MAPPING CYLINDER NEIGHBORHOODS

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1. Let $X$ be a triangulated 3-manifold and $C$ a subcomplex of $X$. A regular
neighborhood of $C$ in $X$ is the union of all simplexes in a second derived sub-
division of $X$ that intersect $C$. Every subcomplex $C$ of $X$ has a regular neighborhood.
We consider the converse using a generalization of regular neighborhoods.

Let $C$ be a closed subset of a space $X$. A subspace $U$ of $X$ is called a mapping
cylinder neighborhood (MCN) of $C$ if $U = f(M \times I) \cup C$ where $f$ is a map of a
space $M \times I$ into $X$ such that $f|_{M \times [0, 1)}$ is a homeomorphism into $X - C$, $f(M \times 1) = C \cap \text{Cl} (X - C)$ and $f(M \times (0, 1]) \cup C$ is open in $X$. As noted in [12], regular
neighborhoods are MCN's.

Suppose $C$ is a closed subset of a 3-manifold $X$ and $U = f(M \times I) \cup C$ a MCN
of $C$. We note some properties of $U$.

(a) Since $M \times (0, 1)$ is a 3-manifold, $M$ is a generalized 2-manifold [17] and
thus a 2-manifold [19]. Hence $U$ is a 3-manifold with boundary.

(b) If $C$ is compact then $U$ is compact (Lemma 1).

(c) If $C$ is compact and $U'$ is another MCN of $C$ then $\text{Int } U$ and $\text{Int } U'$ are
homeomorphic [12]. Thus $U$ and $U'$ are homeomorphic [9, Theorem 3].

Our converse: Suppose $X$ is a 3-manifold and $C \subseteq X$ is a topological complex,
i.e., $C$ is homeomorphic to a locally finite simplicial complex. Suppose also that
$C$ is closed in $X$ and $C$ has a MCN. Then $C$ must be a subcomplex of some triangu-
lation of $X$.

THEOREM 1. If $C$ is a topological complex which is a closed subset of a 3-manifold
$X$, then $C$ is tame if and only if $C$ has a MCN.

Our motivation for Theorem 1 was the special case where $C$ is a 1, 2 or 3-cell
and $M$ is a 2-sphere [6], [10]. An immediate corollary to Theorem 1 is

THEOREM 2. Suppose $C$ is a tame topological complex in a 3-manifold $X$, $g$ is a
map of $X$ into a 3-manifold $Y$ such that $g^{-1}g(C) = C$, $g$ is a homeomorphism on $X - C$,
and $g(C)$ is a topological complex. Then $g(C)$ is tamely embedded in $Y$.

Proof. By Theorem 1, $C$ has a MCN $U$. The conditions on $g$ guarantee that
$g(U)$ is a MCN of $g(C)$.

A special case of Theorem 1 in dimension four is also immediate. Suppose $N$
is a space, $f: N \to N$ an onto map, and $N_f$ the mapping cylinder defined by $N$ and

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f. Let $g: N \times I \to N_f$ be the natural map. If there exists a pseudo-isotopy $k(N) \to N$ such that $k_0 = \text{id}$ and $k_1 = f$, then the map $h: N_f \to N \times I$ defined by $h(g(x, t)) = (k_1(x), t)$ is a homeomorphism. If $G$ is a cellular upper semicontinuous decomposition of a 3-manifold $M$ and $M/G$ is a 3-manifold, then there exists a pseudo-isotopy of $M$ onto itself that shrinks the nondegenerate elements to points [18]. Thus such a pseudo-isotopy exists when $N$ and $f(N)$ are 3-manifolds and $f$ is the projection map of a cellular upper semicontinuous decomposition of $N$. We have

**Theorem 3.** Suppose $N$ is a compact connected 3-manifold in a 4-manifold $Y$. Suppose $M$ is a 3-manifold and $U = f(M \times I)$ is a MCN of $N$ where the restriction of $f$ to each component of $M \times I$ is a cellular map. Then $U$ is a bicollar for $N$ in $Y$.

Is the cellularity condition given in Theorem 3 implied by the fact that $Y$ is a 4-manifold? This is the case in dim 3; see Lemma 3(2).

2. **Proof of Theorem 1.** We have placed the lemmas in the sections following the proof.

