tau$ and also not on the choice of $(a, \alpha)$ and $(b, \beta)$, representing their classes. We can show all these in one step, assuming

$$(a, \alpha) \sim (a', \alpha'), \qquad (b, \beta) \sim (b', \beta')$$

and

$$(a', \alpha')\cdot(b', \beta') \sim (a't', \beta'\tau') \quad \text{with } b'\tau' = \alpha't'$$

in proving $(at, \beta\tau) \sim (a't', \beta'\tau')$. In order to do this we solve $\beta\tau\lambda = \beta'\tau' l$ by $q_r(S, \Sigma)$ and get, always using the second version of (i), $b\tau\lambda = b'\tau' l$, hence $\alpha t\lambda = \alpha' t' l$ and therefore $at\lambda = a't'l$, which gives our statement by the first version of (i).

The associativity of this multiplication follows easily with suitable choices of the elements which occur in the definition of the product:

$$\begin{aligned}
[(a, \alpha)(b, \beta)](c, \gamma) &\sim (at, \beta\tau)(c\gamma) && \text{with } b\cdot\tau = \alpha\cdot t\\
&\sim (ats\rho, \gamma\sigma\rho) && \text{with } c\cdot\sigma\rho = \beta\tau\cdot s\rho,\\
(a, \alpha)[(b, \beta)(c, \gamma)] &\sim (a, \alpha)(b\tau s, \gamma\sigma) && \text{with } c\cdot\sigma = \beta\cdot\tau s\\
&\sim (ats\rho, \gamma\sigma\rho) && \text{with } b\tau s\cdot\rho = \alpha\cdot ts\rho.
\end{aligned}$$

The rest of the proof is routine. It is easily seen that $a \to (a\gamma, \gamma)$ gives an isomorphism of $S$ into $\overline{T}$, and identifying $a$ with the class $\overline{(a\gamma, \gamma)}$ we get a semigroup $T$ from $\overline{T}$ with

$$\overline{(\gamma, \gamma)} = 1 \quad \text{and} \quad \overline{(\alpha, \beta)}^{-1} = \overline{(\beta, \alpha)},$$

and each element of $T$ may be written

$$\overline{(a, \alpha)} = \overline{(a\alpha, \alpha)}\overline{(\alpha, \alpha\alpha)} = a\alpha^{-1}$$

as a right quotient of $a \in S$ and $\alpha \in \Sigma$. This completes our proof.

We remark that generally there are different subsemigroups $\Sigma_i$ of $S$ which give the same s.o.r.q. $T = Q_r(S, \Sigma_i)$. Among these there is a maximal one, $\Sigma^*$, containing all the others. $\Sigma^*$ is the set of all elements of $S$ which have inverses in $T$ and is related to each $\Sigma_i$ by

$$\begin{aligned}
\xi \in \Sigma^* &\Leftrightarrow \xi \text{ left cancellable in } S,\ \xi x \in \Sigma_i \text{ for some } x \in S\\
&\Rightarrow x \in \Sigma^*.
\end{aligned} \tag{2}$$

The proof (cf. [10]) depends on the property that an element $a \in S$ which is left cancellable in $S$ is also left cancellable in each s.o.r.q. of $S$, a statement generally not valid for right cancellability.

We may call $\Sigma^*$ a *relatively maximal subsemigroup of right denominators of* $S$, and there is a 1-1 correspondence between all such $\Sigma^*$ and all s.o.r.q. of $S$. Moreover, among all subsemigroups of right denominators of $S$ (if this set is not empty) there is one, $\Sigma^{**}$, containing all the others, which we call the *absolutely maximal subsemigroup of right denominators of* $S$. Hence all s.o.r.q. $Q_r(S, \Sigma)$ of $S$ are contained in a *unique maximal s.o.r.q.* $Q_r(S, \Sigma^{**})$ of $S$. This statement (Satz 3, §4 in [10]) has a generalization in Theorem 3.

