SOME TAMENESS CONDITIONS INVOLVING
SINGULAR DISKS

BY
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Introduction. A familiar sort of lemma in the study of $E^3$ is the following:

**LEMMA.** Let $D$ and $F$ be two disks in $E^3$ with $\partial D \cap F = D \cap \partial F = \emptyset$, and let $U$ be a neighborhood of $F^\circ$ in $E^3$. Then there is a disk $D'$ in $E^3$ such that $\partial D' = \partial D$, $D' \subset D \cup U$, and $O(D', F) \subset U$ where $O(D', F)$ is $D' \setminus F$ minus the component containing $\partial D$. ($D' \setminus F$ means $D' \setminus D' \cap F$.)

Theorem 4 generalizes this lemma, allowing $E$ to be a singular disk with its "interior" disjoint from its "boundary". It is necessary to redefine $O(D', E)$, and this is done in §2; the new definition is motivated by Lemma 5A.

Applications of Theorem 4 to the study of 2-spheres in $E^3$ are given in §6. Burgess has shown (Theorem 7 in [6]) that a 2-sphere $S$ in $E^3$ is tame from the interior (i.e., $S \cup \mathrm{int} \ S$ is a 3-cell) if it is "locally spanned" by disks in the interior; Theorem 6 partially extends this result, letting the spanning disks be singular but imposing a condition on their boundaries. Corollary 6A notes that $S$ is then tame from the interior if "small loops in $S$ can be shrunk to points in small subsets of the interior." Corollary 6B answers a question raised by Bing [5, §5].

1. Notation and terminology. We use the letter $d$ to denote the Euclidean metric for Euclidean 3-space $E^3$, and let $\rho(f, g) = \sup_{x \in A} d(f(x), g(x))$ for any two maps $f$ and $g$ of a space $A$ into $E^3$. A map $f$ of a subspace of $E^3$ into $E^3$ is a $\delta$-map if $\rho(f, I) < \delta$, where $I$ is the identity map.

An $n$-manifold $N$ is a separable metric space such that each point $p \in N$ has an $n$-cell neighborhood in $N$. $N^\circ = \{ p \in N : p$ has a neighborhood in $N$ homeomorphic to $E^n \}$, and $\partial N = N \setminus N^\circ$. $N$ is an $n$-manifold-with-boundary if $\partial N \neq \emptyset$. A Euclidean neighborhood of a point $p \in N$ is an $n$-cell neighborhood $U$ together with a linear structure on $U$. If $S$ is a connected $(n-1)$-manifold in $N$ which separates $N$, and $V$ is a component of $N \setminus S$, then $S$ is tame from $V$ if $S \cup V$ is an $n$-manifold. All 2-manifolds and 3-manifolds are assumed to be triangulated [2, Theorem 6], and we use the same symbol for both the manifold and its triangulation.

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(1) This paper is essentially the author’s Ph.D. thesis written under Joseph Martin at the University of Wisconsin. The author was supported by a National Science Foundation Graduate Fellowship.
Two subsets $X$ and $Y$ of an $n$-manifold $N$ are in *relative general position* if, for each point $p \in N$, there is a Euclidean neighborhood $U$ of $p$ and triangulations $T_X$ and $T_Y$ of $X \cap U$ and $Y \cap U$ such that

(i) each simplex of $T_X$ and $T_Y$ is a simplex in $U$,
(ii) dimension $(|T_X| \cap |T_Y|) \leq i+j-n$.

A map $f: X \to N$ is in *general position* if, for each point $p \in N$, there is a Euclidean neighborhood $U$ of $p$ and a triangulation $T$ of $f^{-1}(U)$ such that

(i) for each simplex $a \in T$, $f(a)$ is a simplex in $U$,
(ii) for any two distinct simplices $a_1 \in T'$ and $a_2 \in T'$, dimension $(f(a_1) \cap f(a_2)) \leq i+j-n$.

If $X$ and $Y$ are two triangulated spaces, then $X \oplus Y$ denotes the disjoint union of both the spaces $X$ and $Y$ and their triangulations.

A *Dehn disk* $D$ in $E^3$ is the image of a real disk $\Delta$ under a map $f: \Delta \to E^3$ such that, for some subdisk $\Delta_1 \subset \Delta$, $f(\Delta_1) \cap f(\Delta_1) = \emptyset$ and $f|_{\Delta_1}$ is piecewise linear and 1-1. The *singularities of* $f$ are the points of $\Delta$ in the closure of $\{x \in \Delta : f^{-1}(f(x)) \neq x\}$, and the *singular points of* $D$ are the images under $f$ of these singularities. $\partial D = f(\partial \Delta)$.

If $S$ is a 2-sphere in $E^3$, then int $S$ and ext $S$ are, respectively, the bounded and unbounded components of $E^3 \setminus S$. *Sierpinski curve* and *inaccessible point* are as defined in [5].

2. Algebraic separation. Let $N$ be a simply-connected $n$-manifold, $n \leq 3$. An $(n-1)$-polyhedron $K$ is an algebraic separator of $N$ if $K \cap N^\circ$ can be given a triangulation in which each $(n-2)$-simplex is the face of an even number of $(n-1)$-simplices.

