SELF-EQUIVALENCES OF $S^n \times S^k$\(^{(1)}\)

BY

JEROME LEVINE

Introduction. Topology is primarily concerned with classifying various (topological) objects with respect to various equivalence relations. Given two equivalent objects, a secondary problem is then to classify the group (under composition) of equivalences between these objects.

We will deal with the following three categories:

\(\mathcal{H}\): Topological spaces and homotopy classes of maps,

\(\mathcal{P}\): Piecewise-linear manifolds and piecewise-linear maps,

\(\mathcal{D}\): Smooth manifolds and smooth maps.

The equivalences (i.e. invertible maps) in these categories are homotopy equivalences, piecewise-linear homeomorphisms, and diffeomorphisms, respectively. We will be concerned with the group of self-equivalences of an object, with respect to some suitable relation between equivalences. The most interesting relation, aside from the equality relation, is isotopy. Two equivalences are isotopic if they can be joined by a continuous family of equivalences. We will work with the ostensibly weaker relation of concordance or weak isotopy. Two equivalences \(f_0, f_1: X \to Y\) are concordant if there is an equivalence \(F: I \times X \to I \times Y\) such that \(F(t, x) = (t, f_1(x))\) for \(t = 0, 1\) and \(x \in X\) (in \(\mathcal{D}\), some elaboration is needed if \(X\) has a boundary). In \(\mathcal{H}\), both isotopy and concordance are equality.

The simplest nontrivial object in any of the categories is the \(n\)-sphere \(S^n\). The group of concordance classes of self-equivalences is largely determined. In the category \(\mathcal{H}\) and \(\mathcal{P}\), it is \(Z_2\), generated by a reflection. This follows, in \(\mathcal{P}\), from the Alexander process (see [6]). In \(\mathcal{D}\), the group is a semidirect product \(\Gamma_{n+1} \times Z_2\), where \(\Gamma_{n+1}\) is the subgroup of concordance classes of orientation-preserving diffeomorphisms of \(S^n\); the generator of \(Z_2\) acts on \(\Gamma_{n+1}\) by inversion. \(\Gamma_{n+1}\) has been extensively studied by Kervaire and Milnor [12].

In this work we will study the group of concordance classes of self-equivalences of \(S^n \times S^k\). In §1, the induced homological automorphisms are examined; it then suffices to study a certain subgroup \(A^{n,k}\) containing all the homologically trivial self-equivalences. In §2, a group theoretic structure for \(A^{n,k}\) is established, in terms of certain subgroups of \(A^{n,k}\) and group actions involving these subgroups. In §3, the subgroups are identified with certain groups of knots, in the categories \(\mathcal{P}\) and \(\mathcal{D}\), and with the homotopy groups of the space of self-equivalences of a sphere, in

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the category $\mathcal{C}$. In §4, the group actions are investigated. We conclude, in §5, with a number of consequences. For example, in a metastable range of dimensions, we see that there are no unfamiliar self-equivalences of $S^n \times S^k$. Another application is the classification of a certain family of objects in each of our categories, up to equivalence. A special case gives an example of a smooth closed manifold tangential homotopy equivalent, but not piecewise-linearly homeomorphic, to $S^6 \times S^2$.

Of perhaps, independent interest, we obtain an extension of Whitney's technique for removing intersections of submanifolds to the case where one of the manifolds is two-dimensional (see Lemma 3.6).

It is interesting to compare our results with the general obstruction theory developed in [23]. We also refer the reader to [10] for another treatment of the special case of homotopy equivalences of $S^n \times S^n$. Related results have been obtained by Morlet (Comptes Rendus, 1968) and Cerf (Proceedings International Congress of Mathematicians, Moscow 1966), and Lashof-Shaneson [35].

1. Automorphisms of homology.

1.1. $\mathcal{A}$ will denote any of the categories $\mathcal{C}$, $\mathcal{P}$ or $\mathcal{D}$. The group of concordance classes of self-equivalences of $S^n \times S^k$ in the category $\mathcal{C}$, $\mathcal{P}$ or $\mathcal{D}$, will be denoted $H_{n, k} \mathcal{C}$, $H_{n, k} \mathcal{P}$ or $H_{n, k} \mathcal{D}$ will denote any of them. We have natural homomorphisms:

\[
\begin{array}{ccc}
\mathcal{D}_{n, k} & \xrightarrow{\mu_1} & \mathcal{P}_{n, k} & \xrightarrow{\mu_2} & H_{n, k} \\
\end{array}
\]

$\mu_1$ is defined by the process of approximating a diffeomorphism by a piecewise-linear homeomorphism [27]; $\mu_2$ is defined by considering a piecewise-linear homeomorphism as merely a homotopy equivalence.

1.2. Let $\text{Auto} \ H^*(S^n \times S^k)$ be the group of graded ring automorphisms of $H^*(S^n \times S^k)$. If $n > k$, it is isomorphic to $\mathbb{Z}_2 + \mathbb{Z}_2$. If $n = k$, we can obviously identify $\text{Auto} \ H^*(S^n \times S^n)$ with a subgroup of the group of $2 \times 2$-unimodular matrices $GL(2, \mathbb{Z})$, by associating to such an automorphism its matrix representative in $H^*(S^n \times S^n)$ with respect to the natural basis (see [10]). It follows easily from the commutativity of $H^*(S^n \times S^n)$, that $\text{Auto} \ H^*(S^n \times S^n) = GL(2, \mathbb{Z})$, if $n$ is odd, but consists only of the eight matrices:

\[
\begin{pmatrix}
\pm 1 & 0 \\
0 & \pm 1
\end{pmatrix}
\] and
\[
\begin{pmatrix}
0 & \pm 1 \\
\pm 1 & 0
\end{pmatrix}
\]

if $n$ is even.

Let $\Phi: \mathcal{A}_{n, k} \to \text{Auto} \ H^*(S^n \times S^k)$ be the obvious homomorphism.

**Proposition.** If $n > k$, or $n = k = 1, 3$ or 7, or $n = k$ is even, then $\Phi$ is onto. If $n = k$ is odd, but $\neq 1, 3$ or 7, then $\text{Image} \Phi$ is the subgroup of $GL(2, \mathbb{Z})$ consisting of matrices

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]

where $ab \equiv cd \equiv 0 \mod 2$. 

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Proof. If \( n > k \) or \( n = k \) is even, this is easy to verify. If \( n = k \) is odd, but not 1, 3 or 7, this is proven in [24, Lemma 5]. The proof of [24, Lemma 5] can be simplified to also prove the case \( n = k = 1, 3 \) or 7 as follows. It is proved in [13, Appendix B] that \( \text{GL}(2, \mathbb{Z}) \) is generated by

\[
\begin{pmatrix}
\pm 1 & 0 \\
0 & \pm 1
\end{pmatrix}
\text{ and }
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}.
\]

But the corresponding automorphisms of \( H^n(S^n \times S^n) \) are induced by diffeomorphisms; one for

\[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\]

is defined by \((x, y) \mapsto (x, \phi(x) \cdot y)\), where \( \phi : S^n \to SO_{n+1} \) is a map such that \( p \circ \phi \) has degree +1 if \( p : SO_{n+1} \to S^n \) is the usual evaluation map.

1.3. It now seems reasonable to consider the subgroup, Kernel \( \Phi \), of \( A^{n, k} \). But our final results are simpler to state if we, instead consider the somewhat larger subgroup defined by orientation-preserving self-equivalences \( f \) such that \( f|_{x_0 \times S^n} \) is homotopic to the inclusion, for any \( x_0 \in S^n \). We denote this subgroup by \( A^{n, k} \).

Since \( n \geq k \), Kernel \( \Phi \subset A^{n, k} \); if \( n > k \), clearly Kernel \( \Phi = A^{n, k} \). If \( n = k \) is even, it follows from (1.2) that Kernel \( \Phi = A^{n, n} \). But if \( n = k \) is odd, \( A^{n, n} / \text{Kernel} \Phi \) is infinite cyclic; in fact by (1.2) \( \Phi(A^{n, n}) \) is the subgroup of \( \text{GL}(2, \mathbb{Z}) \) consisting of matrices of the form

\[
\begin{pmatrix}
a & 0 \\
0 & 1
\end{pmatrix},
\]

where \( a \) is any integer, if \( n = 1, 3 \) or 7, or \( a \) is even if \( n \) is otherwise odd.

Notice that \( A^{n, n} \) is not normal in \( A^{n, n} \), if \( n \) is odd, because \( \Phi(A^{n, n}) \) is not normal in \( \text{GL}(2, \mathbb{Z}) \). In this case, we leave it to the reader to deduce explicit results on Kernel \( \Phi \) from the results to be obtained concerning \( A^{n, k} \).