**Proof.** Suppose $C$ has a MCN. By the procedure described at the first of the proof of Lemma 6 the union of the 1 and 2-skeleton of $C$ has a MCN. Thus the interior of each 2-simplex is tame by Lemma 3(3). By Lemmas 5 and 6 the 1-skeleton of $C$ is tame. In particular, the boundary of each 2-simplex in $C$ is tame. It is a consequence of Lemmas 5.1 and 5.2 of [13] that a disk is tame if its interior and boundary are tame. Therefore the star of each vertex in $C$ is tame, since each 2-simplex is tame and the 1-skeleton is tame [7, Theorem 3.3]. Thus $C$ is locally tame and hence tame [3]. Suppose $C$ is tame. Then $C$ has a regular neighborhood under some triangulation of $X$. This regular neighborhood is a MCN. This completes the proof.

2.a. Suppose $C$ is a closed subset of a 3-manifold $X$ and $U = f(M \times I) \cup C$ is a MCN of $C$. We let $F = f(M \times 1)$ and $i_t(t \in I)$ denote the identification of $M \times t$ with $M$. For example, if $x \in C \cap C_t(X - C)$ then $f(i_t F^{-1}(x) \times 0) \subset \text{Bd} \ U$.

**Lemma 1.** Suppose $C$ is a closed subset of a 3-manifold $X$ and $U = f(M \times I) \cup C$ is a MCN of $C$. (1) If $A$ is open in $F(M \times 1)$ and contractible then every simple closed curve in $F^{-1}(A)$ separates $F^{-1}(A)$. (2) If $A \subset C$ is compact then $F^{-1}(A)$ is compact.

**Proof.** Suppose not. There exist simple closed curves $S_1$ and $S_2$ in $f(i_t F^{-1}(A) \times 0) \subset \text{Bd} \ U$ which intersect in one point and cross there. Both curves are inessential in $U$ because of the mapping cylinder structure over $A$. A simple closed curve which bounds a singular disk has a neighborhood homeomorphic to a solid torus. Thus $S_1 \cup S_2$ has an orientable neighborhood in $\text{Bd} \ U$. This is a contradiction. A 3-manifold with boundary having orientable boundary cannot contain two inessential simple closed curves in its boundary which cross at an odd number of points. See [11, p. 29] or [15, Lemma 6.1].

(2) The local compactness of the MCN implies that $F$ is a compact map. For let $T$ be any compact subset of $C$. There exists an open set $Q \subset U$ such that $T \subset Q$.
c Q c Int U and Q is compact. For each point \( p \in F^{-1}(T) \), the arc \( f(i_1(p) \times I) \) intersects \( Q - Q \). Since \( F^{-1}(T) \) is closed, \( J = f(i_1 F^{-1}(T) \times I) \cap (\bar{Q} - Q) \) is compact. Thus \( F^{-1}(T) \) is compact, since the projection of \( M \times I \) onto \( M \times 1 \) carries \( f^{-1}(J) \) onto \( F^{-1}(T) \).

2.b. 2-simplexes. First some definitions. Let \( x \in L \) where \( L \) is a 2-manifold with boundary in a 3-manifold \( X \). The local separation theorem \([1, \S 2, \text{Corollary 2}]\) yields: For every \( \varepsilon > 0 \), there exists an \( \varepsilon \)-neighborhood \( N \) of \( x \) in \( X \) such that \( N - L \) has two components \( O_1 \) and \( O_2 \). If \( x \in \partial L \), then \( O_2 = \emptyset \). If \( x \in \text{Int} L \), then \( O_1 \) and \( O_2 \) are nonempty. We say \( U' \subset X \) is a 1-sided neighborhood of \( x \) if there exists a neighborhood \( N \) of \( x \) from the local separation theorem such that \( O_1 \cup (N \cap L) \subset U' \) and \( O_2 \cap U' = \emptyset \).

Let \( C \) be a topological complex which is closed in \( X \) and consists of 1 and 2-simplexes. Let \( U = f(M \times I) \) be a MCN of \( C \) and \( \Delta \) a 2-simplex of \( C \). We say \( U \) contains a 1-sided MCN, \( U' \), of \( x \in \text{Int} \Delta \) if there exists a disk \( D \subset M \) such that \( V = f(D \times I) \) is a 1-sided neighborhood of \( x \). We shall show in Lemma 3 that 1-sided MCN’s always exist. Lemma 2 is a standard type of result for 2-manifolds; we omit a proof.

**Lemma 2.** Let \( M \) be a 2-manifold, \( B \) a nonempty, proper open connected subset of \( M \) such that \( B \) is compact and every simple closed curve in \( B \) separates \( B \). If \( E \) is a continuum in \( B \) and \( K \) is a continuum in \( B \) which separates \( E \) from \( \partial B \), then \( E \) lies in the interior of a disk \( D \subset B \).