In dealing with s.o.r.q. it is sometimes useful that, by $q_r(S, \Sigma)$, a finite number of elements of $T = Q_r(S, \Sigma)$ may be written with the same denominator according to

$$(3)\qquad \begin{aligned} a\alpha^{-1} &= a\sigma(\alpha\sigma)^{-1}, \\ b\beta^{-1} &= b\beta^{-1}(\beta s)(\alpha\sigma)^{-1} = bs(\alpha\sigma)^{-1} \quad \text{with } \alpha\sigma = \beta s. \end{aligned}$$

Finally we shall make use of the left-right-dual concept of a *semigroup of left quotients* $Q_l(S, \Sigma)$ *with respect to* $\Sigma$, which exists if and only if $q_l(S, \Sigma)$ holds: For each $a \in S$ and $\alpha \in \Sigma$ there are $l' \in S$ and $\lambda' \in \Sigma$ with $\lambda' a = l'\alpha$. It may well happen for a subsemigroup $\Sigma$ of $S$ that $q_r(S, \Sigma)$ holds, but $q_l(S, \Sigma)$ not. If $Q_r(S, \Sigma)$ and $Q_l(S, \Sigma)$ both exist, they are equal because $a\alpha^{-1} = \lambda'^{-1}l'$, by $q_l(S, \Sigma)$, and $\alpha^{-1}a = l\lambda^{-1}$, by $q_r(S, \Sigma)$. This happens especially for each subsemigroup $\Sigma$ of cancellable and central elements of $S$ and the elements $a\alpha^{-1} = \alpha^{-1}a$ of $Q_r(S, \Sigma) = Q_l(S, \Sigma) = Q(S, \Sigma)$ turn out to behave like fractions in the familiar sense.

2. **Strict extensions.** Let $T$ be a semigroup and let $S$ be a subsemigroup of $T$. If no confusion is possible, we shall denote a partial order (briefly p.o.) $O_T$ of $T$ by $\leqq^T$ and a p.o. $O_S$ of $S$ merely by $\leqq$. Usually $O_T$ is said to be an extension of $O_S$, if, for all $a, b \in S$, $a \leqq b$ implies $a \leqq^T b$. If $O_S$ happens to be the restriction $O_T|S$ of $O_T$ on $S$, also the converse holds, and we define:

DEFINITION 2. *A p.o. $O_T$ of $T$ is called a strict extension of a p.o. $O_S$ of $S$, if, for all $a, b \in S$, $a \leqq b$ is equivalent to $a \leqq^T b$.*

THEOREM 2. *Let $T = Q_r(S, \Sigma)$ be a s.o.r.q. of a semigroup $S$ with respect to a subsemigroup $\Sigma$. Then a p.o. $O_S$ of $S$ has a strict extension $O_T$ on $T$, if and only if,*

$$\begin{aligned} &(4r)\qquad a\xi \leqq b\xi \\ &(4l)\qquad \xi a \leqq \xi b \end{aligned} \quad \textit{with } a, b \in S,\ \xi \in \Sigma \textit{ implies } a \leqq b.$$

*In this case $O_T$ is uniquely determined by $O_S$ and is given by* (5) *below. Moreover, an element $c \in S$ is positive* (*negative*) *in $S$ for $O_S$*(3), *if and only if, it is positive* (*negative*) *in $T$ for $O_T$.*

We remark that $(4r, l)$ is equivalent to

$$(4)\qquad \xi a\eta \leqq \xi b\eta \quad \text{with } a, b \in S,\ \xi, \eta \in \Sigma \text{ implies } a \leqq b.$$

Further, because the elements of $\Sigma$ are cancellable in $S$ (which is implied in these statements), $(4r, l)$ is equivalent to $a\xi < b\xi$ or $\xi a < \xi b \Rightarrow a < b$, and therefore to $a \| b \Rightarrow a\xi \| b\xi$ and $\xi a \| \xi b$ for all $\xi \in \Sigma$, where $\|$ denotes that the elements are incomparable.

**Proof.** Let $O_T$ be any p.o. of $T$ and let $O_S = O_T|S$ be its restriction on $S$. Then we have

$$(5)\qquad \begin{aligned} a\alpha^{-1} \leqq^T b\beta^{-1} &\Leftrightarrow \text{there exist } l \in S \text{ and } \lambda \in \Sigma \text{ with} \\ &\qquad \alpha\lambda = \beta l \text{ and } a\lambda \leqq bl. \end{aligned}$$

(3) This means $a \leqq ca$ and $a \leqq ac$ ($ca \leqq a$ and $ac \leqq a$) for all $a \in S$.