2. A decomposition of \( A^{n, k} \).

2.1. Recall the notion of semidirect product of groups. If \( G \) is a group with subgroups \( G_0 \) and \( G_2 \) satisfying:

(i) \( G_0 G_2 = G \), \( G_0 \cap G_2 = \{1\} \),

(ii) \( G_0 \) is normal in \( G \),

then the group structure of \( G \) is entirely determined by the left action of \( G_2 \) on \( G_0 \), \( \phi : G_2 \to \text{Auto} G_0 \), defined by \( \phi(g) \cdot g_0 = gg_0 g^{-1} \). Conversely, given groups \( G_0 \) and \( G_2 \) and such an action \( \phi \), the Cartesian product \( G_0 \times G_2 \) is given a group structure \( G \) by the multiplication:

\[(g_0, g_2) \cdot (g'_0, g'_2) = (g_0 \cdot (\phi(g_2) \cdot g'_0), g_2 \cdot g'_2).\]

2.2. Suppose, in addition, that \( G_0 \) is abelian and has a direct sum splitting \( G_0 = G_1 \oplus \gamma \). Also, suppose that the action \( \phi \) is trivial on \( \gamma \), i.e. \( \phi(g)|\gamma \) is the inclusion, for every \( g \in G_2 \).
PROPOSITION. There is a unique left action, i.e. homomorphism \( \phi: G_2 \to \text{Auto } G_1 \) and function \( \tau: G_2 \to \text{Hom}(G_1, \gamma) \) satisfying:

(i) \( \phi(g) \cdot g' = \phi(g) \cdot g' + \tau(g) \cdot g' \), for \( g \in G_2, g' \in G_1 \),

(ii) \( \tau(g_1 g_2) = \tau(g_1) \phi(g_2) + \tau(g_2) \), for \( g_1, g_2 \in G_2 \).

Proof. \( \phi \) and \( \tau \) are defined by (i), but it must be checked that \( \phi(g) \) is an automorphism, \( \phi \) is a left action and that \( \tau(g) \) is a homomorphism. These are straightforward computations, as is the verification of (ii).

It is a simple matter to reverse this procedure. Given abelian groups \( G_1 \) and \( \gamma \), and a group \( G_2 \) with an action \( \phi \) of \( G_2 \) on \( G_1 \) and a function \( \tau \) satisfying (ii), one can define by (i) an action \( \hat{\phi} \) of \( G_2 \) on \( G_1 \oplus \gamma \) and use \( \hat{\phi} \) to construct the semidirect product. We will denote the resulting group by:

\[
(G_1 \oplus \gamma) \times_{\phi, \tau} G_2.
\]

2.3. Let \( A = A^{n,k} \) be as defined in 1.3. Define subgroups \( A_0, A_1, A_2 \) and \( \alpha \) of \( A \) to consist of those elements represented by \( f: S^n \times S^k \to S^n \times S^k \) satisfying:

\[
(A_0) \ f|D_n+1 \times S^k = \text{inclusion},
\]

\[
(A_1) \ f \text{ extends to a self-equivalence of } S^n \times D^{k+1},
\]

\[
(A_2) \ f \text{ extends to a self-equivalence of } D^{n+1} \times S^k,
\]

\[
(\alpha) \text{ For some } (n+k) \text{-disk } D \subset S^n \times S^k, f(D) \subset D \text{ and } f|(S^n \times S^k - D) = \text{inclusion}.
\]

PROPOSITION. \( \alpha = 0 \) in \( \mathcal{H} \) and \( \mathcal{P} \), while \( \alpha \approx \Gamma^{n+k+1} \) in \( \mathcal{D} \).

Proof. In \( \mathcal{H} \) and \( \mathcal{P} \), any two self-equivalences of a disk, which agree on the boundary, are concordant, by the Alexander process in \( \mathcal{P} \) (see [6]). In \( \mathcal{D} \), \( \Gamma^{n+k+1} \) is precisely the group of concordance classes of self-equivalences of the \((n+k)\)-disk which are the identity on the boundary; thus there is an epimorphism:

\[
\eta: \Gamma^{n+k+1} \to \alpha.
\]

If \( f \) represents \( \eta(\sigma) \), for any \( \sigma \in \Gamma^{n+k+1} \), then the smooth manifold \( S^n \times D^{k+1} \cup_{f} D^{n+1} \times S^k \) is a topological \((n+k+1)\)-sphere representing \( \sigma \). This observation implies that \( \eta \) is injective.

Notice that \( \alpha \) is abelian.

2.4 THEOREM. \( A = (A_1 \oplus \alpha) \times_{\phi, \tau} A_2 \), for suitable \( \phi \) and \( \tau \), assuming \( n \geq 3 \) if \( \mathcal{A} = \mathcal{P} \) or \( \mathcal{D} \). Moreover \( A_1 \) and \( A_2 \) are abelian, and, if \( \mathcal{A} = \mathcal{P} \) or \( \mathcal{D} \), \( A_0 = A_1 \oplus \alpha \).

2.5. We first deal with \( \mathcal{A} = \mathcal{H} \). Notice that a homotopy equivalence \( f: S^n \times S^k \to S^n \times S^k \) represents an element of \( H_i \) if and only if \( p_i \circ f \equiv p_0 \), where \( p_i \) is projection on the \( i \)th factor (\( i = 1 \) or \( 2 \)). It follows immediately that \( H_1 \cap H_2 = \{1\} \).

We now show \( H_2 H_1 = H \). Suppose \( f \) represents an element of \( H \) and \( f_i = p_i \circ f \) are the coordinate functions. By definition of \( H \), the map \( y \mapsto f_2(x, y) \) has degree \( +1 \), for any \( x \in S^n \). Therefore the map \( g \) defined by \( g(x, y) = (x, f_2(x, y)) \) represents an element of \( H \) and, by the above paragraph, of \( H_1 \). Now let \( h: S^n \times S^k \to S^n \times S^k \) be a map satisfying \( h \circ g \equiv f \). It follows that \( f_2 \) is homotopic to the map \((x, y)\)
\[ h_2(x, f_2(x, y)) \), where \( h_2 = p_2 \circ h \). Since we can consider \( f_2 \) and \( h_2 \) as representing elements of \( \pi_\infty(G_{k+1}) - G_{k+1} \) is the space of maps \( S^k \rightarrow S^k \) of degree + 1—we can interpret this fact as saying that \( h_2 \) represents the zero element of \( \pi_\infty(G_{k+1}) \). But this means \( h_2 \cong p_2 \), which implies that \( h \) represents an element of \( H_2 \).

We now show \( H_1 \) is normal in \( H \). Let \( g \) represent an element of \( H_1 \)—we may assume \( p_1 \circ g = p_1 \). If \( g_2 = p_2 \circ g \), then the map \( y \mapsto g_2(x, y) \) has degree + 1, for any \( x \in S^n \). Thus we may assume \( g_2|D^n_x \times S^k = p_2|D^n_x \times S^k \) and, therefore, \( g|D^n_x \times S^k \) is inclusion. Now suppose \( f \) represents any element of \( H \). By definition of \( H \) we may assume that \( f|D^n_x \times S^k = \text{inclusion} \). We must show that a map \( h \), satisfying \( f \circ g \cong h \circ f \), represents an element of \( H_1 \), i.e. \( p_1 \circ h \cong p_1 \). Notice that \( p_1 \circ f \circ g = p_1 \circ f \), since \( g|D^n_x \times S^k = \text{inclusion} \), \( p_1 \circ g = p_1 \), and \( f|D^n_x \times S^k = \text{inclusion} \). Now we observe \( p_1 \circ h \circ f \cong p_1 \circ f \circ g \cong p_1 \circ f \); since \( f \) is a homotopy equivalence, we have \( p_1 \circ h \cong p_1 \).

Finally, we show \( H_1 \) and \( H_2 \) are abelian. As pointed out above, any representative \( g \) of an element of \( H_1 \) may be assumed to satisfy \( p_1 \circ g = p_1 \) and \( g|D^n_x \times S^k = \text{inclusion} \). But it follows that another such \( g' \) may be assumed to satisfy \( p_1 \circ g' = p_1 \) and \( g'|D^n_x \times S^k = \text{inclusion} \). Now we observe that \( g \circ g' = g' \circ g \). Similarly for \( H_2 \).

2.6. We now assume \( k \geq 0 \) or \( S^k \) and \( n \geq k \), \( n \geq 3 \).

**Lemma.** Every element of \( A \) can be represented by \( f \) satisfying \( f|D^n_x \times S^k = D^n_x \times S^k \).

**Proof.** By definition of \( A \), \( f|D^n_x \times S^k \) is homotopic to the inclusion. Choose \( x_1 \in S^n - x_0 \); then we may isotopically deform \( f \) so that \( f(x_0 \times S^k) \) is disjoint from \( x_1 \times S^k \). If \( n > k \), this follows from general position; if \( n = k \), we use the technique of Whitney [28] (see [25] or 3.6 below for the piecewise-linear case) since \( n \geq 3 \). Note that the intersection number of \( f(x_0 \times S^k) \) and \( x_1 \times S^k \) is 0.

We may now move \( f(x_0 \times S^k) \) inside \( D^n_x \times S^k \), by another isotopy. In fact we may even move \( f(D^n_x \times S^k) \) inside \( D^n_x \times S^k \), if \( x_0 \in D^n_x \). For \( n + k \geq 6 \), it follows from the \( k \)-cobordism theorem [21] if \( k > 1 \), or the \( s \)-cobordism theorem [17] if \( k = 1 \), that \( \text{Cl}(D^n_x \times S^k - f(D^n_x \times S^k)) \) is equivalent to \( I \times S^{n-1} \times S^k \). Using this equivalence, there is an obvious isotopy to make \( f(D^n_x \times S^k) = D^n_x \times S^k \).

If \( n + k \leq 5 \) (and \( n \geq 3 \), we may assume \( f|D^n_x \times S^k = \text{inclusion} \)—by general position if \( k = 1 \), or [29], [30] if \( k = 2 \). Then \( D^n_x \times S^k \) and \( f(D^n_x \times S^k) \) are both tubular or regular neighborhoods of \( f(x_0 \times S^k) \); therefore, by the tubular or regular neighborhood theorem, an ambient isotopy of \( S^n \times S^k \) will carry \( D^n_x \times S^k \) onto \( f(D^n_x \times S^k) \).

2.7 Lemma. Suppose \( f \) represents an element of \( A_2 \) (\( A_1 \)) and satisfies \( f|D^n_x \times S^k = \text{inclusion} \) (\( f|S^n \times D^n_x = \text{inclusion} \)). Then \( f \) is concordant to the identity.

**Proof.** Suppose \( f|D^n_x \times S^k = \text{inclusion} \) and \( f \) extends to a self-equivalence \( F \) of \( D^{n+1} \times S^k \). We may assume that \( F \) is the "product extension" in a neighborhood of \( S^n \times S^k \) i.e.

\[ F(tx, y) = (tf_1(x, y), f_2(x, y)) \]
for \((x, y) \in S^n \times S^k\) and \(t \) near 1. Therefore \(F|V\) = inclusion, where \(V\) is a neighborhood of \(D^*_+ \times S^k\). If \(D^*_+ \times S^k\) is the disk of radius 1/2 in \(D^{k+1}\), then, by an isotopy, we may move \(D^*_+ \times S^k\) inside \(V\). So we may assume \(F|D^*_+ \times S^k\) = inclusion. But now \(F|(D^{k+1} - D^*_+) \times S^k\) defines a concordance between \(f\) and the identity.

A similar argument works for \(A_1\).

2.8 LEMMA. Any element of \( A_1 \) \((A_2)\) admits a representative \( \bar{f} \) satisfying \( f|D^*_+ \times S^k = \text{inclusion} \) \((f|S^n \times D^*_+ = \text{inclusion})\).

PROOF. Given \( f \) representing an element of \( A_1 \), we will construct a self-equivalence \( g \), concordant to the identity, which agrees with \( f \) on a neighborhood of \( x_0 \times S^k \).

After expanding this neighborhood, by an isotopy, to contain \( D^*_+ \times S^k \) we may assume \( g \) agrees with \( f \) on \( D^*_+ \times S^k \). Then \( g^{-1} \circ f \) is concordant to \( f \) and satisfies the condition of the lemma.

Let \( F \) be a self-equivalence of \( S^n \times D^{k+1} \) extending \( f \). Given \( x_0 \in S^n \), we may assume that \( f|U \) is the inclusion, for some neighborhood \( U \) of \((x_0, 0)\) in \( S^n \times D^{k+1} \).

We now would like to alter \( F \) by an isotopy so that \( F(x_0 \times D^{k+1}) \) meets \( S^n \times 0 \) only at \((x_0, 0)\).

If \( k = 1 \), it follows from general position for \( n \geq 4 \) and [29] or [30] for \( n = 3 \), that \( f|_{x_0 \times D^{k+1}} \) is isotopic to the inclusion. If \( k \geq 2 \), we can apply Whitney’s technique to remove the undesired intersections, since \( S^n \times 0 \) and \( F(x_0 \times D^{k+1}) \) have intersection number \( \pm 1 \) and an intersection of precisely this sign already occurs at \((x_0, 0)\).

It now follows that for sufficiently small disk neighborhoods \( V_1 \) of \( x_0 \) (in \( S^n \)) and \( V_2 \) of \( 0 \) (in \( D^{k+1} \)), \( F(V_1 \times D^{k+1}) \) meets \( S^n \times V_2 \) only at \( F(V_1 \times V_2) = V_1 \times V_2 \), and \( F|V_1 \times V_2 = \text{inclusion} \).

We now begin to define a self-equivalence \( G \) of \( S^n \times D^{k+1} \) by \( G|S^n \times V_2 = \text{inclusion} \), and \( G|V_1 \times D^{k+1} = F|V_1 \times D^{k+1} \). It follows from our considerations that this defines an imbedding \( G_0: S^n \times V_2 \cup V_1 \times D^{k+1} \rightarrow S^n \times D^{k+1} \). If \( k = 1 \), we have shown that \( F(x_0 \times D^{k+1}) = x_0 \times D^{k+1} \) and, therefore, we may assume \( G_0 \) maps onto \( S^n \times V_2 \cup V_1 \times D^{k+1} \). The extension to a self-equivalence of \( S^n \times D^{k+1} \) is, then, formally equivalent to the extension of a self-equivalence of \( S^{n-1} \times I \times S^k \cup D^n \times 0 \times S^k \) to one of \( D^n \times I \times S^k \). But this is always possible.

Now assume \( k \geq 2 \). Let

\[
X = S^n \times D^{k+1} - \text{Image } G_0, \quad X_0 = \text{Cl } (S^n \times S^k - G_0(V_1 \times S^k)), \quad X_1 = G_0(\text{Cl } (S^n - V_1) \times \partial V_2), \quad Y = G_0(\partial V_1 \times \text{Cl } (D^{k+1} - V_2)).
\]

Then \( X \) is an \( h \)-cobordism from \( X_0 \) to \( X_1 \), extending the \( h \)-cobordism \( Y \) from \( \partial X_0 \) to \( \partial X_1 \). If we choose \( V_2 \) to be a concentric disk in \( D^{k+1} \), then \( G_0 \) defines an obvious equivalence of \( Y \) with \( \partial V_1 \times \partial V_2 \times I \). According to the \( h \)-cobordism theorem (dim \( X \geq 6 \) and \( X, X_0, X_1 \) are 1-connected) this extends to an equivalence of \( X \) with \( \text{Cl } (S^n - V_1) \times \partial V_2 \times I \). This equivalence can be used to extend \( G_0 \) to the desired self-equivalence \( G \), since \( \text{Cl } (D^{k+1} - V_2) \) is equivalent to \( \partial V_2 \times I \).
Now define \( g = G|S^n \times S^k \). Since \( G|S^n \times V_2 = \text{inclusion} \), \( G|S^n \times \text{Cl}(D^{k+1} - V_2) \) defines a concordance from \( g \) to the identity. Also \( g|V_1 \times S^k = F|V_1 \times S^k = f|V_1 \times S^k \). Thus \( g \) is as desired at the beginning of the proof.

When \( f \) represents an element of \( A_2 \), the argument is easier. Let \( F \) be a self-equivalence of \( D^{n+1} \times S^k \) extending \( f \). Then \( F|0 \times S^k \) is isotopic to the inclusion (by general position if \( n > k \), and \([29]\) or \([30]\) if \( n = k \)). Furthermore, since \( F \) is orientation-preserving, we may assume that \( F|D \) is the inclusion for any \((n + k + 1)\)-disk \( D \subset \text{interior}(D^{n+1} \times S^k) \). If \( V \) is a concentric disk neighborhood of 0 in \( D^{n+1} \), we may assume by the tubular or regular neighborhood theorem that \( F(V \times S^k) = V \times S^k \). If \( D \) contains \( V \times D_k^+ \), then \( F|\text{Cl}(D^{n+1} - V) \times S^k \) defines a concordance from \( f \) to a self-equivalence with the desired property.

2.9. We are now ready to prove Theorem 2.4 for \( \mathcal{P} \) or \( \mathcal{D} \).

\[ A = A_0A_2. \] Suppose \( f \) represents an element of \( A \). By Lemma 2.6 we may assume \( f(D^*_+ \times S^k) = D^*_+ \times S^k \). Now extend \( f|D^*_+ \times S^k \) to a self-equivalence \( G \) of \( D^{n+1} \times S^k \); the restriction \( g \) to \( S^n \times S^k \) represents an element of \( A_2 \). Clearly \( f \circ g^{-1} \) represents an element of \( A_0 \), and \( f = (f \circ g^{-1}) \circ g \) is the desired factorization of \( f \).

\[ A_0 \cap A_2 = \{1\}. \] This is implied by Lemma 2.7.

\( A_0 \) is normal in \( A \). Let \( f \) represent an element of \( A \) and \( g \) an element of \( A_0 \). Then we may assume \( f(D^*_+ \times S^k) = D^*_+ \times S^k \), by Lemma 2.6, and \( g|D^*_+ \times S^k = \text{inclusion} \), by definition. But then \( f \circ g \circ f^{-1}|D^*_+ \times S^k = \text{inclusion} \), and, therefore, represents an element of \( A_0 \).

\( \alpha \subset A_0 \). This follows from the definitions by choosing \( D \) disjoint from \( D^*_+ \times S^k \).

\[ A_1 \subset A_0. \] This follows from Lemma 2.8.

\[ A_1 \cap \alpha = \{1\}. \] This follows from Lemma 2.7 by choosing \( D \) disjoint from \( S^n \times D^*_+ \).

\[ A_0 = A_1\alpha. \] Suppose \( f \) is a self-equivalence of \( S^n \times S^k \) satisfying \( f|D^*_+ \times S^k = \text{inclusion} \). We can extend \( f \) to a self-equivalence \( F_0 \) of a neighborhood \( U \) of \( S^n \times S^k \cup x_0 \times D^{k+1} \) in \( S^n \times D^{k+1} \) by the “product extension” (see proof Lemma 2.7) near \( S^n \times S^k \) and the identity near \( x_0 \times D^{k+1} \). The complement of \( U \) in \( S^n \times D^{k+1} \) is an \((n + k + 1)\)-disk \( D \). It is well known that, after changing \( F_0 \) on a disk in \( \partial D \), \( F_0|\partial D \) may be extended to a self-equivalence of \( D \). But such a change in \( F_0 \) can be effected by changing \( f \) on a disk in \( S^n \times S^k \).

This argument shows that a representative of any element of \( A_0 \), after being changed on a disk, extends to a self-equivalence of \( S^n \times D^{k+1} \).

\( \alpha \subset \text{center of } A \). Any orientation-preserving self-equivalence of \( S^n \times S^k \) may be assumed to leave a disk \( D \) fixed. It therefore commutes with any representative of an element of \( \alpha \), which is fixed outside of \( D \).

\( A_0 \) and \( A_2 \) are abelian. If \( f, g \) represent elements of \( A_0 \) (\( A_2 \)), then, by definition (Lemma 2.8), we may assume that \( f|D^*_+ \times S^k \) and \( g|D^*_+ \times S^k(f|S^n \times D^*_+ \text{ and } g|S^n \times D^*_+) \) are inclusions.

Then \( f \) and \( g \) commute.

This completes the proof of Theorem 2.4.
2.10. Notice that the homomorphisms:

\[ f_{n,k} \xrightarrow{\mu_1} p_{n,k} \xrightarrow{\mu_2} \bar{H}_{n,k} \]

defined in 1.1 preserve the subgroups \( A, A_0, A_1 \) and \( A_2 \), and, therefore, the action \( \phi \).

3. Determination of the groups \( A_1 \) and \( A_2 \).

3.1. Let \( G_p \) denote the space of maps \( S^p \rightarrow S^{p-1} \) of degree +1. We write \( H(m, p) = \pi_m(G_{p+1}) \). In the categories \( \mathcal{P} \) and \( \mathcal{D} \), let \( A(m, p) \) be the group of concordance classes of framed imbeddings \( S^m \rightarrow S^{m+p+1} \) i.e. imbeddings \( S^m \times R^{p+1} \rightarrow S^{m+p+1} \). Two such \( f_0, f_1 \) are concordant if there is an imbedding \( F: I \times S^m \times R^{p+1} \rightarrow I \times S^{m+p+1} \) such that \( F(t, x) = (t, f_t(x)) \), for \( t = 0, 1 \) and \( x \in S^m \times R^{p+1} \).

Observe that \( D(m, p) = FC_{m+1}^p \), in the notation of [7]. Furthermore, one can show, using the arguments in [8, §3], that \( P(m, p) \approx \pi_{m+1} \), the group of concordance classes of smooth framed submanifolds \( \Sigma \) of \( R^{m+p+1} \), where \( \Sigma \) is piecewise-smoothly homeomorphic to \( S^m \). These groups are extensively studied in [7] and [16]. In particular there are monomorphisms:

\[ D(m, p) \xrightarrow{\mu_1} P(m, p) \xrightarrow{\mu_2} H(m, p). \]

\( \mu_1 \) is defined by the passage from a framed imbedding to the framed submanifold determined by its image; \( \mu_2 \) is defined in [16]. Recall from [7], [16] the exact sequences:

\[ \cdots \rightarrow \Gamma_{m+1} \rightarrow D(m, p) \xrightarrow{\mu_1} P(m, p) \xrightarrow{\theta} \Gamma_m \rightarrow \cdots \]

\[ \cdots \rightarrow P_{m+1} \xrightarrow{\partial} P(m, p) \xrightarrow{\mu_2} H(m, p) \xrightarrow{\omega} P_m \rightarrow \cdots \]

\( P_m \) is defined to be 0 for \( m \) odd, \( Z \) for \( m = 0 \mod 4 \), and \( Z_2 \) for \( m = 2 \mod 4 \). Sequences (a) and (b) are valid for \( p \geq 2, m \geq 1 \) (see [16], [7]).

3.2. We construct homomorphisms:

\[ \lambda_1: A_1^{n,k} \rightarrow A(n, k), \quad \lambda_2: A_2^{n,k} \rightarrow A(k, n). \]

\( \mathcal{A} = \mathcal{H} \). An element of \( H_1 \) is represented by a map of the form \((x, y) \mapsto (x, g(x, y))\), where \( g: S^n \times S^k \rightarrow S^k \) represents an element of \( \pi_n(G_{k+1}) = H(n, k) \). This induces the homomorphism \( \lambda_1 \) and, in a similar fashion, \( \lambda_2 \) is defined.

\( \mathcal{A} = \mathcal{P} \) or \( \mathcal{D} \). Let \( f \) represent an element \( \xi \in A_1^{n,k} \). Then \((x, t) \mapsto (f(x), t)\) defines a self-equivalence \( \hat{f} \) of \( S^n \times S^k \times R \). Choose an orientation-preserving imbedding \( R^k \subset S^k \) and notice the standard imbeddings

\[ S^n \times S^k \times R \subset S^n \times R^{k+1} = S^n \times R \times R^k \subset R^{n+k+1} \times R^k = R^{n+k+1} \]

defined by \((x, y, t) \mapsto (x, e^t y), (x, t, y) \mapsto (e^t x, y)\). Then the composite imbedding:

\[ S^n \times R^{k+1} = S^n \times R^k \times R \subset S^n \times S^k \times R \xrightarrow{\hat{f}} S^n \times S^k \times R \subset S^n \times R^{k+1} \subset \bar{R}^{n+k+1} \]
represents the element $\lambda_1(\xi) \in A(n, k)$. $\lambda_1$ is clearly well defined. $\lambda_2$ is defined similarly.

An alternative description of $\lambda_1(\xi)$ is as follows. Let $F$ be an extension of $f$ to a self-equivalence of $S^n \times S^{k+1}$. The standard equivalence of $R^{k+1}$ with the interior of $D^{k+1}$ induces an imbedding $S^n \times R^{k+1} \subset S^n \times D^{k+1}$. Now, the composition:

$$S^n \times R^{k+1} \subset S^n \times D^{k+1} \xrightarrow{F} S^n \times D^{k+1} \subset S^n \times R^{k+1} \subset R^{n+k+1}$$

represents $\lambda_1(\xi)$. This seems to depend upon $F$, but its equivalence to the first definition is easily verified. A similar description holds for $\lambda_2$.

That $\lambda_1$ is a homomorphism follows by choosing representative self-equivalences which restrict to the inclusion on $D_n^+ \times S^k$ or $D_n^- \times S^k$, by Lemma 2.8. The framed imbeddings constructed are then standard on $D_n^+$ or $D_n^-$. The composition of the self-equivalences then clearly yields the sum of the framed imbeddings.

The functions $\mu_i$ and $\mu_i'$ are related by the $\lambda_i$ according to the commutative diagram:

$$\begin{array}{ccc}
D_1 & \xrightarrow{\mu_1} & P_1 \\
\downarrow \lambda_1 & & \downarrow \lambda_1 \\
D(n, k) & \xrightarrow{\mu'_1} & P(n, k)
\end{array}$$

$$\begin{array}{ccc}
P_1 & \xrightarrow{\mu_2} & H_1 \\
\downarrow \lambda_1 & & \downarrow \lambda_1 \\
P(n, k) & \xrightarrow{\mu'_2} & H(n, k)
\end{array}$$

Similarly for $\lambda_2$.

3.3. The fibration $G_{n+1} \to S^n$ defined by evaluation at a point of $S^n$ induces a homomorphism $H(k, n) = \pi_k(G_{n+1}) \to \pi_k(S^n)$. We define:

$$\epsilon : A(k, n) \to \pi_k(S^n)$$

by composing this with the necessary $\mu'_i$, and write $A_0(k, n) = \text{Kernel } \epsilon$. Notice that $A_0(k, n) = A(k, n)$ unless $k = n$ is odd (see (1.2)).

**Theorem.** (i) $P_{1}^{n-1} = 0$, $D_{1}^{n-1} \cong \theta^{n-1}/bP^{n+2}$ (see [12]) for $n \geq 3$,

(ii) $\lambda_1$ is an isomorphism onto $A(n, k)$ if either $\mathcal{A} = \mathcal{P}$ or $n \geq 3$ and $k \geq 2$,

(iii) $\lambda_2$ is an isomorphism onto $A_0(k, n)$ if either $\mathcal{A} = \mathcal{H}$ or $n \geq 3$.

Compare [1, §5] for $k = 1$.

3.4. The definition of $A$ (see 1.3) implies that $\text{Image } \lambda_2 \subset A_0(k, n)$. The statements of the theorem for $\mathcal{A} = \mathcal{P}$ follow immediately from the previous (2.5) observation that elements of e.g. $H_1$ are represented by maps of the form $(x, y) \mapsto (x, g(x, y))$, where $g$ represents an element of $\pi_n(G_{k+1})$. The correspondence, $\lambda_1$, so defined, is clearly injective. Since any map $S^n \times S^k \to S^n$ extends to a map $S^n \times D^{k+1} \to D^{k+1}$, by radial extension, it follows that $\lambda_1$ is surjective. For $\lambda_2$ the same arguments work except that the definition of $A$ (see above) restricts $\text{Image } \lambda_2$ to $A_0(k, n)$.

3.5. We now restrict ourselves to $\mathcal{A} = \mathcal{P}$ or $\mathcal{D}$. The surjectivity of $\lambda_1$ and $\lambda_2$ is seen as follows. Given a framed imbedding of $S^n$ in $S^{n+k+1}$, we consider the restriction
$F: S^n \times D^{k+1} \to S^{n+k+1}$. By an argument in [16, §3.5] which uses the $h$-cobordism theorem—since $k \geq 2$, $n \geq 3$—we may assume that $F(S^n \times D^{k+1}) = S^n \times D^{k+1}$. It then follows from our second description of $\lambda_1$ that $\lambda_1$ is onto. A similar argument works for $\lambda_2$, if $k > 1$. If $k = 1$, notice that $A(1, n) \approx \mathbb{Z}_2$, and its generator is the image, under $\lambda_2$, of the element of $A^{2, 1}$ represented by $(x, y) \mapsto (\phi(y) \cdot x, y)$, where $\phi: S^1 \to SO_{n+1}$ is essential.

3.6. Before continuing, we digress briefly to prove a slight extension of Whitney’s theorem [28] on removing intersections of submanifolds. This will be needed so that we can apply Whitney’s technique in all the dimensions needed. The result is stated for both the smooth and piecewise-linear category.

**Lemma.** Suppose $M$, $N$ are oriented connected locally flat submanifolds of the oriented simply-connected manifold $V$ where $\dim M + \dim N = \dim V$, $\dim N \geq 2$ $\dim M > 2$, and $M$ meets $N$ in general position. If the intersection number of $M$ and $N$ is zero and, for $\dim N = 2$, $\pi_1(V - M)$ is abelian, there is an isotopy of $V$, stationary on $\partial V$, which separates $M$ and $N$.

**Proof.** We describe the necessary modifications in Whitney’s proof [28] (see also [20])—placing ourselves in the differential situation. Recall that the intersections of $M$ and $N$ are removed by a sequence of local isotopies, each of which removes a pair of oppositely oriented intersections.

The construction of a local isotopy is begun by joining the points by arcs $a_1$ in $M$ and $a_2$ in $N$ and then constructing vector fields $v_1$ along $a_1$ such that $v_1$ is normal to $M$ and inwardly tangent to $a_2$ at $a_1 \cap a_2$—similarly for $v_2$. If we push $a_1$ slightly along $v_1$, we obtain new arcs $a_1'$ in $V$ which, we may assume, again meet at two points. Then $a_1' \cup a_2'$ contains a loop $l$ in $V - (M \cup N)$. If $M$ and $N$ both have dimension $> 2$, it follows by general position that $l$ is null-homotopic in $V - (M \cup N)$ and bounds an imbedded 2-disk $D$. If dimension $N = 2$, it is not necessarily true that $l$ is null-homotopic in $V - (M \cup N)$. Now notice that $H_1(V - M)$ is generated by the element $\alpha$ represented by any small loop linking $M$ once. But $l$ may be modified, e.g. by adding twists to $v_1$ or letting $a_2$ wind around one of its endpoints extra times, to add any multiple of $\alpha$ to its homology class. Thus we may assume $l$ is null-homologous, and so null-homotopic, in $V - M$. Since $\dim M > 2$, a general position argument allows us to complete the construction of $D$.

The next step (see [20]) is to construct a normal frame $w_1, \ldots, w_n$ to $D$, so that $w_1, \ldots, w_k$ is tangent to $N$ along $a_2$ and normal to $M$ along $a_1$, where $\dim V = n + 2$ and $\dim N = k + 1$. This is begun by constructing $w_1, \ldots, w_k$ along $a_1 \cup a_2$ and extending over $D$. The obstruction to the extension is an element of $\pi_k(V_{n,k})$, where $V_{n,k}$ is the Stiefel manifold of $k$-frames in $n$-space; since $n - k = (\dim M) - 1 > 1$, $\pi_k(V_{n,k}) = 0$.

The remainder of the proof is identical to Whitney’s. For the piecewise-linear case, we can construct the arcs $a_1$ as above and then smooth $M$ and $N$ in neighborhoods of $a_1$ and $a_2$, respectively, to enable us to apply the procedures of Whitney.
with the above modifications—see [3] for a description of the necessary smoothing. Alternatively, we could modify the arguments of [25] to obtain the piecewise-linear result.

3.7. It is illuminating to examine two simple examples toward understanding the necessity of the hypotheses of Lemma 3.6.

Example 1. Let \( K_1, K_2 \subseteq S^3 \) be trivial knots with linking number zero, and \( D_1, D_2 \subseteq D^4 \) disks bounded by \( K_1, K_2 \) respectively. Then the intersection number of \( D_1 \) and \( D_2 \) is zero, but, if \( K_1 \) and \( K_2 \) are nontrivially linked we cannot remove the intersections of \( D_1 \) and \( D_2 \). Note that, we may choose \( D_1, D_2 \) so that \( \pi_1(D^4 - D_i) \) is abelian.

Example 2. Let \( K \subseteq S^{n+2} \) be an imbedded \( n \)-sphere such that \( \pi_1(S^{n+2} - K) \) is not abelian. Let \( C \subseteq S^{n+2} - K \) be an imbedded circle representing a nonzero element of the commutator subgroup of \( \pi_1(S^{n+2} - K) \). If \( n > 1 \), \( C \) is unknotted. Let \( V = I \times S^{n+2}, M = I \times K, \) and \( N \) an imbedded 2-disk in \( V \) bounded by \( 0 \times C \). The intersection number of \( M \) and \( N \) is zero, but the intersections cannot be removed since \( C \) is essential in \( S^{n+2} - K \).

3.8. We now return to the proof of Theorem 3.3. It remains to prove the injectivity of \( \lambda_1 \) and \( \lambda_2 \), and assertion (i). We first consider \( A_1^{n,k} \). Suppose \( F_0, F_1 \) are self-equivalences of \( S^n \times D^{k+1} \), and the compositions:

\[
S^n \times R^k + 1 \xrightarrow{F_i} S^n \times D^{k+1} \subseteq S^n \times R^k + 1 \subseteq R^{n+k+1}
\]

are concordant framed imbeddings. Let \( C: I \times S^n \times R^k + 1 \rightarrow I \times S^{n+k+1} \) be the concordance.

Recall the imbedding \( S^n \times R^k + 1 = S^n \times R^k \times R \subseteq R^{n+k+1} \) defined by \((x, y, t) \rightarrow (e^t x, y)\). If \( S^{n+k+1} \) is represented as the one-point compactification of \( R^{n+k+1} \), then the complement of this imbedding is a \( k \)-sphere \( \Sigma \) which links \( S^n \times 0 \) once. Now the intersection number of \( C(I \times S^n \times 0) \) with \( I \times \Sigma \) is 0. If \( k = 1 \), we assume \( \pi_1(I \times S^{n+1} - C(I \times S^n \times 0)) \) is abelian. Therefore we can apply Whitney’s procedure, extended by Lemma 3.6, to change \( C \) so that \( C(I \times S^n \times 0) \subseteq I \times S^n \times R^k + 1 \).

We can then move \( C(I \times S^n \times 0) \) into an arbitrarily small neighborhood of \( I \times S^n \times 0 \). If \( D_0^{k+1} \) is a disk of small radius, we may assume \( C(I \times S^n \times D_0^{k+1}) \subseteq I \times S^n \times D^{k+1} \). The annular region \( C(I \times S^n \times D^{k+1} - C(I \times S^n \times D_0^{k+1})) \) is an \( h \)-cobordism between \( I \times S^n \times S^k \) and \( C(I \times S^n \times \partial D_0^{k+1}) \), and \( F_0, F_1 \) define an equivalence of

\[
I \times S^n \times (S^k \times I) = I \times S^n \times \text{Cl} (D^{k+1} - D_0^{k+1})
\]

with \( \text{Cl} (I \times S^n \times D^{k+1} - C(I \times S^n \times D_0^{k+1})) \). By the \( h \)-cobordism theorem, for \( k > 1 \), and the \( s \)-cobordism theorem [17] if \( k = 1 \) (note that, if \( k = 1 \) and \( C \) as above, \( \pi_1(I \times S^n \times D^{k+1} - C(I \times S^n \times D_0^{k+1})) \approx Z \)), \( F_0, F_1 \) and \( C(I \times S^n \times D_0^{k+1}) \) extend to a self-equivalence of \( I \times S^n \times D^{k+1} \). Thus \( F_0 \) and \( F_1 \) are concordant, and we have proved (ii) when \( k \geq 2 \).
The proof that $\lambda_2$ is injective is the same, except that special consideration when $n=1$ is not required. On the other hand, when $k=1$, it is necessary to point out that $\pi_1(I\times S^{n+1} - I\times\Sigma)\approx\mathbb{Z}$ in order to apply Lemma 3.6.

3.9. We now prove (i). Let $F_0$ be a self-equivalence of $S^n\times D^2$ and $F'_0$ the imbedding

$$S^n\times D^2 \xrightarrow{F_0} S^n\times D^2 \subset S^n\times R^2 \subset R^{n+2} \subset S^{n+2};$$

notice that $S^{n+2} - F'_0(S^n\times 0)$ is a homotopy circle. It is proved in [15] or [33] for $n\geq 4$, and [32] for $n=3$, that any collared imbedded $n$-sphere in the $(n+2)$-sphere whose complement is a homotopy circle bounds a collared $(n+1)$-disk. From this it follows that, for some orientation-preserving self-equivalence $h$ of $S^n$ we can define a self-equivalence $F_1 = h\times 1$ of $S^n\times D^2$ such that the framed imbeddings:

$$S^n\times R^2 \longrightarrow S^n\times D^2 \longrightarrow S^n\times D^2 \subset S^n\times R^2 \subset R^{n+2}$$

are isotopic. Now we can apply the argument in (3.8) to prove that $F_0$ and $F_1$ are concordant, since $C$ can be chosen to be an isotopy from which it follows that $\pi_1(I\times S^{n+2} - C(I\times S^n\times 0))$ is abelian.

In $\mathcal{B}$ all orientation-preserving self-equivalences of $S^n$ are isotopic; thus we have shown that $P_{n+1} = 0$ for $n \geq 3$.

3.10. We can define a homomorphism $\theta: \Gamma^{n+1} \rightarrow D_1^{n+1}$ by associating to a self-equivalence $h$ of $S^n$, the self-equivalence $h\times 1$ of $S^n\times S^1$. The argument of (3.9) shows $\theta$ is onto. Since $\Gamma^n = 0$ ([5]), this proves $D_1^{n+1} = 0$; if $n \geq 4$, $\theta^{n+1} \approx \Gamma^{n+1}$. We will show that Kernel $\theta = bP^{n+2}$, completing the proof of (i).

If $h$ represents an element of Kernel $\theta$, then the imbedding $i \circ h$, where $i: S^n \rightarrow S^{n+2}$ is the standard inclusion, is concordant to $i$. By [31, Theorem III. 3], this proves $h$ represents an element of $bP^{n+2}$. Thus Kernel $\theta = bP^{n+2}$.

Conversely if $h$ represents an element of $bP^{n+2}$, let $C'$ be a concordance between $i$ and $i \circ h$. $C'$ also defines a concordance between the framed imbedding

$$S^n\times R^2 \xrightarrow{h\times 1} S^n\times R^2 \subset S^{n+2}$$

and some framing of $i$. But since $\pi_n(SO_2) = 0$ if $n \geq 3$, this is isotopic to $S^n\times R^2 \subset S^{n+2}$. Now, if we knew that $\pi_1(I\times S^{n+2} - C'(I\times S^n))$ were abelian, we could use the argument of (3.8) to prove that $h\times 1$ (on $S^n\times S^1$) is concordant to the identity (and so $bP^{n+2} \subset \text{Ker} \theta$).

3.11. It remains to construct $C'$ as desired. Recall the argument of [31, p. 262]. Let $\Sigma^{n+1} = D^{n+1} \cup_h D^{n+1}$, representing an element of $bP^{n+2}$; then $\Sigma^{n+1}$ imbeds smoothly in $S^{n+3}$. In fact, if $V^{n+2}$ is a $[(n+1)/2]$-connected parallelizable manifold bounded by $\Sigma^{n+1}$, we can imbed $V^{n+2}$ in $S^{n+3}$. Notice also that, for an imbedding of $\Sigma^{n+1}$ in $S^{n+3}$ constructed this way, $\pi_1(S^{n+3} - \Sigma^{n+1})$ is abelian (see e.g. [15]). Let $B_1$, $B_2$ be disjoint $(n+3)$-balls in $S^{n+3}$ with $B_1 \cap \Sigma^{n+1} = (n+1)$-ball. There is a diffeomorphism $S^{n+3} - (B_1 \cup B_2) \approx I \times S^{n+2}$ which carries $\Sigma^{n+1} - (B_1 \cup B_2)$...
onto the image of a concordance $C'$ between $i \circ h$ and $i$ (see [31, p. 262]). Then
\[\pi_1(I \times S^{n+2} - C'(I \times S^n)) \approx \pi_1(S^{n+3} - \Sigma^{n+1})\]
which is abelian.

This completes the proof of Theorem 3.3.

4. **Group structure of $A^{n,k}$.**

4.1. We now study the action $\phi$, and the function $\tau$, which occur in the description of $A^{n,k}$ given by Theorem 2.4. By Theorem 3.3, $\phi$ corresponds to an action of $A_0(k, n)$ on $A(n, k)$, also called $\phi$, and $\tau$ corresponds to a bilinear pairing $D(n, k) \otimes D_0(n, k) \to \Gamma^{n+k+1}$, also called $\tau$.

We first study $\phi$ in the category $\mathcal{C}$. The fibration $G_{m+1} \to S^m$ (see 3.3) has fiber $F_m$, the space of base-point preserving maps $S^m \to S^m$ of degree $+1$. There is a canonical isomorphism $\pi_p(F_m) \approx \pi_{p+m}(S^m)$ (see [26, p. 465]). Note $F_m$ is not the usual loop space; in the notation of [26], $F_m = F^m(S^m, x)$. There are homomorphisms $e: \pi_p(G_{m+1}) \to \pi_p(S^m)$ and $\nu: \pi_p(F_m) \to \pi_p(G_{m+1})$ induced by the fibration and inclusion of the fiber.

4.2 **Proposition.** If $\xi \in \pi_{n+k}(S^n)$ and $\beta \in \pi_n(G_{k+1})$, then $\phi(\nu(\xi)) \cdot \beta = \beta - \nu(e(\beta) \circ \xi)$. Note $H_0(k, n) = \nu(\pi_{n+k}(S^n))$.

**Proof.** Consider representatives $f: S^n \times S^k \to S^n$ of $\nu(-\xi)$, and $g: S^n \times S^k \to S^k$ of $\beta$. Let $\rho: (S^n, D^+_n) \to (S^n, x_0)$ be a map of degree $+1$; we may assume $f(x, y) = \rho(x)$ for $x \in D^+_n$ or $y \in D^+_k$.

Consider a new map $f': S^n \times S^k \to S^n$ defined by:
\[
f'(x, y) = f(x, y) \quad \text{if } x \in D^+_n
\]
\[
= x_0 \quad \text{if } y \in D^+_k.
\]
Recall from [26, p. 465] that $\nu$ is defined as a composition:
\[\pi_{m+p}(S^m) \xrightarrow{\nu'} \pi_{m+p}(F^0_m) \xrightarrow{\nu''} \pi_{m+p}(F_m),\]
where $F^0_m$ is the space of base-point preserving maps $S^{m-1} \to S^{m-1}$ of degree 0, $\nu'$ is the usual isomorphism and $\nu''$ is induced by “adding” a fixed map of degree $+1$. Then $f'$ represents $\nu'(-\xi)$.

Now it follows that $-\nu'(e(\beta) \circ \xi) = \nu'(e(\beta) \circ (-\xi))$ is represented by a map $h': S^n \times S^k \to S^k$ defined by $h'(x, y) = g(f'(x, y), y_0)$. Notice that $h'(x, y) = g(x_0, y_0)$ if $y \in D^+_k$; we may assume $g(x_0, y_0) = y_0$. Let $\gamma: (S^k, D^+_k) \to (S^k, y_0)$ be a map of degree $+1$; then $-\nu(e(\beta) \circ \xi)$ is represented by $h: S^n \times S^k \to S^k$ defined by:
\[
h(x, y) = h'(x, y) \quad \text{if } y \in D^+_k
\]
\[
= \gamma(y) \quad \text{if } y \in D^+_k.
\]

Now $f'(x, y) = x_0$, if $y \in D^+_k$, and, therefore, the map $(D^+_k, S^{k-1}) \to (S^k, y_0)$, defined by $y \mapsto g(f'(x, y), \gamma(y))$, is independent of $x$. Let $\varphi: (S^k, D^+_k) \to (S^k, y_0)$ be the extension of degree $+1$. Since, in the definition of $h$, $\gamma$ was arbitrary, replace $\gamma$ with $\varphi$. Then it may be checked that:
\[
h(x, y) = g(f'(x, y), \varphi(y)) \quad \text{for } x \in S^n, y \in S^k.
\]
Since \( y \) is homotopic to the identity, we can change the definition of \( h \), without altering its homotopy class, to: 
\[
h(x, y) = g(f'(x, y), y)
\]
for \( x \in S^n, y \in S^k \).

Now \( \beta \) is also represented by the map \( (x, y) \mapsto g(\rho(x), y) \). Therefore, by the definition of addition, \( \beta - \nu(e(\beta) \circ \xi) \) is represented by the map \( l: S^n \times S^k \to S^k \) defined by:
\[
l(x, y) = g(f'(x, g(\rho(x), y)), g(\rho(x), y)).
\]

If \( g \) is chosen to satisfy \( g(x_0, y) = y \), for \( y \in S^k \), then it may be checked that 
\( l(x, y) = g(f(x, y), y) \). But this is also a representative of \( \phi(\nu(\xi)) \cdot \beta \).

4.3. Proposition 4.2, together with Theorems 2.4 and 3.3, completely determine the group \( H^{n,k} \) in terms of standard homotopy groups. Notice also that Proposition 4.2 gives us a great deal of information about \( \phi \) in the categories \( \mathcal{P} \) and \( \mathcal{D} \) (see 2.10).

4.4. We can define suspension homomorphisms:
\[
\begin{align*}
\sigma_1 &: A_1^{n,k} \to A_1^{n,k+1}, & \sigma_2 &: A_2^{n,k} \to A_2^{n+1,k},
\end{align*}
\]
Suppose \( f \) represents an element \( \xi \in A_1^{n,k} \) and \( F \) is an extension of \( f \) to a self-equivalence of \( S^n \times D^{k+1} \). If we choose equivalences \( \epsilon_+: D_+^{k+1} \to D^{k+1} \) and \( \epsilon_-: D_-^{k+1} \to D^{k+1} \) extending the inclusion \( S^k \to D^{k+1} \), then \( F \) determines a self-equivalence \( \tilde{f} \) of \( S^n \times S^{k+1} \), defined as follows:
\[
\begin{align*}
\tilde{f}(x, y) &= F(x, \epsilon_+(y)) & \text{if } y \in D_+^{k+1}, \\
&= F(x, \epsilon_-(y)) & \text{if } y \in D_-^{k+1}.
\end{align*}
\]
The concordance class of \( \tilde{f} \) depends only on that of \( f \). For certainly it depends only on the concordance class of \( F \). But \( F \) is isotopic to \( F' \), where \( F' \) is a “product extension” of \( f \) (see 2.7) in a neighborhood of \( S^n \times S^k \). If we use \( F' \), instead of \( F \), in the construction of \( \tilde{f} \), we see that \( \tilde{f} \) is determined by \( f \) in \( S^n \times U \), where \( U \) is some neighborhood of \( S^k \) in \( S^{k+1} \). It now follows from Lemma 2.7 that \( \tilde{f} \) is determined by \( f \); notice that \( \tilde{f} \) extends to a self-equivalence of \( S^n \times D^{k+2} \). We define \( \sigma_1(\xi) \) to be the concordance class of \( \tilde{f} \). Clearly \( \sigma_1 \) is a homomorphism. \( \sigma_2 \) is defined similarly.

Recall the suspension homomorphisms \( \sigma: A(m, p) \to A(m, p+1) \) (see [7], [16]). The following diagrams are clearly commutative:
\[
\begin{array}{ccc}
A_1^{n,k} & \xrightarrow{\sigma_1} & A_1^{n,k+1} \\
\downarrow \lambda_1 & & \downarrow \lambda_1 \\
A(n, k) & \xrightarrow{\sigma} & A(n, k+1)
\end{array} \quad \begin{array}{ccc}
A_2^{n,k} & \xrightarrow{\sigma_2} & A_2^{n+1,k} \\
\downarrow \lambda_2 & & \downarrow \lambda_2 \\
A(k, n) & \xrightarrow{\sigma} & A(k, n+1)
\end{array}
\]

4.5 PROPOSITION. For any \( \xi \in A_2 \), \( \phi(\xi) \) acts trivially on \( \text{Image } \sigma_1 \).

Proof. If \( A = H \), this follows immediately from Proposition 4.2, since \( \epsilon \circ \sigma = 0 \). We assume \( A = \mathcal{P} \) or \( \mathcal{D} \).
Let $g$ represent $\xi \in A_*^{k \times k}$ and $f$ represent $\beta \in \sigma_1(A_*^{k \times k-1})$. We may, therefore, assume that $f(S^n \times D^k) = S^n \times D^k$. By Lemma 2.8, we may also assume that $g|S^n \times D^k$ is the inclusion. Now $\phi(\xi) \cdot \beta$ is represented by $g \circ f \circ g^{-1} = f'$. But clearly $f'|S^n \times D^k = f|S^n \times D^k$; then Lemma 2.7 implies $f$ and $f'$ are concordant.

4.6. We now restrict ourselves to $\mathcal{A} = \emptyset$. Referring to the exact sequence 3.1(b), the remarks of 4.3 tell us that the action $\phi$ of $P_0(k, n)$ on $P(n, k)$ is determined, by Proposition 4.2, modulo the image of $\varepsilon: P_{n+1} \rightarrow P(m, k)$. Hence, there remains indeterminacy in $\phi$ only if $n$ is odd. If $n \equiv 3 \mod 4$, then $P_{n+1} = \mathbb{Z}$ and $\varepsilon$ is a monomorphism (see [16]). Therefore $\mu_{2|\text{Torsion}} P(n, k)$ is injective and the indeterminacy will be resolved by: (except when $n = k$).

**Proposition.** For any $\xi \in P_0(k, n)$ and $\beta \in P(n, k)$, $\phi(\xi) \cdot \beta \equiv \beta$ is a torsion element of $P(n, k)$, unless $n = k$ is odd.

**Proof.** Suppose $H(n, k) = \pi_n(G_{k+1})$ is finite, i.e. $n \neq 2k-1$ for $k$ even. Then $r\beta \in \partial P_{n+1}$, for some positive integer $r$, and $r(\phi(\xi) \cdot \beta) = \phi(\xi) \cdot r\beta$, by Proposition 4.5 ($\partial P_{n+1} \subset \text{Image} \sigma_1$).

Suppose $n = 2k - 1$, $k$ even. By Proposition 4.2, $\phi(\xi) \cdot \beta \equiv \beta \in P_0(n, k)$ and it follows that $\mu_2(\phi(\xi) \cdot \beta \equiv \beta) \in H_0(n, k)$, which is finite (since $\pi_{n+1}(S^n)$ is finite). Therefore $r(\phi(\xi) \cdot \beta \equiv \beta) \in \partial P_{n+1}$, for some positive integer $r$. We will now prove the formula:

$$\phi(\xi)^r \cdot \beta' - \beta' = s(\phi(\xi) \cdot \beta' - \beta') = rs(\phi(\xi) \cdot \beta - \beta)$$

where $\beta' = r\beta$, for any positive integer $s$. Since $P_0(k, n)$ is finite ($k$ is even, and so $P_{k+1} = 0$), this will show that $\phi(\xi) \cdot \beta - \beta$ is torsion.

For $s = 1$, the formula is obvious. Suppose it is true for $s$. Applying $\phi(\xi)$ to both sides, and observing from Proposition 4.5 that $\phi(\xi)$ acts trivially on the right-hand side, we obtain:

$$\phi(\xi)^{s+1} \cdot \beta' - \phi(\xi) \cdot \beta' = s(\phi(\xi) \cdot \beta' - \beta').$$

Adding $\phi(\xi) \cdot \beta' - \beta'$ to both sides yields the desired formula for $s+1$.

4.7 **Remark.** One can consider the stable suspension:

$$P(m, p) = F\Gamma_{m+1}^p \xrightarrow{\delta} F\Gamma_m \approx \pi_m(PL)$$

where $PL = \lim_{q \to \infty} PL_q$ and $PL_q$ is the semi-simplicial group of germs of piecewise-linear homeomorphisms of $R^n$ onto itself (see e.g. [8]). It should not be difficult to verify the formula

$$\partial(\phi(\xi) \cdot \beta - \beta) = \pm \partial(\xi) \cdot \varepsilon(\beta) \quad \text{for} \quad \xi \in P_0(k, n), \beta \in P(n, k).$$

Since $\partial|\partial P_{n+1}$ is injective, unless, perhaps, $n = 2e - 3$ for some integer $e$, (see [16], [4] and [2]), this would resolve the indeterminacy quite often. Notice, also, the implication that Proposition 4.6 is false when $n = k$ is odd.

4.8. Finally we comment briefly on the pairing $\tau: D_1^{n \times k} \otimes D_2^{n \times k} \rightarrow \Gamma^{n+k+1}$. Recall the pairing

$$\pi_{n}(SO_{k+1}) \otimes \pi_{k}(SO_n) \rightarrow \Gamma^{n+k+1}$$
studied by Milnor in [18]. Also notice that there are homomorphisms:

$$\pi_n(SO_{k+1}) \to D^{n,k}_2$$

and

$$\pi_n(SO_n) \to D^{n,k}_2$$

defined by associating to a smooth map \( f: S^n \to SO_{k+1} \), the diffeomorphism \((x, y) \mapsto (x, f(x) \cdot y)\)—similarly for \( \pi_n(SO_n) \), except that we must suspend \( SO_n \leftarrow SO_{n+1} \) first. It is then clear that these homomorphisms make the pairing of Milnor a special case of \( \tau \).

In [14] a generalization of Milnor’s pairing is also studied. By considerations of the above type, the construction of [14, §7] is also a special case of \( \tau \). Furthermore the results of [14] can be generalized, with little effort, to give analogous results about \( \tau \). We will not, at this time, present any details.

5. Examples and applications.

5.1. We begin by making some specific comparisons of self-equivalences of \( S^n \times S^k \) in the three categories, which amounts to considering the homomorphisms \( \mu_1 \) and \( \mu_2 \) defined in (1.1). We concentrate on \( \mu_2 \), and leave a similar study of \( \mu_1 \) to the reader. By (3.2) and (3.3), it suffices to consider \( \mu_2 \) in sequence 3.1(b).

The homomorphism \( \omega: H(m, p) \to P_m \) is defined, using the Thom construction, by taking the index or Kervaire invariant of a closed framed \( m \)-dimensional submanifold of \( S^n \times S^p \). According to the Index Theorem and results of Brown-Peterson-Browder ([2], [4]), \( \omega = 0 \) unless \( m = 2^e - 2 \), for some \( e \geq 2 \). On the other hand if \( e = 3, 4 \) or \( p \) is large, then \( \omega \neq 0 \) (see [12], [2]). More specifically, \( \omega 
eq 0 \) when \( e = 3 \) and \( p \geq 4 \), or \( e = 4 \) and \( p \geq 8 \) and \( \omega = 0 \) when \( e = 3 \) and \( p \leq 3 \), or \( e = 4 \) and \( p \leq 6 \). These facts follow from the observation that the suspension \( \pi_n(G_{p+1}) \to \pi_m(G) \) is onto (for \( \omega \neq 0 \)) or zero (for \( \omega = 0 \)) in the stated cases.

Now it follows immediately from 3.1(b) that \( \mu_2: P(m-1, p) \to H(m-1, p) \) is not injective when \( \omega = 0 \) and \( P_m \neq 0 \) and \( \mu_2: P(m, p) \to H(m, p) \) is not surjective when \( \omega = 0 \). These phenomena imply, therefore:

1. The existence of piecewise-linear homeomorphisms which are homotopic but not concordant on \( S^n \times S^p \) when \( m \) (or \( p \)) is odd but not of the form \( 2^e - 3 \)—see [1] for \( m \equiv 3 \mod 4 \)—or when \( m = 5 \) and \( p = 2 \) or 3, or \( m = 13 \) and \( 2 \leq p \leq 6 \). The lowest dimensional example is \( S^5 \times S^2 \).

2. The existence of a homotopy equivalence not homotopic to a piecewise-linear homeomorphism on \( S^n \times S^p \) when \( m = 6, p \geq 4 \) or \( m = 14, p \geq 8 \) or \( m = 30, p \) large.

In fact, it follows from work of Sullivan ([23] and [34]) that “piecewise-linear” may be replaced by “topological” in these examples, since \( H^*(S^n \times S^p) \) contains no 2-torsion.

5.2. We will now show that, in a certain metastable range of dimensions, every self-equivalence of \( S^n \times S^k \) (in the categories \( \mathcal{P} \) and \( \mathcal{D} \)) is concordant to a composition of familiar ones.

Consider the following types of diffeomorphisms \( f \) of \( S^n \times S^k \).

(a) \( f(x, y) = (\rho(y), x, y) \), where \( \rho: S^k \to O_n+1 \) is a smooth map,
(b) \( f(x, y) = (x, \rho(x) \cdot y) \), where \( \rho: S^n \to O_{k+1} \) is a smooth map and,
(c) \( f \) is the identity outside an \((n+k)\)-disk.

**Proposition.** If \( n < 2k-1 \), \( n \geq 3 \), then every diffeomorphism of \( S^n \times S^k \) is concordant to a composition of diffeomorphisms of types (a), (b) and (c).

**Proof.** First notice that every element of \( \overline{D}^{n,k}/D^{n,k} \) is representable as desired. This follows immediately from §1 and the diffeomorphisms considered in [24, Lemma 5]. We can, therefore, consider only \( D^{n,k} = (D_1^{n,k} \oplus \alpha) \times_{\varphi_1} D_2^{n,k} \) (see Theorem 2.4).

By definition, every element of \( \alpha \) is represented by a diffeomorphism of type (c). Consider the homomorphisms:

\[
\pi_n(SO_{k+1}) \to D_1^{n,k} \quad \text{and} \quad \pi_n(SO_n) \to D_2^{n,k}
\]

defined in 4.8. It is only necessary to show these are surjective. But they correspond to the homomorphism \( \pi_m(SO_{p+1}) \to D(m, p+1) = FC^{m+1}_m \) defined by twisting the framing on a representative of the zero element in \( FC^{m+1}_m \) (see [7, §5.9]). There is an exact sequence:

\[
\cdots \to \pi_m(SO_{p+1}) \to FC^{m+1}_m \to C^{m+1}_m \to \cdots
\]

where \( C^{m+1}_m \) is the group of concordance classes of smooth imbeddings \( S^m \to S^{m+p+1} \). But if \( m < 2p-1 \), this group is zero (see e.g. [7, §6.6]), and we have the desired surjectivity in the asserted range.

5.3. A similar result is true in the category \( \mathcal{P} \). We define two types of piecewise-linear homeomorphisms of \( S^n \times S^k \):

(d) Let \( \alpha \) be a smoothing of \( S^n \) such that \( S^n \times S^k \) is diffeomorphic to \( S^n \times S^k \). Then consider the composition

\[
S^n \times S^k \xrightarrow{1} S^n \times S^k \xrightarrow{h} S^n \times S^k
\]

where \( 1 \) is the identity and \( h \) is a piecewise-linear approximation to a diffeomorphism.

(e) Similarly, using a smoothing of \( S^k \).

**Proposition.** Every piecewise-linear homeomorphism of \( S^n \times S^k \) onto itself is concordant to a composition of types (d) and (e).

**Proof.** If \( n + k \leq 4 \), this follows from the obstruction theory of Munkres [19], and the fact that \( \Gamma_m = 0 \) for \( n \leq 4 \) [5]. We will show that type (d) generates the cokernel of \( \mu_1: D_1 \to P_1 \) and type (e) generates the cokernel of \( \mu_2: D_2 \to P_2 \). Referring to the sequence 3.1 (a), for \( n = m \) and \( k = p \), \( S^n \) represents an element of the image \( \theta: \Gamma^{k+1}_m \to \Gamma_n \) if and only if \( S^n \times D^{k+1} \) is diffeomorphic to \( S^n \times D^{k+1} \) by the argument in 3.5. If \( \xi \in \Gamma^{k+1}_m \) is represented by a submanifold \( \Sigma \) of \( S^n \times S^{k+1} \), with some framing, then \( \theta(\xi) \) is represented by \( \Sigma \). It follows easily then that if \( \alpha \) is a smoothing of \( S^n \) derived from an element \( \xi \) in the image of \( \theta \), we can use \( \alpha \) to
construct a homeomorphism of type (d) and the image under $\theta$ (using the isomorphism $\lambda_0)$ of the element of $P_1^{n,k}$ it represents is $\xi$. