REARRANGEMENTS OF SERIES OF FUNCTIONS

BY
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1. Introduction and the statement of the main result. It is well known [3, p. 301] that a conditionally convergent series of real numbers can be rearranged in such a way so as to converge to any preassigned value. Suppose now we have a series of functions

\[ \sum_{n=1}^{\infty} x_n(t), \quad 0 \leq t \leq 1. \]

In this paper we shall be concerned with the studies of rearrangements of such series, the convergence being that of \( L_2(0,1) \). The work is motivated by the paper of E. Steinitz [4], who considered the rearrangements of conditionally convergent series of vectors in the finite dimensional Euclidean spaces. We are going to prove the following result:

Theorem 1. Suppose \( x_n(t) \) is a sequence of real valued functions, belonging to the real space \( L_2(0,1) \). Suppose also that:

(a) the series (1) converges in norm to some \( x \in L_2 \),
(b) \( \sum \|x_n\| = +\infty \),
(c) \( \sum \|x_n\|^2 < \infty \),
(d) the linear subspace \( M = \{ y \in L_2 : \sum \langle x_n, y \rangle < \infty \} \) is closed ((a, b) is the real inner product of \( a \) and \( b \)).

Then there exists a closed linear subspace \( N \) and a function \( x_0 \in L_2 \) such that:

I. any rearrangement of (1), which converges in norm, must have the limit of the form \( x_0 + z \), where \( z \in N \);
II. for any \( z \in N \), there exists a rearrangement of (1), which converges in norm to \( x_0 + z \).

In fact \( N = M^\perp \), i.e., \( N \oplus M = L_2 \).

2. Proof of Theorem 1. First we need several lemmas.

Lemma 1. Let \( x_1, \ldots, x_{n+1} \) be \( n+1 \) linearly dependent elements in \( L_2 \) and let

\[ x = \alpha_1 x_1 + \cdots + \alpha_{n+1} x_{n+1}, \quad 0 \leq \alpha_i \leq 1. \]

Then we can express \( x \) as

\[ x = \gamma_1 x_1 + \cdots + \gamma_{n+1} x_{n+1}, \quad 0 \leq \gamma_i \leq 1 \]

and at least one \( \gamma_i = 0 \) or 1.

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Proof. This Lemma is proved in [4, p. 167] for the case when \( x_i \in \mathbb{R}^n \) and the proof carries verbatim to the present situation.

**Lemma 2.** Let \( x_1, x_2, \ldots, x_n \in \mathbb{L}_2 \) and let
\[
x = \lambda_1 x_1 + \cdots + \lambda_n x_n, \quad 0 \leq \lambda_i \leq 1.
\]
Then there exists a vector \( x' \in \mathbb{L}_2 \) of the form
\[
x' = \delta_1 x_1 + \delta_2 x_2 + \cdots + \delta_n x_n, \quad \delta_i = 0 \text{ or } 1
\]
such that \( \| x - x' \|^2 \leq \sum_{i=1}^n \| x_i \|^2 \).

**Proof.** We proceed by induction on \( n \). The case \( n = 1 \) is clear since \( |\lambda_1| \leq 1 \). Suppose then the lemma is true for any \( n \) vectors in \( \mathbb{L}_2 \) and let
\[
x = \lambda_1 x_1 + \cdots + \lambda_n x_n + \lambda_{n+1} x_{n+1}, \quad 0 \leq \lambda_i \leq 1.
\]
We may write \( x_{n+1} = y_{n+1} + z_{n+1} \), where \( y_{n+1} \in \text{sp} \{ x_1, \ldots, x_n \} \) and \( (z_{n+1}, x_i) = 0 \), \( i = 1, 2, \ldots, n \). Clearly
\[
\| x_{n+1} \|^2 = \| y_{n+1} \|^2 + \| z_{n+1} \|^2,
\]
and the vectors \( x_1, \ldots, x_n, y_{n+1} \) are linearly dependent. Using Lemma 1 we can write (4) as
\[
x - \lambda_{n+1} z_{n+1} = \gamma_1 x_1 + \cdots + \gamma_n x_n + \gamma_{n+1} y_{n+1}, \quad 0 \leq \gamma_i \leq 1
\]
and at least one \( \gamma_{i_0} = 0 \) or 1.