Given $\alpha, \beta \in \Sigma$ arbitrarily, by $q_r(S, \Sigma)$ there are $l \in S$ and $\lambda \in \Sigma$ with $\alpha\lambda = \beta l$. When the left side of (5) is multiplied by this element, we obtain $a\lambda \leqq^T bl$, i.e. $a\lambda \leqq bl$, and this multiplied by $\lambda^{-1}\alpha^{-1}$ gives $a\alpha^{-1} \leqq^T b(l\lambda^{-1})\alpha^{-1} = b(\beta^{-1}\alpha)\alpha^{-1} = b\beta^{-1}$. Therefore, $O_T$ is uniquely determined by its restriction $O_S$. Further, $O_S$ obeys $(4r, l)$, because we can use $\xi^{-1} \in T$.

Conversely, let $O_S$ be any p.o. of $S$ which fulfills $(4r, l)$. Then we define a relation $\leqq^T$ on $T$ by (5). We shall prove that this definition does not depend on the choice of $l$ and $\lambda$ and also not on the kind of representation of $a\alpha^{-1}$ and $b\beta^{-1}$. To do this assume $a\alpha^{-1} = a'\alpha'^{-1}$, $b\beta^{-1} = b'\beta'^{-1}$, $\alpha\lambda = \beta l$, $a\lambda \leqq bl$ and $\alpha'\lambda' = \beta' l'$. We have to show $a'\lambda' \leqq b'l'$. For this purpose we take $m \in S$, $\mu \in \Sigma$ according to $q_r(S, \Sigma)$ with $\alpha\lambda m = \alpha'\lambda'\mu$. Hence

$$\beta lm = \alpha\lambda m = \alpha'\lambda'\mu = \beta' l'\mu$$

and applyihg the second version of (i) in Theorem 1 $a\lambda m = a'\lambda'\mu$, $blm = b'l'\mu$. Therefore we have

$$a'\lambda'\mu = a\lambda m \leqq blm = b'l'\mu$$

and hence by $(4r)$ $a'\lambda' \leqq b'l'$. So our relation $\leqq^T$ is by (5) well defined. For the further steps we use that according to (3) a finite number of elements of $T$ can be written with the same denominator $\alpha$, and in this case (5) reduces to

$$(5') \qquad a\alpha^{-1} \leqq^T b\alpha^{-1} \Leftrightarrow a \leqq b.$$

Now obviously $\leqq^T$ is reflexive, antisymmetric and transitive. In order to prove monotony we start with (5′) and take

$$\begin{aligned} a\alpha^{-1}\cdot c\gamma^{-1} &= at(\gamma\tau)^{-1} \\ b\alpha^{-1}\cdot c\gamma^{-1} &= bt(\gamma\tau)^{-1} \end{aligned} \quad \text{both with } c\tau = \alpha t.$$

Because of $at \leqq bt$ we have $at(\gamma\tau)^{-1} \leqq^T bt(\gamma\tau)^{-1}$. A little bit more trouble gives the multiplication from the left side:

$$\begin{aligned} c\gamma^{-1}\cdot a\alpha^{-1} &= ct(\alpha\tau)^{-1} \quad \text{with } a\tau = \gamma t \\ c\gamma^{-1}\cdot b\alpha^{-1} &= cs(\alpha\sigma)^{-1} \quad \text{with } b\sigma = \gamma s. \end{aligned}$$

Here we first can assume $\tau = \sigma$, because the two elements $\gamma^{-1}a = t\tau^{-1}$ and $\gamma^{-1}b = s\sigma^{-1}$ can be written with the same denominator $\tau = \sigma$. From this we have $a\tau \leqq b\tau$, hence $\gamma t \leqq \gamma s$ and by $(4l)$(4) $t \leqq s$ which gives $ct \leqq cs$ as we were to show for the left-sided monotony law.

(4) Only here we need the assumption $(4l)$. If we introduce the concept of a *right-partially-ordered semigroup* (cf. [3]) omitting the left-sided monotony law in the definition of a p.o. semigroup, our Theorem 2 (with some corrections for the last statement) also holds for these structures only with the condition $(4r)$. Moreover, according to Proposition 2 in the next paragraph, each right-p.o. $O_S$ of $S$ has an extension to a right-p.o. $O_T$ for each s.o.r.q. $T = Q_r(S, \Sigma)$.

For the remaining statements about positive (and similarly for negative) elements we have: An element $c \in S$, which is right-positive in $S$ for $O_S$, is also left-positive in $S$ for $O_S$; indeed, $\xi \leqq \xi c$ for one element $\xi \in \Sigma$ implies $\xi a \leqq \xi ca$, hence by (4$l$) $a \leqq ca$, for all $a \in S$. From the latter it follows by (5′) $a\alpha^{-1} \leqq^T ca\alpha^{-1}$ for all $a\alpha^{-1} \in T$, so that such an element $c$ is left-positive in $T$ for $O_T$ and then also right-positive, because $T$ has an identity. The other direction is trivial.