Suppose that $K$ is an algebraic separator of $N$. Any arc $A \subset N$ in general position relative to $K$ hits $K$ at a finite number $\|A \cap K\|$ of points, and standard counting arguments show that:

**Proposition 2A.** If $A \subset N$ and $B \subset N$ are polygonal arcs in general position relative to $K$, and $A$, $B$ have the same endpoints, then $\|A \cap K\| = \|B \cap K\| (\text{mod } 2)$. In particular, if $\|A \cap K\|$ is odd then the endpoints of $A$ are separated in $N$ by $K$.

Suppose that $D$ is a disk and $K \subset D^\circ$ is an algebraic separator of $D$. It follows from Proposition 2A that we can define a map $\phi_{D|K}$ on $D \setminus K$ by setting $\phi_{D|K}(x) = \|A \cap K\| (\text{mod } 2)$, where $A$ is any arc from $\partial D$ to $x$ in general position relative to $K$. We let $O(D, K) = \{x \in D \setminus K : \phi_{D|K}(x) = 1\}$.

Now suppose that $\Delta$ is a disk, $M$ a 3-manifold, and $f: \Delta \to M$ a map such that $f|_{\Delta^\circ}$ is locally piecewise linear and in general position. Let $D \subset M$ be a polyhedral disk in general position relative to $f(\Delta^\circ)$, such that $\partial D \cap f(\Delta) = D \cap f(\partial \Delta) = \emptyset$.

**Proposition 2B.** $f^{-1}(D) = J_1 \cup \cdots \cup J_s$, where the $J_i$ are disjoint simple closed curves. $f(J_i)$ and $f(\bigcup J_i) = D \cap f(\Delta)$ are algebraic separators of $D$, and $O(D, f(\bigcup J_i)) \subset O(D, f(J_i))$. 

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Proof. To check that \( O(D,f(\bigcup J_i)) \subseteq \bigcup O(D,f(J_i)) \), just note that, for any polygonal arc \( A \) in general position relative to \( f(\bigcup J_i) \), \( A \cap f(\bigcup J_i) = A \cap f(J_i) \). The other statements follow from the general position of \( f|_\Delta \), and of \( D \) relative to \( f(\Delta) \).

3. Induction lemma.

**Lemma 3.** Let \( M \) be a 3-manifold-with-boundary, \( D \) and \( \Delta \) disks. Let \( f: \Delta \to M \) be a simplicial map in general position, \( U \) an open neighborhood of \( f(\Delta) \) in \( M \).

Suppose \( i: D \to M \) is a simplicial embedding such that \( i(D) \) is in general position relative to \( f(\Delta) \), \( i(D) \cap \partial M = i(\partial D) \), and \( i(\partial D) \cap U = i(D) \cap f(\partial \Delta) = \emptyset \).

If \( O(i(D),f(J)) \subseteq U \) for some simple closed curve \( J \subseteq f^{-1}(i(D)) \), then there is a polyhedral disk \( D' \) in \( M \) such that:

\( (3.1) \) \( \partial D' = i(\partial D) \),

\( (3.2) \) \( D' \subseteq i(D) \cup U \),

\( (3.3) \) \( (i(D) \cup U) \setminus (D' \cup U) \neq \emptyset \).

**Proof.** Our proof will be analogous to those of Papakyriakopoulos [10] and Stallings [11], but where they dealt with maps of disks, we will be working with the map \( i \oplus f: D \oplus \Delta \to M \) defined by \( i \oplus f|_\Delta = i \), \( i \oplus f|_\Delta = f \). To measure the singularity of this map we use the complex \( S(i \oplus f) \) defined by Stallings in his proof of [11, Lemma 3]; for completeness, we reproduce the definition here.

For any simplicial map \( \gamma \) of a complex \( X \) into a complex \( Y \), a simplicial map \( \gamma \times \gamma : X \times X \to Y \times Y \) can be constructed, where \( X \times X \) and \( Y \times Y \) are the cartesian products of complexes as defined in [7, p. 67]. We define \( S(\gamma) \) to be the inverse image under \( \gamma \times \gamma \) of the diagonal of \( Y \times Y \); since this diagonal is a subcomplex of \( Y \times Y \), it follows that \( S(\gamma) \) is a subcomplex of \( X \times X \). The useful property of \( S(\gamma) \) is that, if \( \Pi : Y \to Z \) is a simplicial map into some complex \( Z \), then \( S(\gamma) \subseteq S(\Pi \gamma) \), and \( S(\gamma) = S(\Pi \gamma) \) if and only if \( \Pi \) is 1-1.

We will induct on the number \( \mathcal{H}(i,f) \) of simplices in \( S(i \oplus f) \); assume that \( O(i(D),f(J)) \subseteq U \) for some simple closed curve \( J \subseteq f^{-1}(i(D)) \).

To simplify notation, we will identify \( D \) with \( i(D) \) from this point on in the proof of Lemma 3.

Through standard combinatorial techniques and the ideas Stallings uses in proving Lemma 3 of [11], one can show:

**Proposition 3A.** There is a regular neighborhood \( N \) of \( D \cap f(\Delta) \) in \( M \), a closed neighborhood \( V \) of \( D \cap f(\Delta) \) in \( D \), and a piecewise linear embedding \( h: D \times [-1, 1] \to N \) such that

\( (i) \) \( h(x \times 0) = x \) and \( h(x \times \pm 1) \in \partial N \) for all \( x \in D \setminus V \),

\( (ii) \) \( N \subseteq U \cup h((D \setminus V) \times [-1, 1]) \) and \( f(\Delta) \cap h((D \setminus V) \times [-1, 1]) = \emptyset \),

\( (iii) \) the maps \( i: D \to N \) and \( f: \Delta \to N \) are simplicial.