We divide the remaining proof into several cases.

**Case 1.** \( i_0 = n+1 \), \( \gamma_{n+1} = 0 \). By the inductive hypothesis there exists a vector
\[
x' = \delta_1 x_1 + \cdots + \delta_n x_n, \quad \delta_i = 0 \text{ or } 1, \text{ such that}
\]
\[
\| x - \lambda_{n+1} z_{n+1} - x' \|^2 \leq \sum_{i=1}^n \| x_i \|^2.
\]
Since \( x = (x - \lambda_{n+1} z_{n+1}) + \lambda_{n+1} z_{n+1} \) and \( z_{n+1} \) is orthogonal to \( x - \lambda_{n+1} z_{n+1} \) (equation 5 and the conditions of this case) and \( x' \), we get from (6) and (3)
\[
\| x - x' \|^2 = \| x - \lambda_{n+1} z_{n+1} - x' \|^2 + \| \lambda_{n+1} \|^2 \| z_{n+1} \|^2 \leq \sum_{i=1}^n \| x_i \|^2.
\]

**Case 2.** \( i_0 < n+1 \) and \( \gamma_{i_0} = 0 \). We may assume without the loss of generality that \( i_0 = 1 \). By the inductive hypothesis there is a vector \( x'' = \delta_2 x_2 + \cdots + \delta_{n+1} y_{n+1}, \delta_i = 0 \text{ or } 1 \), such that
\[
\| x - \lambda_{n+1} z_{n+1} - x'' \|^2 \leq \sum_{i=1}^n \| x_i \|^2 + \| y_{n+1} \|^2.
\]
Let \( x' = x'' + \delta_{n+1} z_{n+1} = \delta_2 x_2 + \cdots + \delta_{n+1} x_{n+1} \). Since \( z_{n+1} \) is orthogonal to \( x_1, x_2, \ldots, x_n, y_{n+1} \), and \( |\lambda_{n+1} - \delta_{n+1}| \leq 1 \) we obtain from (7) and (3)
\[
\| x - x' \|^2 = \| x - \lambda_{n+1} z_{n+1} - x'' + (\lambda_{n+1} - \delta_{n+1}) z_{n+1} \|^2
\]
\[
= \| x - \lambda_{n+1} z_{n+1} - x'' \|^2 + |\lambda_{n+1} - \delta_{n+1}|^2 \| z_{n+1} \|^2
\]
\[
\leq \sum_{i=1}^n \| x_i \|^2 + \| y_{n+1} \|^2 + \| x'' \|^2 + \| z_{n+1} \|^2.
\]
Since the last 2 terms add up to \( \| x_{n+1} \|^2 \) we get the result.
Case 3. $i_0 = n + 1$, $y_{n+1} = 1$. Here

(8) \[ x - \lambda_{n+1} z_{n+1} = \gamma_1 x_1 + \cdots + \gamma_{n+1} x_n + y_{n+1}. \]

Let $x'' = \delta_1 x_1 + \cdots + \delta_n x_n$, $\delta_i = 0$ or 1, be such that

(9) \[ \| x - \lambda_{n+1} z_{n+1} - y_n - x'' \|_2 \leq \| x_1 \|^2 + \cdots + \| x_n \|^2 \]

and put $x' = x'' + x_{n+1} = x_{n+1} + y_{n+1} + z_{n+1}$. Since $|\lambda_{n+1} - 1| \leq 1$ and $z_1, z_2, \ldots, z_{n+1}$, we get from (9) and (8)

\[
\| x - x' \|^2 = \| x - \lambda_{n+1} z_{n+1} - y_n - x'' + (\lambda_{n+1} - 1) z_{n+1} \|^2
\]
\[
\leq \| x_1 \|^2 + \cdots + \| x_{n+1} \|^2.
\]

Case 4. $i < n + 1$. Here again we assume that $i_0 = 1$ and consequently $y_1 = 1$. We write now

\[ x - \lambda_{n+1} z_{n+1} - x_1 = \gamma_2 x_2 + \cdots + \gamma_n x_n + y_{n+1}. \]

