UNIFORM APPROXIMATION ON A REAL-ANALYTIC MANIFOLD

BY

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1. Introduction. Let $M$ be a compact subset of a real-analytic manifold of dimension $n$ and $F$ a set of real-analytic complex-valued functions on $M$ which separates $M$, meaning that for each pair $p, q$ of distinct points in $M$ there is a function $f$ in $F$ with $f(p) \neq f(q)$. We wish to study the Banach algebra $A$ obtained as the closure in the norm of uniform convergence on $M$ of the algebra of all polynomials in the functions of $F$ (including constants). Thus $A$ is the smallest closed subalgebra of $C(M)$ with identity which contains $F$, where $C(M)$ is all continuous complex-valued functions on $M$. Closed, separating subalgebras of $C(M)$ with identity are frequently called function algebras, and the term is used elsewhere in much more general circumstances, where arbitrary compact Hausdorff spaces are admitted for $M$.

As we note below, the study of this type of function algebra includes the classical problem of uniform polynomial approximation on certain polynomially convex subsets of complex Euclidean space $\mathbb{C}^n$.

Our study of these algebras continues a program initiated by Wermer [12], [13], and treated by the author [2], Wells [11], Nirenberg and Wells [9], [10], and very recently by Hörmander and Wermer [6]. In all of this work it has been shown that the set

$$E = \{p \in M : df_1 \wedge \cdots \wedge df_n(p) = 0 \text{ for all } n\text{-tuples } \{f_1, \ldots, f_n\} \text{ of functions in } F\}$$

plays a major role in determining the structure of $A$. Our principal object in this paper is to prove a result announced earlier [3].

**Theorem 1.** If $M_A = M$ then $A$ contains the ideal of all continuous complex-valued functions which vanish on $E$.

Here $M_A$ is the spectrum or maximal ideal space of the Banach algebra $A$, and consists of all algebra homomorphisms of $A$ onto $\mathbb{C}$. Each point $p$ of $M$ provides such a homomorphism, defined by sending a function $f$ in $A$ into $f(p)$. The hypothesis $M_A = M$ means that all homomorphisms arise in this manner. It is a necessary condition for the conclusion when $E$ is empty and in certain other cases [2]. We refer to [2] for further properties of $E$.

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The theorem says nothing about the behavior of $A$ on $E$, a problem of great interest since the theorem shows it equivalent to the problem of describing $A$ (it is easy to see that a continuous function $f$ on $M$ is in $A$ if and only if its restriction to $E$ coincides with that of some function in $A$). In the case where $M$ is contained in a real-analytic submanifold of $C^n$, and with the usual coordinate functions comprising $F$, $A$ is the algebra of all continuous functions on $M$ which can be approximated uniformly by polynomials. Here the condition $M_A = M$ is equivalent to the assertion that $M$ is polynomially convex [5], [14]. In this case Hörmander and Wermer [6], and earlier Wermer [13] in a special case, have shown that $A$ is the set of all continuous functions on $M$ which admit uniform approximation on $E$ by functions holomorphic in a neighborhood of $E$. Their result of course contains Theorem 1 in this case. Wermer [13] also gives a differential description of $A$ in a special case. A description of this type is conjectured in [3] for the more general situation treated here, but we have not proved it.

The author proved Theorem 1 when $n=2$ in [2], by extending techniques first used by Wermer [13]. The same basic ideas are used here, but a number of modifications and extensions have been necessary to adapt the earlier proof to manifolds of dimension greater than two.

A simple example which verifies Theorem 1 is obtained when $$M = \{(x, y, t) \in R^3 : x^2 + y^2 + t^2 \leq 1\}$$ is the closed unit ball in real Euclidean space $R^3$ and $F = \{f, g, h\}$ where

$$f(x, y, t) = x + iy = z, \quad g(x, y, t) = tz, \quad h(x, y, t) = t.$$

Then the function $(f, g, h): M \rightarrow C^3$ with the indicated coordinates maps $M$ homeomorphically onto a compact polynomially convex [4], [5] subset of $C^3$, so $M_A = M$. Clearly, $E = \{(x, y, t) : t = 0\}$. Moreover, $A$ contains $fh$, $g$, and $h$, which together separate $M - E$, which have no common zero there, and which generate an algebra closed under complex conjugation. An application of the Stone-Weierstrass theorem now shows that $A$ contains every continuous function which vanishes on $E$.

2. Proof of Theorem 1. The argument is similar and in some places identical to that used before [2]. We show again that every bounded regular complex Borel measure which annihilates $A$ also annihilates every continuous function which vanishes on $E$. This follows from Theorem 2 below in exactly the same way as it did in [2]. Theorem 2 reduces the study of such a measure $\mu$ to the study of the family of bounded, regular, compactly supported Borel measures $f^*\mu$ induced on the plane from $\mu$ by certain functions in $A$. These measures are defined for each Borel set $E$ by $f^*\mu(E) = \mu(f^{-1}(E))$ and their relevant elementary properties are listed in [2].

We write $\mu \perp f$ if $\int f \, d\mu = 0$ and write $\mu \perp A$ if this holds for all functions in $A$. If $f_1, \ldots, f_k$ are functions on $M$ we denote by

$$(f_1, \ldots, f_k): M \rightarrow C^k$$

the map with these functions as coordinates.
The following well-known facts are collected as lemmas for future use.

**Lemma 1.** If $\nu$ is a bounded regular complex Borel measure with compact support in $C$, then

1. $\int d|\nu|(\lambda)|\lambda - z|$ is finite for almost all $z$ in $C$ (in the sense of Lebesgue measure), and

2. if $K$ is compact in $C$ and $\int d\nu(\lambda)|(\lambda - z)=0$ for almost all $z$ in $C - K$ (in the same sense), then support $\nu \subset K$.

In particular, if (1) holds for almost all $z$ in $C$, then it follows from (2) that $\nu = 0$.

For a proof, the reader is referred to [14].

**Lemma 2.** If $M$ is a compact subset of a complex manifold and $G$ is a holomorphic map of a neighborhood of $M$ into $C^n$ which is injective and nonsingular on $M$, then $G$ is injective and nonsingular on some neighborhood of $M$.

**Lemma 3.** Let $X$ be a compact Hausdorff space and $F$ a subset of $C(X)$ satisfying

3. $F$ separates $X$, and

4. for each $x$ in $X$ there exists a finite subset of $F$ which separates some neighborhood of $x$.

Then there exists a finite subset of $F$ which separates $X$.

This type of result has been used by Narasimhan [8], and was brought to my attention by H. Rossi.

**Proof.** Property (4) and compactness yield open sets $U_1, \ldots, U_k$ and a finite subset $\{g_1, \ldots, g_l\}$ of $F$ which separates each $U_i$. Property (3) and the compactness of $X \times X$ provide open sets $V_1, \ldots, V_p$, $W_1, \ldots, W_p$ and functions $g_{i+1}, \ldots, g_{i+p}$ in $F$ such that

$$X \times X - \bigcup_{i=1}^k U_i \times U_i \subset \bigcup_{i=1}^p V_i \times W_i,$$

and

$$g_{i+j}(V_i) \cap g_{i+j}(W_j) = \emptyset, \quad j = 1, \ldots, p.$$

It follows easily that $\{g_1, \ldots, g_{l+p}\}$ separates $X$.

**Lemma 4.** Let $M$ be a compact Hausdorff space and $F$ a separating subset of $C(M)$. If $U$ is open in $M$ and $\{f_1, \ldots, f_k\}$ is a subset of $F$ which separates $U$, then for each compact subset $K$ of $U$ there exist functions $f_{k+1}, \ldots, f_m$ in $F$ such that $(f_1, \ldots, f_m)(K)$ is disjoint from $(f_1, \ldots, f_m)(M - K)$.

**Proof.** Since $F$ separates $M$, a standard compactness argument shows that there exist functions $f_{k+1}, \ldots, f_m$ in $F$ such that

$$(f_{k+1}, \ldots, f_m)(K)$$

is disjoint from $(f_{k+1}, \ldots, f_m)(M - U)$.

It is straightforward to verify that $(f_1, \ldots, f_m)$ has the stated property.
Theorem 2. If \( M_A = M, \mu \bot A, \) and \( f \) is a polynomial in the functions of \( F, \) then

\[
\int_C \frac{d(f \ast \mu)(\lambda)}{\lambda - a} = 0
\]

for almost all points \( a \) in \( C - f(E). \) Thus by Lemma 1, support \( f \ast \mu \subset f(E). \)

Proof. As in [2], it will suffice to show that \( \int d\mu/(f - a) = 0 \) for all points \( a \) in \( C - f(E) \) for which the integral is absolutely convergent. For each such \( a, \) we construct a sequence \( \{f_n\} \) of functions in \( A \) such that

\[
f_n \to 1/(f - a) \text{ a.e. } |\mu|, \text{ and } |f_n| \leq 2/|f - a| \text{ a.e. } |\mu|.
\]

These functions are obtained as before from the solution of a certain Cousin Problem I on a domain in \( C^p. \) This problem has the same basic structure as it did in [2], but is somewhat more complicated because of the higher dimension. To set it up, we appeal to a result of Whitney and Bruhat [15], which states that there exists a complex manifold \( \tilde{M} \) in which the ambient manifold of \( M \) can be imbedded as a real-analytic submanifold in such a way that every real-analytic function on \( M \) can be extended to a holomorphic function on an \( \tilde{M} \)-neighborhood of \( M. \) While some of the constructions below could be executed on \( \tilde{M}, \) it is more convenient to transfer immediately to complex Euclidean space. The basic idea of the proof is clearest when \( F \) is finite, so we present that case first. Technical modifications required to handle the general situation are given afterwards.

Case of finite \( F. \) Here the functions in \( F \) comprise the coordinates of a map

\[
H = (f_1, \ldots, f_p): M \to C^p.
\]

By our assumption, \( f = q \circ H \) for some polynomial \( q \) in \( p \) variables. Given a point \( a \) in \( C - f(E) \) we claim that:

There exists an open set \( W \supset H(M) \cap q^{-1}(a) \) and a function \( k \) holomorphic on \( W \) such that \( k|H(M) \cap W = (q - a)^*|H(M) \cap W. \) Here the * denotes complex conjugation.

There is an \( \tilde{M} \)-open set \( \tilde{V} \) which contains \( f^{-1}(a) \) and to which \( H \) has a holomorphic extension \( \tilde{H} = (\tilde{f}_1, \ldots, \tilde{f}_p). \) Since \( a \) is not in \( f(E) \) the \( p \)-form \( df_1 \wedge \cdots \wedge df_p \) has no zeros on \( f^{-1}(a) \) so the same is true of \( d\tilde{f}_1 \wedge \cdots \wedge d\tilde{f}_p. \) Since \( \tilde{H} \) is injective on \( M \) we can use Lemma 2 to choose \( \tilde{V} \) small enough so that \( \tilde{H} \) imbeds \( \tilde{V} \) as a complex submanifold \( V \) of \( C^p. \)

We may assume that \( (f - a)^* \) has a holomorphic extension to \( \tilde{V}. \) Now \( V \) is a closed submanifold of some open set \( U \) in \( C^p \) (for instance, a union of ambient coordinate neighborhoods whose slices define \( V \) locally), and since \( H(M \cap \tilde{V}) \) is disjoint from \( H(M - \tilde{V}) \) we can remove the compact set \( H(M - \tilde{V}) \) from \( U \) to obtain \( V \) as a closed submanifold of the open set \( U \) and satisfying

\[
U \cap H(M) = V \cap H(M) = H(M \cap \tilde{V}).
\]
The set $H(M) \cap q^{-1}(a)$ is polynomially convex [5], as a consequence [14] of our assumption that $M_A = M$. Therefore there exists [5] a domain of holomorphy $W$ such that $H(M) \cap q^{-1}(a) \subset W \subset U$. We have an extension of $(f - a)^* = (q - a)^* \circ H$ to a holomorphic function on $\tilde{V}$. Because of (5), composition of this extension with $(\tilde{H}^{-1} : V \to \tilde{V})$ yields a holomorphic extension to $V$ of $(q - a)^* | H(M) \cap V$. The theorem of Grauert and Docquier [4, Theorem 8, pp. 257–258], which says that $W \cap V$ is a holomorphic retraction of $W$, finally yields the desired extension $k$ on $W$.

We wish now to proceed along lines similar to [2], and construct functions $h$ and $h_1$ in $A$ such that $h = (f - a)h_1$ and $h(M) \subset \{w : |w - 1| > 1\} \cup \{0\}$. A “local” solution to this problem is given by $h_1 = -k \circ H$ and $h = (f - a)h_1$. However $k \circ H$ is not in $A$, and we wish to use $k$ to obtain a holomorphic function $\psi$ with appropriate divisibility properties in a neighborhood of $H(M)$. Since $H(M)$ is polynomially convex, the Oka-Weil theorem [5, Theorem 2.7.7, p. 55] will imply that $h = -\psi \circ H$ is in $A$. Let $g = (q - a)k$, a function holomorphic on $W$.

Then there exists a function $\psi$ holomorphic on a neighborhood of $H(M)$ such that

1. $\psi$ has no zeros on $H(M) \cap q^{-1}(a)$,
2. $\psi$ is divisible by $g$ in a neighborhood of $H(M) \cap q^{-1}(a)$ and the holomorphic quotient $\psi/g$ has the value 1 everywhere on $H(M) \cap q^{-1}(a)$.

This function $\psi$ is exhibited as the solution to a Cousin Problem I [5] with data determined from $g$ as follows. Since $g|H(M) \cap W = |q - a|^2|H(M) \cap W|$, it follows that

$$H(M) \cap \{\Re g = 0\} = H(M) \cap q^{-1}(a).$$

Therefore by shrinking $W$ if necessary, it can be assumed that $\{\Re g = 0\}$ is disjoint from $H(M) - W$ (here the superscript bar denotes closure).

Thus $C^p - q^{-1}(a) - \{\Re g = 0\}$ is an open set containing $H(M) - W$, so that $W$ and this set constitute an open cover of $H(M)$. Since $H(M)$ is polynomially convex, there exists an open domain of holomorphy $S$ in $C^p$ such that

$$H(M) \subset S \subset W \cup [C^p - q^{-1}(a) - \{\Re g = 0\}].$$

Replacing $W$ by its intersection with $S$, we have $W \subset S$. Since $\{\Re g = 0\}$ is closed in $S$, the set $T = S - q^{-1}(a) - \{\Re g = 0\}$ is open and $S = W \cup T$.

We have designed $W$, $T$, and $g$ so that $g$ has a holomorphic logarithm $\log g$ on $W \cap T = W - q^{-1}(a) - \{\Re g = 0\}$, and so that $(\log g)/(q - a)$ is holomorphic on $W \cap T$. Therefore the Cousin Problem I defined on $S$ for the covering $\{W, T\}$ by $(\log g)/(q - a)$ has a solution [5]; that is, there exist functions $g_1$ holomorphic on $T$ and $g_2$ holomorphic on $W$ such that

$$g_1 - g_2 = (\log g)/(q - a) \text{ on } W \cap T.$$

Thus $\log g + (q - a)g_2 = (q - a)g_1$ on $W \cap T$ so the holomorphic functions

$$g \exp ((q - a)g_2) \text{ on } W \quad \text{and} \quad \exp ((q - a)g_1) \text{ on } T$$
coincide on $W \cap T$. Hence they define a holomorphic function $\psi$ on $S$ with the desired properties.

This result is used to construct functions $h$ and $h_1$ in $A$ such that

(8) $h$ has no zeros on $H(M) - q^{-1}(a)$,

(9) $h = (f - a)h_1$, and

(10) $h(M) = \{ w : |w - 1| > 1 \} \cup \{0\}$.

From (6) and (7) it follows that $\psi_1 = \psi/(q - a)$ is holomorphic in a neighborhood of $H(M)$. By (7) there exists an $H(M)$-neighborhood $P$ of $H(M) \cap q^{-1}(a)$ on which $\text{Re}(\frac{\psi}{g}) > 0$. It follows from this and the positivity of $g$ on $H(M) - q^{-1}(a)$ that $\text{Re} \psi > 0$ on $P - q^{-1}(a)$. The function $\psi$ has no zeros on the compact set $H(M) - P$, so after multiplication of $\psi$ by a suitable positive constant, it will satisfy $|\psi| \geq 2$ on $H(M) - P$. We have already noted that $h = -\psi \circ H$ and $h_1 = -\psi_1 \circ H$ are in $A$, and they clearly have the properties (8), (9), and (10).

These functions are used exactly as in [2, p. 54] to construct the sequence $\{f_n\}$. There we defined the rational functions $\phi_n$ by

$$\phi_n(w) = \frac{1}{w} \left(1 - \frac{1}{(w - 1)^2}\right), \quad n = 1, 2, \ldots$$

and showed easily that the sequence $\{f_n\}$ of functions in $A$ defined by $f_n = (\phi_n \circ h)h_1$, $n = 1, 2, \ldots$ has the properties set forth at the beginning of the proof. Theorem 2 is thereby proved when $F$ is finite.

**Proof of Theorem 2 for arbitrary $F$.** The proof has the same basic structure as before, but it must surmount two additional difficulties. Since we cannot expect to find any finite subset of $F$ to provide the coordinates of a homeomorphism of $M$ into a complex Euclidean space, the separation arguments made at the beginning are somewhat more involved. A more serious problem is that the image of $M$ under a map $G = (f_1, \ldots, f_m)$ with the $f_i$’s in $F$ will not necessarily be polynomially convex. Because of this, domains of holomorphy corresponding to $W$ and $S$ will be harder to find. To construct them, we will use a standard technique due to Arens and Calderón [1], [5].

We again choose a point $a$ in $C - f(E)$ and claim that there exist functions $\{f_1, \ldots, f_m\}$ in $F$, an open set $U$ in $C^n$, and a closed complex submanifold $V$ of $U$ such that if $G = (f_1, \ldots, f_n)$ then

(11) $f = q \circ G$ for some polynomial $q$ in $m$ variables,

(12) $V \supseteq G(M) \cap q^{-1}(a)$,

(13) $U \cap G(M) = V \cap G(M) = G(M \cap \bar{V})$, and

(14) $(q - a)^k$ extends from $G(M) \cap V$ to a function $k$ holomorphic on $V$.

To begin the construction of $G$, $U$, and $V$ we note that for each point of $f^{-1}(a)$ there can be found functions $f_1, \ldots, f_n$ in $F$ and a neighborhood of the point on which $df_1 \wedge \cdots \wedge df_n$ has no zeros. These functions extend to holomorphic functions $f_1, \ldots, f_n$ on an $M$-neighborhood of the point on which the extended form $df_1 \wedge \cdots \wedge df_n$ has no zeros. Since $f^{-1}(a)$ is compact it can be covered by finitely
many such neighborhoods with the result that there exist functions $f_1, \ldots, f_l$ in $F$ whose holomorphic extensions provide a map $(\tilde{f}_1, \ldots, \tilde{f}_l)$ with maximum rank $n$ on some $\tilde{M}$-neighborhood of $f^{-1}(a)$. Since $F$ separates $M$, Lemma 3 applied to $F$ and $f^{-1}(a)$ yields the existence of functions $f_{l+1}, \ldots, f_m$ in $F$ whose adjunction provides a map

$$G = (f_1, \ldots, f_m): M \to C^m$$

which is injective and of rank $n$ in an $M$-neighborhood of $f^{-1}(a)$. By Lemma 2, the map $(\tilde{f}_1, \ldots, \tilde{f}_m)$ imbeds an open $\tilde{M}$-neighborhood $\tilde{V}$ of $f^{-1}(a)$ as a complex submanifold $V$ of $C^m$. This property is clearly unaffected by the adjunction of more coordinate functions, so we may assume that (11) is true. Statement (12) is then immediate.

By applying Lemma 4 to $M$ and $M \cap \tilde{V}$ and passing from $F$ to a relatively compact subset of it which contains $f^{-1}(a)$, we can assume that $G(M \cap \tilde{V})$ and $G(M - \tilde{V})$ are disjoint. Just as before we can arrange that $V$ is a closed submanifold of an open set $U$ in $C^m$ which satisfies (13), and find an extension $k$ verifying (14).

However, $U$ need not contain a domain of holomorphy containing $G(M) \cap q^{-1}(a)$, since the latter set is not necessarily polynomially convex. Hence the theorem of Grauert and Docquier cannot yet be used to extend $k$ to an open set in $C^m$.

To effect this extension and thus prepare the way for the Cousin I construction above, we shall use a technique of Arens and Calderón [1], [5]. In fact, we assert that there are functions $f_{m+1}, \ldots, f_p$ in $F$ such that if $H = (f_1, \ldots, f_p)$ and $B$ is the closed subalgebra of $A$ with identity generated by $\{f_1, \ldots, f_p\}$, then there exists a domain of holomorphy $W$ in $C^p$ such that $M_B \cap q^{-1}(a) \subset W \subset U \times C^{p-n}$ and $k$ extends to a holomorphic function on $W$. Here we have made the usual identification of the maximal ideal space $M_B$ of $B$ with the polynomially convex hull of $H(M)$ in $C^p$, and the functions $k$ and $q$ are transferred in the obvious way to functions on $V \times C^{p-n}$ and $C^p$, respectively.

These additional functions are obtained by means of the Lemma of Arens and Calderón [5], which says that there can be found $f_{m+1}, \ldots, f_p$ in $F$ such that with $H$ and $B$ as defined above and $\sigma_B(f_1, \ldots, f_m)$ the joint spectrum [5] of the indicated functions relative to $B$, we have

$$\sigma_B(f_1, \ldots, f_m) \subset U \cup [C^m - q^{-1}(a)].$$

Their lemma is applicable because $U \supset G(M) \cap q^{-1}(a)$, so the right side of (15) is an open neighborhood of $G(M)$.

Now since $\sigma_B(f_1, \ldots, f_m)$ is the projection of $M_B$ on $C^m$, we have

$$M_B \subset [U \times C^{p-n}] \cup [C^p - q^{-1}(a)], \quad \text{so}\quad M_B \cap q^{-1}(a) \subset U \times C^{p-n}.$$

Moreover, $M_B \cap q^{-1}(a)$ is polynomially convex, since $M_B$ has this property. Thus there exists a domain of holomorphy $W$ in $C^p$ with

$$M_B \cap q^{-1}(a) \subset W \subset U \times C^{p-n}.$$
Finally, \((V \times C^{p-m}) \cap W\) is a closed submanifold of \(W\), and the function \(k\) extends to \(W\) exactly as it did before.

We can now construct a function \(\psi\) holomorphic in a neighborhood of \(M_B\) with no zeros on \(M_B - q^{-1}(a)\), divisible by \(g = (p-a)k\) in a neighborhood of \(M_B \cap q^{-1}(a)\) and such that the holomorphic quotient \(\psi/g\) has the value 1 everywhere on \(H(M) \cap q^{-1}(a)\).

For we have again that
\[
\{\text{Re } g = 0\} \cap H(M) = q^{-1}(a) \cap H(M),
\]
and we may therefore assume that \(\{\text{Re } g = 0\}^-\) is disjoint from \(H(M) - W\). However, it may not be the case that \(\{\text{Re } g = 0\}^-\) is disjoint from \(M_B - W\). If not, this separation may be achieved by another application of the Arens-Calderón Lemma, noting that
\[
W \cup [C^p - q^{-1}(a) - \{\text{Re } g = 0\}^-]
\]
is an open neighborhood of \(H(M)\) and proceeding as above. In other words, we can assume that
\[
M_B \subset W \cup [C^p - q^{-1}(a) - \{\text{Re } g = 0\}^-],
\]
which enables the construction of \(\psi\) as the solution to the same Cousin Problem I that we have already described. The proof is then completed exactly as it was when \(F\) is finite.

3. Some conditions for \(A = C(M)\).

**Corollary 1.** If \(M_A = M\) and \(E\) is totally disconnected, then \(A = C(M)\).

This result appears in [2]. Since it depends solely on the conclusion of Theorem I and not on the dimension of the ambient manifold of \(M\), the proof given there holds without modification.

In [2] we also deduced for the two-dimensional case that \(A = C(M)\) if \(M_A = M\) and \(E\) has Lebesgue measure zero. The example presented in §1 shows that this result fails in higher dimensions. In this example, \(E\) has three-dimensional Lebesgue measure zero but every function in \(A\) is a uniform limit on \(E\) of polynomials in \(f\), with \(f(x, y, t) = x + iy\). Thus each function in \(A\) is holomorphic on \(E\) (in the obvious sense), so that \(A \neq C(M)\). However, a stronger measure-theoretic restriction on \(E\) will still yield the same result:

**Corollary 2.** If \(M_A = M\) and \(E\) has two-dimensional Hausdorff measure zero, then \(A = C(M)\).

**Proof.** It is easily seen that the image by a continuously differentiable map of a set of two-dimensional Hausdorff measure zero also has this property. Because of the relation [7, p. 104] between Hausdorff two-dimensional measure and plane Lebesgue measure, we then have for any polynomial \(f\) in the functions of \(F\) that
$f(E)$ has measure zero. From Theorem 2 it follows that $f_{*\mu} = 0$, which implies that $\mu = 0$ [2, p. 56].

We remark that it is clear how, by adjoining more coordinates, an example of the type presented in §1 can be constructed where $M$ has any dimension greater than two but $E$ has Hausdorff three-dimensional measure zero and $A \neq C(M)$.

**References**

9. R. Nirenberg and R. O. Wells, Jr., *Holomorphic approximation on real submanifolds of a complex manifold*, Bull. Amer. Math. Soc. 73 (1967), 378–381. (This is an announcement of some results in reference [10].)

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