From (5) it is clear that in Theorem 2 $O_T$ is a full order of $T$, if and only if, $O_S$ is a full order of $S$. Since for a full order $O_S$ of $S$ the conditions (4$r$, $l$) hold, we have the

COROLLARY. *A full order $O_S$ of a semigroup $S$ always has a unique extension to a full order $O_T$ on each s.o.r.q. $T = Q_r(S, \Sigma)$.*

In the case $\Sigma = S$ the statement of this Corollary was given in [4], and the general case is contained in the proof of a theorem about semirings in [11].

[As we already remarked in §1, there may be different subsemigroups $\Sigma_i$ of $S$ with $T = Q_r(S, \Sigma_i)$. It is a consequence of Theorem 2 that, concerning a certain p.o. $O_S$ of $S$, for those subsemigroups $\Sigma_i$ (4$r$, $l$) is simultaneously valid or not. The way this works becomes clearer if we use the relatively maximal subsemigroup $\Sigma^*$ with $T = Q_r(S, \Sigma^*)$. Then, by (2), we have immediately that (4$r$) with respect to one $\Sigma_i$ implies (4$r$) with respect to $\Sigma^*$ and of course conversely. In order to see the same for (4$l$) we have to use $q_r(S, \Sigma_i)$ and (4$r$). If we let $\xi$ be any element of $\Sigma^*$, then by (2) $\xi x \in \Sigma_i$, and we solve $\xi a\tau = \xi xt$, $\xi b\sigma = \xi xs$ with $\tau = \sigma \in \Sigma_i$. From $\xi a \leqq \xi b$ we get, using (4$l$) with respect to $\Sigma_i$, $t \leqq s$, hence $xt \leqq xs$, $a\tau \leqq b\tau$, and finally $a \leqq b$, which proves (4$l$) with respect to $\Sigma^*$.]

We conclude this paragraph with some results about those s.o.r.q. of $S$, for which a certain p.o. $O_S$ of $S$ has a strict extension.

PROPOSITION 1. *Let $S$ be a semigroup and $O_S$ a p.o. of $S$. If there is any subset $M = \{\alpha, \xi, \ldots\}$ of $S$ which obeys $q_r(S, M)$ and (4$r$, $l$) with respect to $M$, then the subsemigroup $\Sigma$ of $S$ generated by $M$ fulfills $q_r(S, \Sigma)$ and (4$r$, $l$) with respect to $\Sigma$.*

Since each element of $\Sigma$ is a product of a finite number of elements out of $M$, the first statement is clear, and the second follows by induction concerning the number of these factors similarly as was done in the proof of Satz 3, §4 in [10].

THEOREM 3. *If for a p.o. $O_S$ of a semigroup $S$ there exists at least one s.o.r.q. $T = Q_r(S, \Sigma)$ such that $O_S$ has a strict extension $O_T$ on $T$, then there is a unique maximal s.o.r.q. $T' = Q_r(S, \Sigma')$ for which a strict extension of $O_S$ exists, and $T'$ contains all s.o.r.q. of $S$ with this property.*

**Proof.** We apply Proposition 1, taking for $M$ the union of all subsemigroups $\Sigma_i$ of $S$ with $q_r(S, \Sigma_i)$ and (4$r$, $l$) with respect to $\Sigma_i$. Then we get $\Sigma'$ as the subsemigroup generated by $M$ and we have $T' = Q_r(S, \Sigma') \supseteq T_i = Q_r(S, \Sigma_i)$ for all $T_i$ in discussion.

## 3. Arbitrary extensions.

THEOREM 4. *Let $T=Q_r(S, \Sigma)$ be a s.o.r.q. of $S$ with respect to $\Sigma$. Then a p.o. $O_S$ ($\leqq$) of $S$ can be extended to a p.o. $O_T$ of $T$, if and only if, there is already in $S$ an extension $O'_S$ ($\leqq'$) of $O_S$ which obeys* (4$r$, $l$) *with respect to $\Sigma$, i.e.*

$$\begin{array}{ll} (6r) & a\xi \leqq' b\xi \\ (6l) & \xi a \leqq' \xi b \end{array} \quad \textit{with } a, b \in S,\ \xi \in \Sigma \textit{ implies } a \leqq' b.$$