Let $x'' = \delta_2 x_2 + \cdots + \delta_n x_n + \delta_{n+1} y_{n+1}$, $\delta_i = 0$ or 1, be such that

(10) \[ \| x - \lambda_{n+1} z_{n+1} - x_1 - x'' \|_2 \leq \| x_2 \|^2 + \cdots + \| x_{n+1} \|^2. \]

Put

\[ x' = x_1 + x'' + \delta_{n+1} z_{n+1} = x_1 + \delta_2 x_2 + \cdots + \delta_{n+1} x_{n+1}. \]

Noting again that $z_{n+1} \perp x_1, \ldots, y_{n+1}$, $|\lambda_{n+1} - \delta_{n+1}| \leq 1$ we get

\[
\| x - x' \|^2 = \| x - \lambda_{n+1} z_{n+1} - x_1 - x'' + (\lambda_{n+1} - \delta_{n+1}) z_{n+1} \|^2
\]
\[
\leq \| x_2 \|^2 + \cdots + \| y_{n+1} \|^2 + |\lambda_{n+1} - \delta_{n+1}|^2 \| z_{n+1} \|^2 \leq \| x_1 \|^2 + \cdots + \| x_{n+1} \|^2.
\]

This proves Lemma 2.

**Lemma 3.** Let $X = \{x_n\}$ be a sequence of elements of $L_2$. Let

\[ P(X) = \{x_{i_1} + x_{i_2} + \cdots + x_{i_k} : i_1 < i_2 < \cdots < i_k \}, \]

\[ Q(X) = \text{conv} P(X) \] (convex hull of $P(X)$).

\[ R(X) = \{y_1 x_{i_1} + \cdots + y_k x_{i_k} : 0 \leq y_i \leq 1, i_1 < i_2 < \cdots < i_k \}. \]

Then $Q(X) \subset R(X)$.

**Proof.** It is enough to show that $R(X)$ is convex, since $R(X) \supset P(X)$. Let now $y, z \in R(X)$. We may assume

\[ y = y_1 x_{i_1} + y_2 x_{i_2} + \cdots + y_k x_{i_k}, \quad z = \delta_1 x_{i_1} + \delta_2 x_{i_2} + \cdots + \delta_k x_{i_k} \]

by inserting the terms $0 \cdot x_j$ if necessary.

Let now $0 \leq \lambda \leq 1$. We have

\[ \lambda y + (1 - \lambda z) = \sum_j [\lambda y_j + (1 - \lambda) \delta_j] x_{i_j} \]

and since $0 \leq \lambda y_j + (1 - \lambda) \delta_j \leq \lambda + (1 - \lambda) = 1$ we get the result.

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Lemma 4. Let $N$ be a closed linear subspace of $L_2$ and let $B$ be a convex subset of $N$. Suppose that for any $x \in N$ and any $T > 0$, there exist elements $b_1$ and $b_2$ in $B$ so that $(x, b_1) \leq -T$ and $(x, b_2) \geq T$. Then $B$ is dense in $N$. (We recall that our $L_2$ is a real space.)

Proof. Consider $N$ as a Hilbert space and suppose the closure of $B$ is different from $N$. Let $K = \{x \in N : \|x - x_0\| < r\} \subset N \setminus \bar{B}$, for some $x_0 \in N$ and $r > 0$. Since $K$ and $\bar{B}$ are convex and $K$ has an interior point (in the relative topology of $N$), there exists a continuous linear functional $x'$ such that $x'(x) \leq c \leq x'(y)$, for all $x \in \bar{B}$, $y \in K$ for some constant $c$. (See [2, p. 412].) By the Riesz representation theorem for the Hilbert spaces, $x'(x) = (x, x')$ for some $x' \in N$. But this implies that $(x, x') \leq c$ for all $x \in B$, which contradicts the hypothesis of the theorem.