Consider a fixed \( x \in \text{Int} \Delta \). We distinguish two sets, \( H \) and \( L \), in \( M \times 1 \) which correspond to the two sides of \( \Delta \) near \( x \). Let \( N, O_1 \) and \( O_2 \) be given for \( x \) by the local separation theorem. Let \( P \) be a disk such that \( x \in \text{Int} P \subset \text{Int} (N \cap \text{Int} \Delta) \) and \( z \in \text{Int} P \). For \( y \in F^{-1}(z) \), let \( A_y \) denote the arc \( f(i_1(y) \times I) \). There exists a first point \( p \) from \( z \) in \( A_y \cap (\bar{N} - N) \). Then \( [z, p) \subset N \) and \( (z, p) \subset O_1 \) or \( O_2 \). We say \( A_y \) ends through \( O_1 \) or \( O_2 \), respectively. Let \( H(A_y) \) be the set of all points \( y \) such that \( A_y \) ends through \( O_1 \) or \( O_2 \). Let \( H = \bigcup H_A, L = \bigcup L_A, z \in \text{Int} P \).

We show \( H \) and \( L \) are open and separated. We assume the neighborhood \( N \) was chosen to lie inside a neighborhood \( Q \) of \( x \) homeomorphic to \( E^3 \). Suppose there exist \( y \in H \) and \( b \in L \) lying in the same component of \( F^{-1}(\text{Int} P) \). There exist an arc \( by \subset F^{-1}(\text{Int} P) \) and an arc \( F(b)F(y) \subset \text{Int} P \). There exists \( 0 < t < 1 \) so that the arc \( f(i_1(by) \times t) \) together with \( F(b)F(y) \) and subarcs of \( A_y \) and \( A_b \) form a simple closed curve \( S \subset Q \). Since \( A_y \) ends through \( O_1 \) and \( A_b \) ends through \( O_2 \), \( S \) links \( \partial P \) (homology linking mod 2; see [4]). But since \( by \subset F^{-1}(\text{Int} P) \), \( S \) can be shrunk to a point in \( Q - P \) by first pulling it into \( \text{Int} P \) using the mapping cylinder. Contradiction. Therefore \( H \) and \( L \) are the union of components of \( F^{-1}(\text{Int} P) \). Thus they are open and separated.

**Lemma 3.** Suppose \( C \) is a topological complex which is a closed subset of a 3-manifold \( X \). Suppose \( C \) consists of 1 and 2-simplexes and \( U = f(M \times I) \) is a MCN of \( C \). Also suppose \( \Delta \) is a 2-simplex in \( C \) and \( x \in \text{Int} \Delta \). Then (1) \( U \) contains a 1-sided MCN.
of x on each side of Δ and the two disks defining the MCN’s are disjoint, (2) Hx and $L_x$ are cellular in M, and (3) Int Δ is locally tame.

**Proof.** (1) Let x be a distinguished point in Int Δ and N, O1 and O2 be given for x by the local separation theorem such that $N$ lies in a neighborhood of x homeomorphic to $E^3$. Let $P$ be a disk such that $x \in Int P \subseteq (N \cap Int Δ)$ and let $H$ and $L$ be given as in the discussion preceding the lemma. We consider only $H$. By Lemma 1, $F^{-1}(x)$, and hence $H_x$, is compact. If $H_x$ were not connected we could separate two of its components, say $T_1$ and $T_2$, in $M \times 1$ with a finite number of simple closed curves $S_i \subseteq H$. But points in $f(T_1 \times I)$ and $f(T_2 \times I)$ can be joined by small arcs in $O_1$. A contradiction is reached since $f((\bigcup S_i \times [0, 1]))$ separates $f(M \times [0, 1])$ and x is not a limit point of $f(\bigcup S_i \times I)$. Thus $F|H$ is monotone.