The proof of the lemma splits into two parts, depending on whether or not \( N \) is simply connected.
Case I. $N$ simply connected:

PROPOSITION 3B. The components $S_1, \ldots, S_r$ of $\partial N$ are spheres.

Proof. See 7.2 in [10].

PROPOSITION 3C. If $p$ is a point of $O(D, f(J)) \setminus U$, then $h(p \times -1)$ and $h(p \times 1)$ lie in different spheres $S_i$.

Proof. If $E$ is the disk in $\Delta$ bounded by $J$, then it follows from the general position of $f$, and of $D$ relative to $f(\Delta)$, that $O(D, f(J)) \cup f(E) = K$ is an algebraic separator of $N$. $h(p \times [-1, 1])$ is a polyhedral arc in general position relative to $K$ which hits $K$ once; by Proposition 2A, $h(p \times -1)$ and $h(p \times 1)$ are separated in $N$ by $K \subset N^c$, and must therefore lie in different components of $\partial N$.

PROPOSITION 3D. If $p$ is a point of $O(D, f(J)) \setminus U$, then $\partial D$ bounds a polyhedral disk $D'$ in $(D \setminus p) \cup U$. ($D'$ satisfies (3.1)-(3.3)).

Proof. Suppose $\partial D$ lies in $S_1$. By Proposition 3C, $S_1$ does not contain both $h(p \times -1)$ and $h(p \times 1)$. $\partial D$ misses $h(p \times \pm 1)$, so $\partial D$ bounds a disk $D_1$ in $S_1$ missing $h(p \times \pm 1)$. For any $x \in D \setminus V$, $D_1$ contains at most one of the two points $h(x \times -1)$ and $h(x \times 1)$, since $D$ is an algebraic separator of $N$ separating them. Using the embedding of $D \times [-1, 1]$ in $N$ given by Proposition 3A, we can therefore draw $D_1$ homeomorphically into $(D \setminus p) \cup U$.

Case II. $N$ not simply connected:

Let $(N_1, p)$ be the universal (simply connected) covering space for $N$. $N_1$ is a 3-manifold-with-boundary, and we triangulate $N_1$ so that $p : N_1 \to N$ is simplicial. Let $U_1 = p^{-1}(U \cap N)$.

Let $f_1 : \Delta \to N_1$ be a lifting of $f$, and let $i_1, i_2, \ldots, i_k, \ldots : D \to N_1$ be the distinct liftings of $i$.

It is easy to check that

PROPOSITION 3E. The hypotheses of Lemma 3 are satisfied by the substitution:

\[
\begin{array}{ccc}
\text{for} & \text{substitute} & \text{for} & \text{substitute} \\
M & N_1 & U & U_1 \\
\Delta & \Delta & D & D \\
f & f_1 & i & \text{any } i_k
\end{array}
\]
Proposition 3F. For any \( ik, \mathcal{H}(ik, f_1) < \mathcal{H}(i,f) \).

Proof. Consider the commutative diagram:

\[
\begin{array}{ccc}
\pi_1(i_k(D) \cup f_1(\Delta)) & \xrightarrow{\psi_1} & \pi_1(N_1) = 0 \\
(p|_{i_k(D) \cup f_1(\Delta)})_\ast & \downarrow & p_\ast \\
\pi_1(D \cup f(\Delta)) & \xrightarrow{\psi} & \pi_1(N) \neq 0,
\end{array}
\]

where \( \psi_1 \) and \( \psi \) are induced by inclusions. \( \psi \) is onto since \( N \) is a regular neighborhood of \( D \cup f(\Delta) \).

Now, \( S(i_k \oplus f_1) \subset S(p \circ (i_k \oplus f_1)) = S(i \oplus f) \). If \( S(i_k \oplus f_1) = S(i \oplus f) \), then \( p|_{i_k(D) \cup f_1(\Delta)} \) is 1-1 and hence a homeomorphism, so \( (p|_{i_k(D) \cup f_1(\Delta)})_\ast \) is onto. But then

\[
0 \neq \pi_1(N) = \psi(p|_{i_k(D) \cup f_1(\Delta)} \ast \pi_1(i_k(D) \cup f_1(\Delta)) = p_\ast \psi_1 \pi_1(i_k(D) \cup f_1(\Delta)) = 0,
\]

a contradiction. Thus, \( S(i_k \oplus f_1) \) is properly contained in \( S(i \oplus f) \), and \( \mathcal{H}(ik, f_1) < \mathcal{H}(i,f) \).

Proposition 3G. For some \( K, J \subset f_1^{-1}i_k(D) \) and \( O(i_k(D), f_1(J)) \neq U_1 \).

Proof. \( J \subset f_1^{-1}(D) = (p|_{f_1(D)})^{-1}(D) = f_1^{-1}(p^{-1}(D)) = f_1^{-1}(\bigcup i_k(D)) = \bigcup f_1^{-1}i_k(D) \); since the disks \( i_k(D) \) are disjoint, \( J \subset f_1^{-1}i_k(D) \) for some \( K \). \( p|_K = i \) is a homeomorphism, so \( p(O(i_k(D), f_1(J)) = O(D, f(J)) \); if \( O(i_k(D), f_1(J)) \subset U_1 \), then \( O(D, f(J)) \subset p(U_1) = U \cap N \), a contradiction to our assumption that \( O(D, f(J)) \neq \emptyset \).