Lemma 5. Suppose $x_i \in L_2$, $\|x_i\| \leq M$, $i = 1, 2, \ldots, n$. Let

$$x_1 + x_2 + \cdots + x_n = a.$$  

Then we can rearrange the order of $x_i$'s, say into $\{x'_1, x'_2, \ldots, x'_n\}$, such that

$$\|x'_1 + x'_2 + \cdots + x'_{n-p}\|^2 \leq \|x'_1\|^2 + \cdots + \|x'_n\|^2 + \|a\|\|a\| + 2M, p = 1, 2, \ldots, n.$$  

Proof. First assume that $a = 0$ and call the right hand side of (12) $K$. We proceed to rearrange $x_1, \ldots, x_n$ as follows. Let $x'_1 = x_1$. Clearly $\|x'_1\|^2 \leq K$. On account of (11) we have $\sum (x'_1, x_i) = (x'_1, 0) = 0$ and the first term of the sum is equal to $\|x'_1\|^2 = 0$. Hence for some $x'_2$ among $x_2, \ldots, x_n$ we must have $(x'_1, x'_2) \leq 0$. From this it follows that

$$\|x'_1 + x'_2\|^2 = \|x'_1\|^2 + 2(x'_1, x'_2) + \|x'_2\|^2 \leq \|x'_1\|^2 + \|x'_2\|^2 \leq K.$$  

Consider next $\sum (x'_1 + x'_2, x_i) = (x'_1 + x'_2, 0) = 0$. The first 2 terms add up to $\|x'_1 + x'_2\|^2 \geq 0$; hence for some $x'_3$ among $x$'s different from $x'_1, x'_2$ we must have $(x'_1 + x'_2, x'_3) \leq 0$. So

$$\|x'_1 + x'_2 + x'_3\|^2 = \|x'_1 + x'_2\|^2 + 2(x'_1 + x'_2, x'_3) + \|x'_3\|^2 \leq K.$$  

Continuing in this fashion we get the result.

Suppose now $a \neq 0$. Then $\sum (x_i - n^{-1}a) = 0$; so we can order $x_i$'s in such a way that

$$\left\| \sum_1^n (x_i - \frac{1}{n}a) \right\|^2 \leq \sum_1^n \left\| x_i - \frac{1}{n}a \right\|^2 \leq \sum_1^n \left( \|x_i\|^2 + 2\frac{1}{n}\|x_i\|\|a\| + \frac{1}{n^2}\|a\|^2 \right) \leq (2M + \|a\|)\|a\| + \sum_1^n \|x_i\|^2.$$  

Q.E.D.

We are now ready to prove Theorem 1. Let $X = \{x_n : n = 1, 2, \ldots\}$ be the sequence of functions satisfying the conditions a–d. Each $x_n$ can be written as $x_n = y_n + z_n$, $y_n \in M$, $z_n \in M^\perp = N$. It is clear that if $\{x'_n\}$ is a rearrangement of $X$, then $\sum x'_n$
converges in $L^2$ if and only if $\sum y'_n$ and $\sum z'_n$ converge in $L^2$. We shall show first that $\sum y'_n$ converges for any rearrangements of $y$'s. For any $y \in M$ we have

\begin{equation}
\sum |(y_n, y)| = \sum |(x_n, y)| < \infty.
\end{equation}

Hence $\sum (y_n, y)$ converges absolutely, so every rearrangement of $\sum (y_n, y)$ will converge to the same limit, say $W(y)$. This shows that $\sum y_n$ converges weakly, and for every $y \in M$, every subseries of $\sum (y_n, y)$ will also converge. This implies [1, p. 60] that every subseries of $\sum y_n$ will converge, which in turn implies [1, p. 59 (1-b)] that every rearrangement of $\sum y_n$ converges in norm. The strong limit of $\sum y_n$ must be equal to the weak limit of $\sum y_n$. The weak limit of $\sum y_n$ is the same for every rearrangement, hence the strong limit must be independent of the rearrangement. This shows that $\sum y_n$ converges unconditionally in norm, i.e. for every rearrangement it converges in norm to the same limit, say $x_0$. This proves the first part of the theorem, since if a rearrangement of $\sum x_n$ converges in norm then the limit must be of the form $x_0 + \sum z'_n$ where $\sum z'_n \in N = M^1$.

What remains to be shown is that for every $w \in N$ there exists a rearrangement of $\sum z_n$ which converges in norm to $w$. We introduce the following notation.