There exist disks $D_1$ and $D_2$ such that $x \in Int D_1 \subseteq D_1 \subseteq Int D_2 \subseteq Int P$. Let $E = H_x$ and $B = \bigcup H_y$, $y \in Int D_2$. Let $K = \bigcup H_y$, $y \in Bd D_1$. The map $F$ is closed on $F^{-1}(D_2)$ and $H$. The inverse image of a connected set is connected under a monotone closed map. Thus the sets $E$, $K$ and $B$ satisfy the hypothesis of Lemma 2. Let $\tilde{U} = f(i_x(D) \times I)$ be a 1-sided MCN of x. For since $D \subseteq H$, there exists $t < 1$ such that $f(i_x(D) \times \{t, 1\}) \subseteq O_1$. Picking a neighborhood $N(q)$ of x by the local separation theorem such that $\tilde{U} \subseteq N(q)$ and $N(q) \cap f(i_x(D) \times \{0, 1\}) = \emptyset$, we have $O_2(q) \cap f(i_x(D) \times I) = \emptyset$. Since $H_x \cap Bd D = \emptyset$ and $i_x(Bd D) \times I$ separates $M \times I$, there exists a neighborhood $N(r)$ of x from the local separation theorem such that $O_1(r) \subseteq U'$. For a neighborhood $N(s)$ of x from the local separation theorem contained in $N(q) \cap N(r)$ we have $O_1(s) \subseteq U'$ and $O_2(s) \cap U' = \emptyset$. Therefore $U'$ is a 1-sided MCN of x. A similar argument using $L$ yields a disk disjoint from $D$ and a 1-sided MCN of x on the $O_2$ side of $Δ$.

(2) Let $N(s)$ be the neighborhood of x given above and $D_3$ a disk such that $x \in Int D_3 \subseteq D_3 \subseteq (N(s) \cap Int Δ)$. Then $H \cap F^{-1}(Int D_3)$ is an open connected subset of Int D and not separated by $H_x$. Thus $H_x$, and similarly $L_x$, is cellular.

(3) We shall show that $X - Int Δ$ is locally simply connected at x. Consider the 1-sided MCN of x, $U' = f(i_x(D) \times I)$ and let $\epsilon > 0$. There exists an $\epsilon$-neighborhood $N(\epsilon)$ of x from the local separation theorem such that $O_1(\epsilon) \subseteq U'$ and $O_2(\epsilon) \cap U' = \emptyset$. Since $F^{-1}(N(\epsilon) \cap U')$ is open in $i_x(D) \times I$ and $H_x$ is cellular, there exists a disk $G \subseteq i_x(D)$ and a number $t < 1$ such that $G \times \{t, 1\} \subseteq F^{-1}(N(\epsilon) \cap U')$ and $H_x \subseteq Int G \times 1$. Let $T = (Int G) \times \{t, 1\}$. There exists a neighborhood $Q$ of x from the local separation theorem such that $Q \subseteq N(\epsilon)$ and $Q \cap f((D \times I) \setminus T) = \emptyset$. The component $O_1(q)$ of $Q - Int Δ$ lies in $f(T)$. Let J be any simple closed curve in $O_1(q)$. There exists $r < 1$ such that $f(D \times r)$ separates $J$ from $f(D)$ in $f(i_x(D) \times I)$. Thus J can be shrunk to a point in the interior of the 3-cell $f(G \times [t, r]) \subseteq N(\epsilon)$. Using the 1-sided MCN of $x$ on the other side of Int $Δ$, we have that $X - Int Δ$ is locally simply connected at x. Since $X - Int Δ$ is locally simply connected at each $x \in Int Δ$, Int Δ is locally tame [5].
2.c. l-complexes. Let \( n \) be a positive integer. An \( n \)-frame \( T \) is the union of \( n \)-arcs \( A_t = [p, a_t] \) such that \( A_t \cap A_s = p \). The points \( a_t \) are the endpoints of \( T \). The interior of \( T \), \( \text{Int} \ T \), is \( T \) minus its endpoints. We define a MCN of the interior of \( T \). No confusion should result from this different use of MCN. Let \( S^2 \) denote the 2-sphere and \( D_i, i = 1, \ldots, n \), be disjoint disks in \( S^2 \). Let \( M = S^2 - \bigcup D_i \) and consider \( M \times I \) as a subspace of \( S^2 \times I \). If \( T \) is an \( n \)-frame in a 3-manifold \( X \) then \( \text{Int} \ T \) is said to have a MCN, \( U = f(M \times I) \), if there exists a map \( f \) of \( M \times I \) into \( X \) such that (1) \( f|_{M \times [0, 1)} \) is a homeomorphism into \( X - T \), (2) \( f(M \times 1) = \text{Int} \ T \), (3) \( U \) is a neighborhood of \( \text{Int} \ T \) in \( X \), and (4) for any sequence \( \{b_i\} \) in \( M \times I \) which converges to a point of \( \text{Bd} D_i \times 1 \), \( \{f(b_i)\} \) converges to the endpoint \( a_t \) of \( T \).

**Lemma 4.** Suppose \( T \) is an \( n \)-frame in a 3-manifold \( X \). If there exists a MCN, \( f(M \times I) \), of \( \text{Int} \ T \) then \( \text{Int} \ T \) is locally tame.

**Proof.** The proof of Lemma 4 follows the procedure used to prove Theorem 1 in [6]. We partition a neighborhood of \( \text{Int} \ T \) and a neighborhood of the interior of a standard \( n \)-frame in \( E^3 \) into homeomorphic pieces. We then obtain a homeomorphism between the neighborhoods which carries \( T \) onto the standard \( n \)-frame.

Since for each \( t \in (0, 1) \), \( f(M \times t) \) is bicollared, we may assume that \( f(M \times 0) \) is locally tame. Let \( C \) be a circle, \( A = C \times (0, 1) \times I \) and \( (x, y, z) \in A \) such that \( x \in C \), \( y \in (0, 1) \) and \( z \in I \). Let \( B \) denote the half-open annulus in \( A, B = \{(x, y, z) : y = 1/2z + 1/2\} \).

The properties given in Lemma 1 also hold for a MCN of \( \text{Int} \ T \). It therefore follows that \( F \) is closed and monotone. Thus the inverse image under \( F \) of any connected subset of \( \text{Int} \ T \) is connected. For each \( i \), \( F^{-1}(\text{Int} A_i) \) is a component of \( (M \times I) - F^{-1}(p) \) and \( F^{-1}(\text{Int} A_i) \cup D_i \) is a component of \( (S^2 \times I) - F^{-1}(p) \). Since each component of \( S^2 \) minus the continuum \( F^{-1}(p) \) is homeomorphic to \( E^2 \), subtracting the disk \( D_i \) yields that \( F^{-1}(\text{Int} A_i) \) is an open annulus. Thus there exist homeomorphisms \( k_1 \) and \( k_2 \) of \( A \) into \( i_1 F^{-1}(\text{Int} A_i) \times I \) such that \( B(a_t) = \text{Cl} (f(k_1(B)) \) is a disk with \( B(a_t) \cap T = a_t \), \( B(p_t) = \text{Cl} (f(k_2(B)) \) is a disk with \( B(p_t) \cap T = p \), \( \text{Int} B(a_t) - a_t \), and \( \text{Int} B(p_t) - p \) are locally tame, and \( B(p_t) \cap B(a_t) = \emptyset \). Similarly, for each \( x \in \text{Int} A_i \), \( F^{-1}(\text{Int} A_i) - F^{-1}(x) \) is the union of two disjoint open annuli. By mapping \( A \) homeomorphically into one of these annuli we can define a disk \( B(x) \) such that \( B(x) \cap T = x \) and \( B(x) - x \) is locally tame. For distinct \( x \) and \( y \) in \( A_i \), there exist numbers \( t_1 < t_2 < 1 \) such that if \( W = \text{Cl} (f(M \times [t_1, 1]) - f(M \times [t_2, 1]) \) and if \( p \) is the closure of the component of \( f(M \times I) - (B(x) \cup B(y)) \) that intersects \( \text{Int} A_i \) then \( W \cap p \) is a tame solid torus. Let \( O \) be the closure of the component of \( f(M \times I) - \bigcup B(a_t) \) that contains \( \text{Int} T \). Let \( L_i \) be the closure of the component of \( O - \bigcup B(p_t) \) that contains \( \text{Int} A_i \) and \( L = \text{Cl} (O - \bigcup L_i) \). Let \( O' \) be the unit ball in \( E^3 \) and \( T' \) an \( n \)-frame whose vertex is the origin, whose endpoints lie in \( \text{Bd} O' \) and which is composed of straight line segments. Partition \( O' \) into regions \( L_i' \) corresponding to \( L_i \) and \( L_i \). It follows from the proof of Theorem 1 of [6] that we may partition \( L_i - A_i \) and \( L_i' - A_i \) into tame solid tori as above whose diameters
go to zero as the tori approach $T$ in such a way that a homeomorphism $R_i : L_i \to L'_i$ can be obtained by defining homeomorphisms on corresponding tori. Let $J_1 = f(M \times [0, 1/2]) \cap L$, $J_j = f(M \times [|j|, 1/|j| + 1]) \cap L$, $j \geq 2$. Each $J_i$ ($j \geq 1$) has tame boundary and is homeomorphic to $M \times I$. There exists a collection of regions $\{J'_i\}$ in $L'$ and a sequence of onto homeomorphisms $S_j (j \geq 1)$ such that $S_i : J_i \to J'_i$ and $S_j$ extends $R_i$ ($i = 1, \ldots, n$) and $S_k$ ($k < j$). The union of the $S_i$ and the $R_i$ can be extended to a homeomorphism of $O$ onto $O'$ that carries $T$ onto $T'$. Thus $\text{Int } T$ is locally tame.

**Lemma 5.** Suppose $C$ is a topological 1-complex which is a closed subset of a 3-manifold $X$. Suppose $C$ has a MCN, $f(M \times I)$. Then $C$ is tame.

**Proof.** Let $p$ be a vertex of $C$ and $T$ the n-frame consisting of all simplexes in $C$ containing $p$. We shall show that $f(i_1 F^{-1} (\text{Int } T) \times I)$ is a MCN of $\text{Int } T$. It follows from Lemma 1 that $F$ is closed and monotone. Let $[p, a_i]$ be a 1-simplex in $T$. Let $x, y \in (p, a_i)$ and $q \in (x, y)$. Let $K_z = f(i_1 F^{-1} (z) \times 0)$, $z = x, y$ and $q$. There exist simple closed curves $S$ and $S'$ which separate $K_z$ from $K_x$ and $K_q$ from $K_y$ in $f(i_1 F^{-1} [x, y] \times 0)$, respectively. The curves $S$ and $S'$ can be shrunk to points on disjoint subsets of $f(M \times (0, 1]) \cup S \cup S'$. They therefore bound disjoint disks there by Dehn’s Lemma [16]. Let $K$ be the union of the two disks and the component of $f(i_1 F^{-1} [x, y] \times 0) - (S \cup S')$ that contains $K_x$. Since a simple closed curve in $K$ can be pushed off of the two disks, we can obtain from Lemma 1 that every simple closed curve in $K$ separates $K$. Thus $K$ is a 2-sphere [2]. It follows that $F^{-1}(p, a_i)$ is an open annulus. Thus for each $i$, there exists a disk $B_i$ in $f(i_1 F^{-1}(p, a_i) \times I) \cup a_i$ constructed as in the proof of Lemma 4 such that $B_i \cap T = a_i$. The component of $f(M \times 0) - \bigcup \text{Bd } B_i$ which contains $f(i_1 F^{-1}(p) \times 0)$ is a sphere with $n$-holes since its union with the disks $B_i$ is a 2-sphere (again by Lemma 1 and [2]). By the construction of the $B_i$, $F^{-1}(\text{Int } T)$ is a sphere with $n$-holes. It follows that $f(i_1 F^{-1}(\text{Int } T) \times I)$ is a MCN of $\text{Int } T$. By Lemma 4, $C$ is locally tame and hence tame [3].

**Lemma 6.** Suppose $C$ is a topological complex which is a closed subset of a 3-manifold $X$. If $C$ has a MCN then the 1-skeleton of $C$ has a MCN.

**Proof.** Let $M'$ be a 2-manifold and $f'$ a map of $M' \times I$ into $X$ such that $f'(M' \times I) \cup C$ is a MCN of $C$. Let $K_i$ be the collection of all 3-simplexes in $C$. Let $N_i$ be a layer in the collar for $\text{Bd } K_i$ in $K_i$. Let $M = M' \cup \{N_i\}$ and $f : M \times I \to X$ be such that $f|M' \times I = f'$, $f|N_i \times I$ is a homeomorphism onto the region between $N_i$ and $\text{Bd } K_i$, and $f(N_i \times 0) = N_i$. Then $f(M \times I) \cup C$ is a MCN of $C_0$, the union of the 1- and 2-skeleton of $C$. Having removed the 3-simplexes we proceed to eliminate the 2-simplexes. Let $\Delta$ be a 2-simplex in $C_0$. We shall show there exists a 2-manifold $M_1$ and a map $H_1$ of $M_1 \times I$ into $X$ such that $H_1(M_1 \times I)$ is a MCN of $C_0 - \text{Int } \Delta$ and $H_1(M_1 \times I)$ agrees with $f(M \times I)$ outside of $f(i_1 F^{-1}(\text{Int } \Delta) \times I)$. Defining such a map for each 2-simplex in $C_0$ will yield a MCN of the 1-skeleton of $C$. Let
By Lemma 3 there exist disjoint disks $D_i$ ($i=1, 2$) in $M$ such that $F(D_i \times I) \subseteq \text{Int } \Delta$ and $U_i = f(D_i \times I)$ are 1-sided MCN's of $p$ on opposite sides of $\Delta$. Let $\Delta_1$ and $\Delta_2$ be disks lying in the intersection of the interiors of $F(D_1 \times I)$ and $F(D_2 \times I)$ such that $p \in \text{Int } \Delta_1 \subseteq \Delta_1 \subseteq \text{Int } \Delta_2$.

The MCN of $C_0 - \text{Int } \Delta_1$. Intuitively, we bore a hole through the MCN. Let $A = F^{-1}(\text{Bd } \Delta_1) \cap (D_1 \times I)$ and $J = F^{-1}(\text{Bd } \Delta_2) \cap (D_1 \times I)$. The region between $A$ and $J$ is an open annulus. Let $B$ be a simple closed curve lying in this region and concentric to $A$ and $J$. There exists a map $k$ of $D_1 \times I$ onto itself that carries each region in Figure 1 onto the corresponding region (labeled with a prime) in Figure 2, $k$ is fixed on the boundary of $D_1 \times I$ and the region labeled $h$ is collapsed into
The map $k$ may be extended to map $D_3 \times I$ onto itself in the same manner as $k$ maps $D_3 \times I$ onto itself. The spaces $f_k(\text{Int } D_i \times I)$, $i=1/2$, $1/4$; $i=1, 2$, are each homeomorphic to $E^2$ because they are homeomorphic to spaces of cellular upper semicontinuous decompositions of $E^2$, by Lemma 3(2) and [14]. Let $E_i$ denote the annulus

$$f([(\text{Bd } D_1, i_1(B)] \times 0) \cup (i_1(B) \times [0, 1/4]) \cup k([i_1(B), i_1(A)] \times 1/4)) \cup \text{Bd } \Delta_1$$

and $T_1$ the torus

$$f([(\text{Bd } D_1) \times [0, 1/2]) \cup k([\text{Bd } D_1, i_1(J)] \times 1/2]) \cup [\text{Bd } \Delta_1, \text{Bd } \Delta_2] \cup E_1.$$

It follows from Lemmas 5.1 and 5.2 of [13] that $T_1$ is tame. Let $E_2$ and $T_2$ be the corresponding annulus and torus in $U_2$. Let $M_1$ be $f(M \times 0)$ minus $f(\text{Int } D_1 \times 0) \cup f(\text{Int } D_2 \times 0)$ plus $E_1 \cup E_2$. A homeomorphism $\beta$ may be defined to map $(E_1 \cup E_2) \times I$ onto the tori $T_1$ and $T_2$ plus their interiors such that the extension of $\beta$ on $M_1 \times I$ agrees with $\gamma$ and yields a MCN of $C_0 - \text{Int } A_2$.

The MCN of $C_0 - \text{Int } \Delta$. We show there exists a map $P$ of $X$ onto itself which collapses the annulus $[\text{Bd } \Delta_2, \text{Bd } \Delta]$ onto $\text{Bd } \Delta$, $\gamma$ is a homeomorphism on $X - [\text{Bd } \Delta_2, \text{Bd } \Delta]$, and $\gamma$ moves no point of $X - f(F^{-1}(\text{Int } \Delta) \times I)$. Letting $H_1 = P\beta$ will give us that $H_2(M_1 \times I)$ is a MCN of $C_0 - \text{Int } \Delta$. The following spaces are described in cylindrical coordinates in $E^3$. Let $S$ be the simple closed curve $(r=1/4, z=0)$. Let $L$ be the solid annulus $(1/4 \leq r \leq 1$, $-1 \leq z \leq 1)$. Let $P'$ be the map of $L$ onto itself defined by

$$P'(r, \theta, z) = \begin{cases} (r + r(1 - |z|), \theta, z), & 0 \leq r \leq 1/2, \\ (r + (1-r)(1-|z|), \theta, z), & 1/2 \leq r \leq 1. \end{cases}$$

The map $P'$ is a homeomorphism on $\text{Bd } L$ and collapses the annulus $(1/2 \leq r \leq 1, z=0)$ into the simple closed curve $(r=1, z=0)$. Let $S_1$ be a simple closed curve lying in the annulus $[\text{Bd } \Delta_1, \text{Bd } \Delta_2]$ concentric to $\text{Bd } \Delta_1$. There exists a homeomorphism $\alpha$ of the annulus $(1/4 \leq r \leq 1, z=0)$ onto the annulus $[S_1, \text{Bd } \Delta]$ in $\Delta$ such that $\alpha(1/4, \theta, 0) \in S_1$, $\alpha(1/2, \theta, 0) \in \text{Bd } \Delta_2$ and $\alpha(1, \theta, 0) \in \text{Bd } \Delta$, for every $\theta$. Since $\text{Int } \Delta$ is locally tame there exists a homeomorphism $g$ of $L$ into $X$ such that $g$ extends $\alpha$ and $(g(L) - \text{Bd } \Delta) \cap \beta(M_1 \times (0, 1)] \cap f(i_1F^{-1}(\text{Int } \Delta) \times I)$. The required map $P$ is: $P(w) = gP'g^{-1}(w)$ for $w \in g(L)$ and the identity elsewhere. This completes the proof.

3. 1-sided MCN's. Let $L$ be a 2-manifold with boundary in a 3-manifold $X$. Let $x \in L$, $U$ a 1-sided neighborhood of $x$ and $N$, $O_1$ and $O_2$ given for $U$. We say $X - L$ is locally simply connected on the $U$ side of $L$ at $x$ if for every $\varepsilon > 0$, there exists a neighborhood $N(\varepsilon)$ of $x$ from the local separation theorem such that $N(\varepsilon) \subset N$ and any simple closed curve in $O_1(\varepsilon)$ can be shrunk to a point in $X - L$ on a set of diameter less than $\varepsilon$. If $x \in \text{Int } L(x \in \text{Bd } L)$ then $L$ is said to be locally tame from the $U$ side at $x$ if $x$ has a neighborhood in $U$ homeomorphic to a 3-cell.
(C is locally tame at x). It follows from Theorems 4 and 8 of [5] that L is locally tame from the U side at x if L is locally simply connected on the U side at each point in a neighborhood of x. Let D be a disk. A point x ∈ L is said to have a 1-sided MCN, \( U = f(D \times I) \), if there exists a map \( f : D \times I \to X \) such that \( f(D \times 1) \subset L \), \( f[D \times [0, 1)] \) is a homeomorphism into \( X - L \), and \( U \) is a 1-sided neighborhood of x.

**Theorem 4.** Suppose L is a 2-manifold with boundary in a 3-manifold X. If \( x \in L \) and x has a 1-sided MCN then L is locally tame from the U side at x.

**Proof.** The proof of Lemma 3(3) essentially shows that \( X - L \) is locally simply connected on the U side at x for \( x \in \text{Int } L \). The case for \( x \in \text{Bd } L \) is a consequence of Theorem 1. Consider a small neighborhood of x in L as the topological complex and let X be a properly chosen subset of the 1-sided MCN.

We give a short proof of a result which is part of the folklore of upper semi-continuous decompositions.

**Theorem 5.** Suppose L is a 3-manifold with boundary and G an upper semi-continuous decomposition of L all of whose nondegenerate elements lie in \( \text{Bd } L \) and are cellular in \( \text{Bd } L \). Then \( L/G \) is a 3-manifold with boundary.

**Proof.** If G is an upper semicontinuous decomposition of \( E^3_+ \), all of whose nondegenerate elements lie in \( E^2 \) and are cellular in \( E^2 \), then \( E^2/G \) has a neighborhood in \( E^2/G \) homeomorphic to \( E^2_+ \). For consider \( E^2_+ \subset E^3 \); then \( E^2/G \) is homeomorphic to \( E^2 \) by [14] and \( E^0/G \) is homeomorphic to \( E^3 \) by [8]. Let P denote the projection map of \( E^3 \) onto \( E^3/G \). For each \( x \in E^2/G \), there exists a disk D such that \( P^{-1}(x) \subset \text{Int } D \). Let \( U = \{(x, y, z) : (x, y, 0) \in D, 0 \leq z \leq 1\} \). Then \( P(U) \) is a 1-sided MCN of x in \( E^2/G \). By Theorem 4, x has a 3-cell neighborhood in \( P(U) \). Hence \( E^2_0/G \) contains a neighborhood of \( E^2/G \) homeomorphic to \( E^2_+ \).

Let h be the projection map of L onto \( L/G \) and \( x \in \text{Bd } L/G \). There exists a neighborhood \( Q \) of \( h^{-1}(x) \) in \( \text{Bd } L \) which is homeomorphic to \( E^2 \) and is the union of elements of G. There exists a neighborhood \( B \) of \( Q \) in \( L \) homeomorphic to \( E^2_+ \). By the above, \( h(B) \) contains a neighborhood of x in \( L/G \) homeomorphic to \( E^2_+ \). Thus \( L/G \) is a 3-manifold with boundary.

**References**


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