Proposition 3H. There is a polyhedral disk \( D' \) in \( M \) satisfying (3.1)–(3.3).

Proof. Let \( D_1 = i_k(D) \), where \( K \) is given by Proposition 3G. By our induction, there is a polyhedral disk \( D'_1 \) in \( N_1 \) such that:

(i) \( \partial D'_1 = \partial D_1 \),
(ii) \( D'_1 \subset D_1 \cup U_1 \),
(iii) \( (D_1 \setminus U_1) \setminus (D'_1 \setminus U_1) \neq \emptyset \).

Since \( p|_{D_1} \) is a homeomorphism, the singularities of \( p: D'_1 \to M \) all lie in \( U_1 \); \( p(U_1) \cap p(\partial D'_1) \subset U \cap \partial M = \emptyset \), so we can apply Dehn's lemma [9, Theorem IV.3] to get a polyhedral disk \( D' \) in \( M \) such that:

(iv) \( \partial D' = p(\partial D'_1) \),
(v) \( D' \subset p(D'_1) \cup U \).

It is easy to check that (i)–(v) imply that \( D' \) satisfies (3.1)–(3.3).

4. Using a singular disk to "cut back" a real disk.

Theorem 4. Let \( U_0 \) be an open subset of \( E^3 \), \( \Delta_0 \) a disk, and \( f_0: \Delta_0 \to E^3 \) a map such that \( f_0(\Delta_0) \cap U_0 = f_0(\Delta^o_0) \) and \( f_0|_{\Delta^o_0}: \Delta^o_0 \to E^3 \) is locally piecewise linear and in general position.
Suppose that \( D \subset E^3 \) is a polyhedral disk such that \( D \cap f_0(\partial \Delta_0) = \partial D \cap f_0(\Delta_0) = \emptyset \). Then there is a polyhedral disk \( D' \) in \( E^3 \) such that

\[
\begin{align*}
\text{(4.1)} & \quad \partial D' = \partial D, \\
\text{(4.2)} & \quad D' \subset D \cup U_0, \\
\text{(4.3)} & \quad D' \text{ is in general position relative to } f_0(\Delta_0), \\
\text{(4.4)} & \quad O(D', D' \cap f_0(\Delta_0)) \subset U_0.
\end{align*}
\]

**Proof.** We may assume that \( \overline{U}_0 \) is locally polyhedral \( \text{mod } f_0(\partial \Delta_0) \), and that \( U_0 \cap \partial D = \emptyset \). For any disk \( D' \) satisfying (4.1) and (4.2), let \( \mathcal{H}(D') \) be the number of components of \( D' \setminus U_0 \); \( \mathcal{H}(D') \) is finite because \( \overline{U}_0 \) is polyhedral near \( D' \).

\( D \) satisfies (4.1) and (4.2); we will induct on \( \mathcal{H}(D) \). By adjusting \( D \) within \( U_0 \), if necessary, we may assume that \( D \) is in general position relative to \( f_0(\Delta_0) \); if \( O(D, D \cap f_0(\Delta_0)) \subset U_0 \), as is the case when \( \mathcal{H}(D) = 1 \), then we have nothing to prove. Suppose that \( O(D, D \cap f_0(\Delta_0)) \subset U_0 \).

Since \( (f \cup U_0) \cap f_0(\partial \Delta_0) = \emptyset \), we can choose a disk \( \Delta \subset \Delta_0 \) such that \( f = f_0|\Delta : \Delta \to E^3 \) is piecewise linear and in general position, and

\[
D \cap f_0(\Delta_0) = f(\Delta_0)|f(\partial \Delta).
\]

Using standard Euclidean-space techniques, together with the fact that

\[
f_0(\Delta_0) \cap f_0(\partial \Delta_0) = \emptyset,
\]

one can show:

**Proposition 4A.** There is a 3-manifold-with-boundary \( M \subset E^3 \) such that

\[
\begin{align*}
\text{(i)} & \quad D \cup f(\Delta) \subset M, \\
\text{(ii)} & \quad D \cap \partial M = \partial D, \\
\text{(iii)} & \quad M \cap f_0(\partial \Delta_0) = \emptyset.
\end{align*}
\]

Furthermore, \( M \), \( D \), and \( \Delta \) may be triangulated so that the hypotheses of Lemma 3 are satisfied by \( M \), \( D \), \( \Delta \), \( f \), \( U = U_0 \cap M \), and the natural injection \( i : D \to M \).

Since \( O(D, D \cap f_0(\Delta_0)) \subset U_0 \), we have also \( O(D, D \cap f(\Delta)) \subset U \). By Proposition 2B, \( O(D, D \cap f(J)) \subset U \) for some simple closed curve \( J \subset f^{-1}(D) \). Lemma 3 then gives us a polyhedral disk \( D' \) such that

\[
\begin{align*}
\text{(4.1)} & \quad \partial D' = \partial D, \\
\text{(4.2)} & \quad D' \subset D \cup U \subset D \cup U_0, \\
\text{(3.3)} & \quad (D \cup U) 
\] \( \cup (D' \cup U) \neq \emptyset.
\]

To show that (3.3) implies \( \mathcal{H}(D') < \mathcal{H}(D) \), we note

**Proposition 4B.** \( D' \setminus U \subset D' \setminus U_0 \), \( D \setminus U \subset D \setminus U_0 \), and each component of \( D' \setminus U \) is a component of \( D \setminus U \).

**Proof.** \( D^* \setminus U = D^*(U_0 \cap M) = (D^* \setminus U_0) \cup (D^* \setminus M) = D^* \setminus U_0 \), where \( D^* \) is either \( D' \) or \( D \). That the components of \( D' \setminus U \) are components of \( D \setminus U \) follows from (4.1) and (4.2) above.

**Remark.** The proof of Theorem 4 shows that we can actually have \( D' \) satisfy

\[
O(D', D' \cap f_0(J)) \subset U_0, \text{ for each simple closed curve } J \subset f_0^{-1}(D').
\]

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5. Applying Theorem 4. Throughout the remainder of the paper, $\Delta$ will represent a standard disk. Let $M$ be a 3-manifold, $S$ a 2-manifold in $M$, and $F \subset S$ a disk.

**Proposition 5A.** Let $\mathcal{G}$ be the class of all maps $g: \partial \Delta \to F^o$ which are piecewise linear into $F$ and in general position. Then, for an arbitrary map $f: \partial \Delta \to F$, $\phi_{F|f(\partial \Delta)} = \lim_{g \in \mathcal{G}, g(\partial \Delta) \to f(\partial \Delta)} \phi_{F|g(\partial \Delta)}$ exists on $F \setminus f(\partial \Delta)$, and $\phi_{F|g(\partial \Delta)}(x) = \phi_{F|f(\partial \Delta)}(x)$ for any map $g \in \mathcal{G}$ which is homotopic to $f$ in $F \setminus x$.

**Proof.** Both assertions follow from the easily demonstrated fact that if two maps $g_1$ and $g_2: \partial \Delta \to F$ are piecewise linear into $F$, in general position, and homotopic in $F \setminus x$, then $\phi_{F|g_1(\partial \Delta)}(x) = \phi_{F|g_2(\partial \Delta)}(x)$.

If $V$ is a component of $M \setminus S$, then a blister of $F$ in $V$ is a map $f: \Delta \to F \cup V$ such that $f(\Delta) \cap S = f(\partial \Delta)$. We let $O(F,f) = \{ x \in F \setminus f(\partial \Delta) : \phi_{F,(\partial \Delta)}(x) = 1 \}$, and denote $f(\Delta) \cup O(F,f)$ by $\langle f \rangle_F$.

**Lemma 5A.** Let $S$ be a 2-sphere in $E^3$, $F \subset S$ a disk. Let $f$ be a blister of $F$ in $\text{int } S$, and $B$ a 3-cell in $E^3$, such that $(f)_p \subset B^o$ and $f|_{\Delta^o}: \Delta^o \to E^3$ is locally piecewise linear and in general position.

Suppose $p$ is a point of $O(F,f)$, $q$ is a point of $\text{int } S \setminus B$, and $qp \subset E^3$ is a polygonal arc in general position relative to $f(\Delta^o)$ such that $qp \cap p \subset \text{int } S$. Then $||qp \cap f(\Delta)||$ is odd.

**Proof.** Pick a point $r$ in $\text{ext } S \setminus B$. We can use Theorem 5.37 of [12] to extend $qp$ to a polygonal arc $qp \subset E^3$ such that $pr$ misses $S \setminus O(F,f)$ and $f(\Delta)$.

Let $\delta = d(f(\partial \Delta), (E^3 \setminus B) \cup qpr)$, and triangulate $S$ so that $F$ is a polyhedron in $S$. Using Bing’s approximation theorem [1, Theorem 1], we can find a piecewise linear $\delta/2$-homeomorphism $h: S \to E^3$ such that

1. $qpr \cap h(S) = h(O(F,f))$ (where $h(O(F,f)) = \{ x \in h(F) | hf(\partial \Delta) : \phi_{h(F)|h(\partial \Delta)}(x) = 1 \}$,
2. $q \in h(S)$, $r \in h(S)$,
3. $h(S)$ is in general position relative to $f(\Delta) \cup qpr$.

**Proposition 5B.** There is a polyhedron $K \subset B \subset E^3$ such that:

(i) $K$ is an algebraic separator of $E^3$ in general position relative to $qpr$.
(ii) $qpr \cap (f(\Delta) \cup h(S)) = qpr \cap K$.

**Proof.** Let $g: \partial \Delta \to E^3$ be a piecewise linear map in general position, such that:

(i) $g(\partial \Delta) \subset h(F^o)$,
(ii) $g$ and $h|_{\partial \Delta}$ are homotopic in $h(F)|qpr$,
(iii) $\rho(g, h|_{\partial \Delta}) < \delta/2$.

Let $\gamma: \Delta \to \Delta^o$ be a homeomorphism such that

(iv) $\rho(\gamma f, f) < \delta$,
(v) $f\gamma: \Delta \to E^3$ is piecewise linear and in general position, and $f\gamma(\Delta)$ is in general position relative to $h(S)$.
As a result of our care with $\delta$, we can get a piecewise linear homotopy $G: \partial \Delta \times [0, 1] \to E^3$ such that

(vi) $G_0 = g, \quad G_1 = f_\gamma|_{\partial \Delta}$,
(vii) $G(\partial \Delta \times [0, 1]) \subset B|_rpr$,
(viii) $G(\partial \Delta \times (0, 1))$ is in general position relative to $h(S)$ and $f_\gamma(\Delta)$.

It is simple to check that $K = O(h(F), g(\partial \Delta)) \cup G(\partial \Delta \times [0, 1]) \cup f_\gamma(\Delta)$ satisfies the requirements.

**Proposition 5C.** $\| qpr \cap f(\Delta) \|$ is odd.

**Proof.** $K$ is contained in $B$, which does not separate $q$ and $r$ in $E^3$, so by Proposition 2A $\| qpr \cap K \|$ is even. $\| qpr \cap K \| = \| qpr \cap (f(\Delta) \cup h(S)) \| = \| qpr \cap f(\Delta) \| + \| qpr \cap h(S) \|$, by condition (3) on $h$. $\| qpr \cap h(S) \|$ is odd since $h(S)$ is a manifold separating $q$ and $r$ in $E^3$, so $\| qpr \cap f(\Delta) \|$ is also odd.

**Lemma 5B.** Let $S$ be a 2-sphere in $E^3$, $F \subseteq S$ a disk. Let $f_1, \ldots, f_s$ be blisters of $F$ in int $S$, and $B_1, \ldots, B_s$ 3-cells in $E^3$ such that $(f_i)\cap B_i = B_i$ for each $i$.

Suppose $D$ is a polyhedral disk in $E^3$ such that $\partial D \subset \text{int } S \cup \bigcup B_i$, $\text{int } S \cup (F \setminus \bigcup f_i(\partial \Delta)) \cup D$ retracts to $\text{int } S \cup (F \setminus \bigcup f_i(\partial \Delta))$, and $D \cap S \subseteq \bigcup O(F, f_i)$.

Then there is a disk $D'$ in $E^3$ such that

(5.1) $\partial D' = \partial D$,
(5.2) $D' \subset D \cup (\bigcup B_i)$,
(5.3) $D' \subset \text{int } S$.

**Proof.** Suppose that we have a polyhedral disk $D_j$ in $E^3$ which satisfies the following conditions:

(1) $\partial D_j = \partial D$,
(2) $D_j \subset D \cup (\bigcup B_i)$,
(3) $\text{int } S \cup (F \setminus \bigcup f_i(\partial \Delta)) \cup D_j$ retracts to $\text{int } S \cup (F \setminus \bigcup f_i(\partial \Delta))$, and $D_j \cap S \subseteq \bigcup O(F, f_i)$.

We can choose $D_0 = D$, for example, and if we had $D_s$ we could choose $D' = D_s$.

For the proof of Lemma 5B it is, therefore, sufficient to produce $D_{j+1}$.

We may assume that, for each $i, f_i|_{\Delta i}$ is locally piecewise linear and in general position. The hypotheses of Theorem 4 are then satisfied by the following substitutions:

<table>
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<th>substitute</th>
<th>for</th>
<th>substitute</th>
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</thead>
<tbody>
<tr>
<td>$\Delta_0$</td>
<td>$\Delta$</td>
<td>$U_0$</td>
<td>$B_{j+1} \cap \text{int } S$</td>
</tr>
<tr>
<td>$f_0$</td>
<td>$f_{j+1}$</td>
<td>$D$</td>
<td>$D_i$</td>
</tr>
</tbody>
</table>

There is, therefore, a polyhedral disk, which we shall call $D_{j+1}$, satisfying:

(4.1) $\partial D_{j+1} = \partial D_j$,
(4.2) $D_{j+1} \subset D_j \cup (B_{j+1} \cap \text{int } S)$,
(4.3) $D_{j+1}$ is in general position relative to $f_{j+1}(\Delta^\circ)$,
(4.4) $O(D_{j+1}, D_{j+1} \cap f_{j+1}(\Delta)) \subset B_{j+1} \cap \text{int } S$. 
From (4.1) and (4.2) it follows that \( D_{j+1} \) satisfies conditions (1)-(3); it remains to check (4).

**Proposition 5D.** \( D_{j+1} \cap S \subset \bigcup_{j+1 \in S} O(F, f_j) \).

**Proof.** (4.2) implies that \( D_{j+1} \cap S \subset \bigcup_{j+1 \in S} O(F, f_j) \), so all we need check is \( O(F, f_{j+1}) \). Suppose that \( D_{j+1} \cap O(F, f_{j+1}) \neq \emptyset \), and use (4.3) to choose a polygonal arc \( A \) with endpoints \( p \) and \( q \), such that

(i) \( A \subset D_{j+1} \),
(ii) \( q \in \partial D_{j+1} \), \( p \in O(F, f_{j+1}) \),
(iii) \( A \) is in general position relative to \( f_{j+1}(\Delta) \).

Using the facts that \( D_{j+1} \) satisfies (3) and \( \text{int } S \) is locally 0-connected [12, Theorem 5.35], we can get a polygonal arc \( A' \) with endpoints \( p' \) and \( q' \), such that

(iv) \( A' \not\subset \text{int } S \),
(v) \( p' \in O(F, f_{j+1}) \),
(vi) \( A' \cap W = A \cap W \), for some neighborhood \( W \) of \( f_{j+1}(\Delta) \) in \( \text{int } S \).

\( A' \) is in general position relative to \( f_{j+1}(\Delta) \) since \( A \) is, so Lemma 5A tells us that \( \| A' \cap f_{j+1}(\Delta) \| \) is odd. Since \( \| A' \cap f_{j+1}(\Delta) \| = \| A \cap f_{j+1}(\Delta) \| \), this means that \( p \in O(D_{j+1}, D_{j+1} \cap f_{j+1}(\Delta)) \). According to (4.4), \( p \) then lies in \( \text{int } S \); but we assumed that \( p \in S \), which is a contradiction. Therefore, \( D_{j+1} \cap O(F, f_{j+1}) = \emptyset \).

6. 2-spheres in \( E^3 \). Let \( M \) be a 3-manifold, \( S \) a 2-manifold in \( M \), and \( V \) a component of \( M \setminus S \). \( S \) satisfies condition (1) toward \( V \) at a point \( p \in S \) if, for any neighborhood \( B \) of \( p \) in \( M \), and any Cantor set \( C \) in \( S \), there is a disk \( F \subset S \cap B \) and a blister \( f \) of \( F \) in \( V \) such that \( p \in O(F, f) \subset (f)_p \subset B \) and \( f(\partial A) \cap C = \emptyset \).

**Theorem 6.** Let \( S \) be a 2-sphere in \( E^3 \) which satisfies condition (1) toward its interior at every point. Then \( S \) is tame from the interior.

**Proof.** Let \( F \subset S \) be a disk, \( U \) a neighborhood of \( F \) in \( E^3 \), and \( D \subset E^3 \) a polyhedral disk with \( \partial D \subset \text{int } S \) and \( D \cap S \subset F^o \).

**Proposition 6A.** To prove Theorem 6, it is sufficient to show that there is a disk \( D' \) in \( E^3 \) such that

(i) \( \partial D' = \partial D \),
(ii) \( D' \subset D \cup U \),
(iii) \( D' \subset \text{int } S \).

**Proof.** As Hempel has noted (in the proof of [8, Theorem 1]), this is a consequence of Bing’s proof of Theorem 1 in [3].

**Proposition 6B.** There is a Dehn disk \( D_0 \) in \( E^3 \) and a Cantor set \( C \) in \( S \) such that

(i) \( \partial D_0 = \partial D \),
(ii) \( D_0 \subset D \cup U \), with the singular points of \( D_0 \) contained in \( U \),
(iii) \( D_0 \subset \text{int } S \cap (F^o \cap C) \).
Proof. We can use the Tietze extension theorem to get a Dehn disk $D'_0$ such that $\partial D'_0 = \partial D$ and $D'_0 \subset (D \cap \text{int } S) \cup F^{\circ}$, with the singular points of $D'_0$ contained in $F$. Theorem 2.1 of [5] then gives us $D_0$.

**Proposition 6C.** There are blisters $f_1, \ldots, f_s$ of $F$ in int $S$, 3-cells $B_1, \ldots, B_s$ in $U \setminus \partial D$, and disjoint disks $G_1, \ldots, G_s$ in $S$, such that $(f_i)_p \subset B'_i$ for each $i$, and $D_0 \cap S = \bigcup G_i \subset \bigcup G_i \subset (\bigcup O(F, f_i)) \cup f_i(\partial \Delta)$.

**Proof.** For any point $p$ in $F^{\circ}$, we can choose a 3-cell neighborhood $B$ in $U \setminus (\partial D \cup (S \setminus F))$. Let $C$ be the Cantor set described in Proposition 6B; since $S$ satisfies condition (1) toward int $S$ at $p$, there is a disk $F_p \subset S \cap B \subset F$ and a blister $f_p$ of $F_p$ in int $S$ such that $p \in O(F_p, f_p) = O(F, f_p) \subset (B^{\circ} \setminus f_p(\partial \Delta)) \cap C = \emptyset$. $D_0 \cap S$ is compact, so we can pick $p_1, \ldots, p_s \in F^{\circ}$ such that $D_0 \cap S \subset (\bigcup O(F, f_p)) \cup f_p(\partial \Delta)$. We let $\{f_1, \ldots, f_s\} = \{f_{p_1}, \ldots, f_{p_s}\}$, $\{B_1, \ldots, B_s\} = \{B_{p_1}, \ldots, B_{p_s}\}$, and use the fact that $D_0 \cap S \subset C$ is 0-dimensional to choose the disks $G_1, \ldots, G_s$.

We can use the Tietze extension theorem to show:

**Proposition 6D.** int $S \cup (\bigcup G_i)$ is a retract of some neighborhood $V$ of int $S \cup (\bigcup G_i)$ in $E^3$.

**Proposition 6E.** There is a polyhedral disk $D_1$ in $E^3$ such that

(i) $\partial D_1 = \partial D$,

(ii) $D_1 \subset D \cup U$,

(iii) $S, F, \{f_1, \ldots, f_s\}, \{B_1, \ldots, B_s\}$, and $D = D_1$ satisfy the hypotheses of Lemma 5B.

**Proof.** The singular points of $D_0$ are contained in $U \cap V$, where $V$ is as in Proposition 6D, so Dehn’s Lemma gives us a polyhedral disk $D_1 \subset D_0 \cup (U \cap V)$ with $\partial D_1 = \partial D_0$. It is simple to check that $D_1$ meets the requirements.

If we apply Lemma 5B to $D_i$, we obtain a disk $D'$ in $E^3$ such that

(5.1) $\partial D' = \partial D_1 = \partial D$,

(5.2) $D' \subset D_1 \cup (\bigcup B_i) \subset D \cup U$,

(5.3) $D' \subset \text{int } S$.

$D'$ satisfies conditions (i)-(iii) of Proposition 6A, and the proof is therefore complete.

Let $M$ be a 3-manifold, $S$ a 2-manifold in $M$, $V$ a component of $M \setminus S$, and $B$ a subset of $M$. A loop $f: \partial \Delta \to S$ can be shrunk to a point through $B \cap V$ if there is a homotopy $H_t: \partial \Delta \to M$ such that $H_0 = f$, $H_1$ is constant, and $H_t(\partial \Delta) \subset B \cap V$ for all $t > 0$. $S$ is 1-LC through $V$ at a point $p$ in $S$ if, for any neighborhood $B$ of $p$ in $M$, there is a neighborhood $B_1$ of $p$ in $B \cap S$ such that any loop $f: \partial \Delta \to B_1$ can be shrunk to a point through $B \cap V$.

**Corollary 6A.** Let $S$ be a 2-sphere in $E^3$ which is 1-LC through its interior at every point. Then $S$ is tame from the interior.

**Proof.** This follows from the observation:
Proposition 6F. Let $B$ be a subset of $E^3$, and suppose that $i: \Delta \to S$ is an embedding such that $i(\partial \Delta)$ can be shrunk to a point through $B \cap \text{int } S$. Then there is a blister $f$ of $i(\Delta)$ in int $S$ such that $f(\partial \Delta) = i(\partial \Delta)$ and $f(\Delta) \subseteq B$, and we have $O(F,f) = i(\Delta^0)$, for any disk $F \subseteq S$ containing $i(\Delta)$.

Let $M$ be a 3-manifold, $S$ a 2-manifold in $M$, and $V$ a component of $M \setminus S$. A set $X$ in $S$ can be deformed into $V$ if there is a homotopy $H_t: X \to M$ such that $H_0 = I$ and $\forall t > 0. \ H_t(X) \subseteq V$.

Corollary 6B. Let $S$ be a 2-sphere in $E^3$ such that every Sierpinski curve in $S$ can be deformed into int $S$. Then $S$ is tame from the interior.

Proof. The following proposition is an adaptation of Theorem 14 in [6].

Proposition 6G. Let $E_1, E_2, \ldots, E_k, \ldots$ be a decreasing sequence of disks in $S$ whose intersection is a point $p \in \bigcap E_t$, and suppose that $p \cup (\bigcup \partial E_t)$ can be deformed into int $S$.

Then for any open set $B \subseteq E^3$ containing $E_1$, there is a blister $f$ of $E_1$ in int $S$ such that $p \in O(E_1,f) = (f)_{E_1} \subseteq B$ and $f(\partial \Delta) = \partial E_k$ for some $K$.

Proof. Let $H_t: p \cup (\bigcup \partial E_t) \to E^3$ be a homotopy such that $H_0 = I$ and $\forall t > 0. \ H_t(p \cup (\bigcup \partial E_t)) \subseteq \text{int } S$. We may assume that $H_t(p \cup (\bigcup \partial E_t)) \subseteq B$ for each $t$; let $B'$ be an open $3$-cell in $B \setminus S$ containing $H_1(p)$. $H_t(\partial E_k)$ lies in $B'$ for large enough $K$, and can be shrunk to a point in $B'$. By Proposition 6F, there is then a blister $f$ of $E_k$ (and hence of $E_1$) in int $S$ such that $p \in E_k^0 = O(E_1,f) = (f)_{E_1} \subseteq B$ and $f(\partial \Delta) = \partial E_k$.

Proposition 6H. Let $p$ be a point of $S$, $C$ a Cantor set in $S$. Then $p$ is an inaccessible point of some Sierpinski curve in $S$ which misses $C$.

Proof. We just construct such a Sierpinski curve, using the fact that $C$ is 0-dimensional.

If $p$ is an inaccessible point of a Sierpinski curve $X$ in $S$, then there is a decreasing sequence of disks $E_1, E_2, \ldots, E_k, \ldots$ in $S$ such that $\partial E_k \subseteq X$ and $\bigcap E_k = \bigcap E_k^0 = p$. Therefore, Propositions 6G and 6H together imply that $S$ satisfies condition (1) toward its interior at every point.

Remarks. (1) The hypothesis of Corollary 6B requires that any Sierpinski curve in $S$ can be continuously approximated from int $S$. For any 2-sphere $S$ in $E^3$, any Sierpinski curve $X$ in $S$, and any $\delta > 0$, it follows from Bing's side approximation theorem [4, Theorem 16] that there is a $\delta$-homeomorphism $h: X \to \text{int } S$.

(2) Theorem 6 is stated for 2-spheres in $E^3$, but its proof is based on a local criterion for tameness [8, Condition A], so Theorem 1 of [6] can be used with Lemma 5B to extend our results to two-sided 2-manifolds in 3-manifolds.

Bibliography


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