This is an immediate consequence of Theorem 2, since if $O_T$ exists, the restriction $O_T|S$ is such an extension $O'_S$ of $O_S$. It reduces the question for the existence of an extension of $O_S$ on $T$ to the question, whether or not $O_S$ can already be extended in $S$ in a suitable way. A first step in this direction is the following:

PROPOSITION 2. *Under the assumptions of Theorem* 4, *i.e. if $\Sigma$ fulfills $q_r(S, \Sigma)$, each p.o. $O_S$ of $S$ can be extended to a p.o. $O'_S$ which obeys* (6$r$), *and the smallest extension of this kind is uniquely determined by*

$$(7r) \qquad a \leqq' b \Leftrightarrow \textit{there exists } a\ \xi \in \Sigma \textit{ with } a\xi \leqq b\xi.$$

**Proof.** We shall show that (7$r$) defines a p.o. $O'_S$ of $S$, if and only if, for $a, b \in S$, $\xi \in \Sigma$

(8$r$) $a\xi \leqq b\xi$ implies that for each $x \in S$ there is a $\lambda \in \Sigma$ with $ax\lambda \leqq bx\lambda$.

This condition is necessary since $a \leqq' b$ implies $ax \leqq' bx$ for $O'_S$. Conversely, a relation $\leqq'$ defined by (7$r$) clearly is an extension of $O_S$, hence reflexive. In order to check transitivity, assume $a \leqq' b$ and $b \leqq' c$ with $a\xi \leqq b\xi$, $b\eta \leqq c\eta$. From (8$r$) with $x=\eta$ we obtain $a\eta\lambda \leqq b\eta\lambda$, hence $a\eta\lambda \leqq b\eta\lambda \leqq c\eta\lambda$ which gives $a \leqq' c$. The same calculation with $c=a$ shows $a\eta\lambda = b\eta\lambda$, hence $a=b$, which proves antisymmetry (and that (7$r$) does not depend on the choice of $\xi \in \Sigma$). The left-sided monotony law is clear; the right-sided one is essentially (8$r$). Therefore (7$r$) defines a p.o. $O'_S$ of $S$, which clearly is the smallest extension of $O_S$ obeying (6$r$)([5]). Since $q_r(S, \Sigma)$ implies (8$r$) by $\xi l = x\lambda$, our proposition is established.

We remark that if $\xi a \leqq \xi b \Rightarrow a \leqq b$ already holds for $O_S$, then also $\xi a \leqq' \xi b \Rightarrow a \leqq' b$ for $O'_S$.

THEOREM 5. *Let $S$ be a semigroup and let $T=Q_r(S, \Sigma)=Q_l(S, \Sigma)$ be a semigroup of right as well as left quotients of $S$ with respect to $\Sigma$. Then any p.o. $O_S$ of $S$ has an extension to a p.o. $O_T$ of $T$, the smallest of which is uniquely determined. Moreover, an element $c \in S$ which is positive* (*negative*) *in $S$ for $O_S$ is positive* (*negative*) *in $T$ for $O_T$, but not conversely.*

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([5]) Observe that the statement formulated at the beginning of this proof is a condition for the existence of an extension $O'_S$ of $O_S$ with (6$r$) independent of $q_r(S, \Sigma)$. But it does not solve this problem at all, because even the smallest $O'_S$ with (6$r$) need not obey (7$r$).

**Proof.** By the remark above, Proposition 2 and its left-right-dual are applicable one after the other and we obtain an extension $O'_S$ of $O_S$ which obeys (6$r$) and (6$l$), given by

$$a \leqq' b \Leftrightarrow \text{there exist } \xi, \eta \in \Sigma \text{ with } \xi a\eta \leqq \xi b\eta. \tag{9}$$

Hence we get the existence of $O_T$ and the uniqueness of the smallest one by Theorem 2. Also the first statement about positive (negative) elements comes from Theorem 2, since an element $c$ which is positive (negative) in $S$ for $O_S$ preserves this property for any extension $O'_S$ of $O_S$ within $S$. But the converse of the latter fails to be true even in the commutative case (cf. [13]).

We are now going to give an *example of a p.o. $O_S$ of a semigroup $S$ which can not be extended to a p.o. $O_T$, where $T$ is any s.o.r.q. of $S$.* It is relatively complicated, but we have the conjecture that for a finitely generated semigroup $S$ each p.o. $O_S$ is extendable to a p.o. $O_T$ of $T$.

Let $S$ be the semigroup generated by the elements $a_1, a_2, \ldots, b_1, b_2, \ldots, c_1, c_2, \ldots, \xi$ with the defining relations

$$\begin{aligned} a_i\xi &= \xi a_{i+1} \\ b_i\xi &= \xi b_{i+1} \qquad i = 1, 2, \ldots. \\ c_i\xi &= \xi c_{i+1} \end{aligned} \tag{10}$$

It is easily checked that $S$ contains the free semigroup $F$ generated by $a_i$, $b_i$, $c_i$, $i=1, 2, \ldots$ and that each element of $S$ has the unique representation (using $S^1 = S \cup \{1\}$ for simplicity of notation)

$$\xi^i f \text{ with } i \geqq 0 \text{ and } f \in F^1, \text{ at least one factor } \neq 1. \tag{11}$$

Moreover, we have a monomorphism $A$ of $F$ defined by $a_i^A = a_{i+1}$, $b_i^A = b_{i+1}$, $c_i^A = c_{i+1}$. Therefore we have $f\xi = \xi f^A$ and the multiplication of $S$ is ruled by

$$\xi^i f \cdot \xi^j g = \xi^{i+j} f^{A^j} g. \tag{12}$$

From this it follows that $S$ is a cancellative semigroup. We omit the straight-forward calculations that the subsemigroup $\Sigma$ of $S$, generated by $\xi$, is the absolutely maximal subsemigroup of right denominators of $S$. Moreover, $T = Q_r(S, \Sigma)$ is the only s.o.r.q. of $S$ which exists at all, because a subsemigroup $\Sigma'$ of right denominators of $S$ is contained in $\Sigma$ and hence the corresponding relative maximal subsemigroup of right denominators coincides with $\Sigma$ by virtue of (2). On the other hand, $q_l(S, \Sigma)$ does not hold (we need not check this because it would give by Theorem 5 a contradiction to the further statements about our example), but there is even no subsemigroup $\Sigma''$ of left denominators in $S$. Let $\xi^i f$ according to (11) be an element of $\Sigma''$, this element and $c_1 f c_1$ have no left common multiple at all since $F^A$ does not contain an element with $c_1$ as a factor.

Now we introduce a p.o. on $S$ by defining the obviously irreflexive and asymmetric relation

$$(13) \qquad \xi^i f_1 < \xi^i f_2 \Leftrightarrow \begin{array}{l} \text{either } f_1 = fa_n g_1, f_2 = fb_n g_2,\ i \geqq n \text{ and } f \in (F^{A^n})^1 \\ \text{or } f_1 = fc_n b_n g_1, f_2 = fc_n a_n g_2 \end{array}$$

where $f$, $g_1$ and $g_2$ are suitable elements of $F^1$. One can look at this relation as a certain kind of lexicographic partial ordering by $c_n b_n < c_n a_n$ and $a_n < b_n$ for all $n$, where the latter only works with some supplementary conditions. From this point of view the proof of transitivity for the relation (13) becomes routine, though it is a little bit tedious.

In order to prove the monotony, no difficulties arise when multiplying (13) by $\xi$ from the left or by an element $f \in F$ from the right side. Also the multiplication with $\xi$ from the right side preserves (13), before $f_1^A$ and $f_2^A$ are in the same way related to each other as $f_1$ and $f_2$, observing $i \geqq n \Rightarrow i+1 \geqq n+1$ and $f \in F^{A^n} \Rightarrow f^A \in F^{A^{n+1}}$. Only the multiplication with an element $f \in F$ from the left side becomes problematic in the case where $f = \cdots c_k$ and $f_1 = a_n g_1, f_2 = b_n g_2$. But here the condition $i \geqq n$ works since $f^{A^i} = \cdots c_{k+i} \in F^{A^n}$ and we have $k+i > n$, which avoid the change from the first case in (13) to the second.

Now it is easy to see that this p.o. $O_S$ of $S$ can not be extended to a p.o. of $T = Q_r(S, \Sigma)$. From (13) we get especially $\xi a_1 < \xi b_1$, but $c_1 a_1 > c_1 b_1$. For an extension $O_T$ on $T$ or even for an extension $O'_S$ with $(6l)$ according to Theorem 4 we would obtain $a_1 <' b_1$ and $c_1 a_1 >' c_1 b_1$, so that the left-sided monotony law can not hold.

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