Let $W = \{w_i : i=1, 2, \ldots\}$ be an arbitrary sequence of elements in $N$. Put

- $P(W) = \{w_{i_1} + w_{i_2} + \cdots + w_{i_k} : i_1 < i_2 < \cdots < i_k\}$,
- $Q(W) = \mathrm{co} P(W)$ (convex hull of $P(W)$),
- $R(W) = \{w_{i_1} + w_{i_2} + \cdots + w_{i_k} : 0 \leq w_{i_1} \leq 1, i_1 < i_2 < \cdots < i_k\}$.

Denote the elements of $P(W)$ by $p$. If $p = w_{i_1} + w_{i_2} + \cdots + w_{i_k}$ denote by $W-p$ the sequence $\{w_i : i \neq i_1, i_2, \ldots, i_k\}$. Put now $W$ to be the sequence $\{z_i : i=1, 2, \ldots\}$. Let $z \in N, z \neq 0$. Then

\begin{align}
& (14) \quad |\sum (z, z_n)| = |\sum (z, x_n)| < \infty \quad \text{since } \sum x_n \text{ converges} \\
& (15) \quad \sum |(z, z_n)| = \sum |(z, x_n)| = +\infty \quad \text{since } z \notin M.
\end{align}

Hence for any $z \neq 0$, $z \in N$, and any $T > 0$, there exist $p_1, p_2$ in $P(W)$ such that $(z, p_1) \geq T$ and $(z, p_2) \leq -T$. Lemma 4 shows then that $Q(W)$ is dense in $N$. Since $R(W) \supset Q(W)$ (Lemma 3), $R(W)$ is also dense in $N$. It follows then that the set $z + R(W) = \{z + r : r \in R(W)\}$ is also dense for every $z \in N$. The equations (14) and (15) also show that the sets $Q(W-p)$ and $z + R(W-p)$ are dense for every $p \in P(W)$ and $z \in N$. Hence we have shown

(A) for every $w \in N, z \in N, p \in P(W)$ and every $\varepsilon > 0$ there exists $r \in z + R(W-p)$ so that

\begin{equation}
\|w-r\| < \varepsilon.
\end{equation}

Choose $w \in N$. We shall now construct a rearrangement of $\sum z_n$ which converges to $w$. Let

\begin{align}
& (16) \quad \varepsilon_k \downarrow 0, \quad A_k^2 = \sum_{k} \|z_n\|^2 \quad \text{and} \quad B_k = \sup \{\|z_n\| : n \geq k\}.
\end{align}
By the hypothesis of the theorem $A_k \downarrow 0$, $B_k \downarrow 0$. We shall define a sequence of elements $w_k \in N$ as follows. Let $p_1 = z_1$. Choose $q_1 \in p_1 + R(z - p_1)$ such that $q_1 = z_1 + u_1$, $u_1 \in R(W - p_1)$ and so that

$$
\|w - p_1 - u_1\| < \epsilon_1.
$$

This is possible by (A). We have

$$
u_1 = \gamma_1 z_1 + \cdots + \gamma_j z_j, \quad 0 \leq \gamma_i \leq 1, \quad 1 < i_1 < \cdots < i_j.
$$

Using Lemma 2 we can find $t_1 = \delta_1 z_1 + \cdots + \delta_j z_j$, $\delta_i = 0$ or 1, such that

$$
\|u_1 - t_1\|^2 \leq \sum_{i=1}^{j} \|z_i\|^2 \leq A_1^2.
$$

Let $w_1 = z_1 + t_1 \in P(W)$. Clearly from (18), (19) we have

$$
\|w - w_1\| \leq \|w - z_1 - u_1\| + \|u_1 - t_1\| \leq \epsilon_1 + A_1.
$$

We now put $p_2$ as the first $z$ not used as a summand in $w_1$. Choose $q_2 \in p_2 + R(W - w_1 - p_2)$ such that $q_2 = p_2 + u_2$, $u_2 \in R(W - (p_1 + w_1))$ and

$$
\|w - w_1 - p_2 - u_2\| \leq \epsilon_2.
$$

Using Lemma 2 we choose $t_2 \in P(W - (p_2 + w_1))$ so that

$$
\|u_2 - t_2\|^2 \leq \sum_{i=2}^{\infty} \|z_i\|^2 = A_2^2.
$$

Let $w_2 = p_2 + t_2 \in P(W - w_1)$. We get again

$$
\|w - w_1 - w_2\| \leq \|w - w_1 - p_2 - u_2\| + \|u_2 - t_2\| \leq \epsilon_2 + A_2.
$$

Inductively it goes as follows. Suppose $w_1, w_2, \ldots, w_n$ are already defined and satisfy

$$
\|w - (w_1 + \cdots + w_k)\| \leq \epsilon_k + A_k,
$$

$$
w_k \in P(W - (w_1 + \cdots + w_{k-1})), \quad k = 1, 2, \ldots, n,
$$

$$
z_1, z_2, \ldots, z_k \text{ are included as summands in } w_1 + w_2 + \cdots + w_k, k = 1, 2, \ldots, n.
$$

Choose $p_{n+1}$ to be the $z$ with the smallest subscript not included in the sum $w_1 + \cdots + w_n$.

Let $q_{n+1} \in p_{n+1} + R(W - (p_{n+1} + w_1 + \cdots + w_n))$ be such that $q_{n+1} = p_{n+1} + u_{n+1} \in R(W - (p_{n+1} + \cdots + w_n))$ and

$$
\|w - (w_1 + \cdots + w_n + p_{n+1} + u_{n+1})\| \leq \epsilon_{n+1}.
$$

This is possible by (A). Using Lemma 2 choose $t_{n+1} \in P(W - (w_1 + \cdots + w_n + p_{n+1}))$ so that

$$
\|u_{n+1} - t_{n+1}\|^2 \leq \sum_{n=1}^{\infty} \|z_i\|^2 \leq A_{n+1}^2.
$$
Put \( w_{n+1} = p_{n+1} + t_{n+1} + f_{n+1} \in P(W - (w_1 + \cdots + w_n)) \). We have from (24), (25)

\[
(26) \quad \| w - (w_1 + \cdots + w_{n+1}) \| \leq e_{n+1} + A_{n+1}.
\]

Clearly \( w_1, w_2, \ldots, w_{n+1} \) satisfy (21), (22), (23). Thus the sequence \( w_n \) is defined, every \( z_i \) is included as a summand and in exactly one \( w_n \) and

\[
(27) \quad \left| w - \sum_{i=1}^{n} w_i \right| \leq e_n + A_n.
\]

This shows that \( \sum w_n \) defines a rearrangement of \( \sum z_n \), say \( \sum z_n^* \) such that

\[
(28) \quad w_n = z_{i_1}^* + \cdots + z_{i_{n+1}}^*
\]

and

\[
(29) \quad \left| w - \sum_{i=1}^{n+1} z_i^* \right| \leq e_n + A_n.
\]

Now we rearrange each of the sums \( w_n \) in such a way that the resulting rearrangement \( \sum z_i^* \) will converge to \( w \). Let

\[
(30) \quad w_n = z_{i_1}^* + \cdots + z_{i_n}^*, \quad n \leq i_1 < i_2 < \cdots < i_k.
\]

Using Lemma 5 we can rearrange \( z_i^* \)'s in such a way that

\[
(31) \quad \left\| z_{i_1}^* + \cdots + z_{i_p}^* \right\|^2 \leq \left\| z_{i_1}^* \right\|^2 + \cdots + \left\| z_{i_p}^* \right\|^2 + \left\| w_n \right\| \left( \left\| w_n \right\| + 2B_n \right) \leq A_n^2 + \left\| w_n \right\| \left( \left\| w_n \right\| + 2B_n \right), \quad p = 1, 2, \ldots, k.
\]

Since

\[
\left\| w_n \right\| \leq \left\| w - \sum_{i=1}^{n} w_i \right\| + \left\| w - (w_1 + \cdots + w_n) \right\| \leq e_{n-1} + A_{n-1} + e_n + A_n = d_n
\]

we have

\[
(32) \quad \left\| z_{i_1}^* + \cdots + z_{i_p}^* \right\|^2 \leq A_n^2 + d_n(d_n + B_n), \quad p = 1, 2, \ldots, k.
\]

Let now \( \sum z_i^* \) be the resulting rearrangement. Let \( N \) be a positive integer and choose \( n = n(N) \) such that

\[
(33) \quad w_{n+1} = z_{i_1}^* + z_{i_{n+1}}^* + \cdots + z_{i_k}^*, \quad i_1 \leq N \leq i_k.
\]

Then

\[
\left| w - \sum_{i=1}^{N} z_i^* \right| \leq \left| w - \sum_{i=1}^{n} w_i \right| + \left| z_{i_1}^* + \cdots + z_{i_k}^* \right| \leq e_n + A_n + [A_n^2 + d_n(d_n + B_n)]^{1/2}.
\]

It is clear that \( n(N) \to \infty \) as \( N \to \infty \); so the right hand side of (33) tends to 0 as \( N \to \infty \). This completes the proof of Theorem 1.
3. Examples.

1. \( M = \{ y : \sum |(x_n, y)| < \infty \} \) need not be closed, even if the conditions a–c of Theorem 1 hold.

Let \( \{ e_n : n = 1, 2, \ldots \} \) be an orthonormal basis of \( L_2 \). Put

\[
E_n = [(n+2) \log (n+2)]^{-1} \quad x_n = (-1)^n + e_n + \cdots + e_n
\]

(34)

\[
x = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (-1)^{j+1} l_j e_i = \sum_{i=1}^{\infty} \alpha_i e_i.
\]

It follows that

\[
(35) \quad \|x_n\|^2 = n l_n^2 < \frac{1}{[n \log n]^2}
\]

so \( \sum \|x_n\|^2 < \infty, \sum \|x_n\| = +\infty. \) Since \( \sum (-1)^{n+1} l_n \) is an alternating series, we have the following estimate

\[
(36) \quad \left| \sum_{i=p}^{\infty} (-1)^{n+1} l_n - \sum_{i=p}^{N} (-1)^{n+1} l_i \right| < |l_{N+1}|.
\]

Hence

\[
(37) \quad \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (-1)^{n+1} l_n^2 = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (-1)^{n+1} l_n^2 - \sum_{j=1}^{N} l_n \leq \sum_{i=1}^{\infty} |l_i|^2 < \infty
\]

so \( x \in L_2 \). Moreover using (36) we obtain

\[
(38) \quad \left\| \sum_{1}^{N} x_n - x \right\| = \left\| \sum_{1}^{N} x_n - \sum_{1}^{N} \alpha_n e_n \right\| + \left\| \sum_{N+1}^{\infty} \alpha_n e_n \right\|
\]

The last term clearly goes to 0 as \( N \to \infty. \) The first term is equal to

\[
\sum_{n=1}^{N} \sum_{i=1}^{N} (-1)^{n+1} l_i - \sum_{i=1}^{\infty} (-1)^{n+1} l_i^2 = \sum_{n=1}^{N} |l_{N+1}| = N l_{N+1} \to 0 \quad (N \to \infty).
\]

Hence \( \sum x_n \) converges to \( x \) in norm. Put now

\[
t_i = [\log (\log (i+2))]^{-1}
\]

(39)

\[
y_1 = 0, \quad y_k = t_i e_1 + \sum_{i=1}^{k-1} (t_{i+1} - t_i) e_i - t_k e_k, \quad k \geq 2,
\]

\[
y = t_i e_1 + \sum_{i=1}^{\infty} (t_{i+1} - t_i) e_i.
\]

Since

\[
t_{i+1} - t_i = \int_{t}^{t+1} \frac{d}{dx} \log (\log (x+2)) \, dx
\]

(40)

\[
= -\int_{t}^{t+1} [(x+2) \log (x+2)(\log (\log (x+2)))^2]^{-1} \, dx
\]
so

\[ |t_{i+1} - t_i| \leq \frac{1}{(i+2) \log (i+2)[\log (i+2)]^2} \]

and hence \( y \in L_2 \). Moreover

\[ \|y - y_k\|^2 \leq |t_k|^2 + |t_{k+1} - t_k|^2 + \sum_{i=k}^{\infty} |t_{i+1} - t_i|^2 \to 0 \quad (k \to \infty) \]

so \( y_k \) converges in norm to \( y \). We also note that \( |(x_n, y_k)| = 0 \) for \( n > k \), hence \( \sum |(x_n, y_k)| < \infty \) for all \( k \). However \( |(x_n, y)| = l_n t_n \) and since

\[ \sum \frac{1}{(i+2) \log (i+2) \log (i+2))} = \infty \quad y \notin M, \]

this shows that \( M \) need not be closed.

2. Next we give an example of a series \( \sum x_n \) for which \( N = L_2 \), or equivalently \( M = \{0\} \). Let \( \{e_k\} \) be an orthonormal basis of \( L_2 \). Define

\[ x_n^{(k)} = (-1)^{n+1} 2^{-n} e_k. \]

Since \( \|x_n^{(k)}\|^2 = 2^{-2n} n^{-2} \) then \( \sum \sum_k \|x_n^{(k)}\|^2 < \infty \). It is also clear that \( \sum_n k \|x_n^{(k)}\| = \infty \). We can order \( \{x_n^{(k)}\} \) into a single sequence \( \{x_n\} \) such that \( \sum x_n \) converges. We proceed by induction as follows. Let \( x_1 = x_1^{(1)} \), \( x_2 = x_2^{(2)} \) and \( x_3 = x_3^{(3)} \). Suppose we have ordered all the elements \( x_n^{(k)} \), \( n + k \leq m \), into a sequence \( x_1, x_2, \ldots, x_N \) such that if \( x_i \) corresponds to \( x_n^{(k)} \), \( x_j \) corresponds to \( x_n^{(k+1)} \) then \( i < j \). We put

\[ x_{N+1} = x_1^{(m)}, \quad x_{N+2} = x_2^{(m-1)}, \ldots, x_{N+j} = x_j^{(m-j+1)}, \quad x_{N+m} = x_m^{(1)}. \]

In this way we have ordered all the elements \( \{x_n^{(k)}\} \) in such a way that if \( x_i \) corresponds to \( x_n^{(k)} \) and \( x_j \) corresponds to \( x_n^{(k+1)} \) then \( i < j \). Hence for each \( k \)

\[ \sum_i (x_i, e_k) = \frac{1}{2^k} \sum_j \frac{(-1)^{j+1}}{j} = \frac{1}{2^k} \log 2 = y_k. \]

Put

\[ x = \sum \frac{1}{2^k} \log 2 e_k = \sum k y_ke_k. \]

We have for any positive integer \( N \)

\[ \left| \sum_{i=1}^{N} (-1)^{j+1} \frac{1}{2^j} \frac{1}{2^k} \log 2 \right| \leq \frac{1}{2^k N}. \]

Let \( N \) be a positive integer and choose \( p \) to be the largest integer such that \( x_1^{(p)} \) is included among \( x_1, x_2, \ldots, x_N \). Then by the construction of \( x_i \)'s

\[ \left| \sum_{i=1}^{N} x_i - x \right| \leq \left| \sum_{p=1}^{N} x_i - \sum_{p=1}^{N} y_i e_i \right| + \left| \sum_{p+1}^{N} x_i - \sum_{p+1}^{N} y_i e_i \right| + \left| \sum_{N+1}^{\infty} y_i e_i \right|. \]
The last sum clearly goes to 0 as \( N \to \infty \). By (44) the first and the second sum are each smaller than
\[
\frac{1}{p} \frac{1}{p-1} + \frac{1}{p-2} \frac{1}{2} + \frac{1}{2} \frac{1}{2^2} + \cdots + \frac{1}{2^{p-1}} = S_p
\]
which goes to 0 as \( p \to \infty \). Since \( p \to \infty \) as \( N \to \infty \) we have shown that \( \sum x_i \) converges to \( x \) in norm. Let now \( y \neq 0 \), \( y \in L_2 \), \( y = \sum b_k e_k \). Then
\[
\sum |(x_i, y)| \geq \sum |(x_i^{(k)}, y)| = \sum \frac{b_k}{2^k} = \infty
\]
unless \( b_k = 0 \). Hence \( M = \{0\} \) and every function in \( L_2 \) can be obtained as a limit of some rearrangement of \( \sum x_i \).

REFERENCES


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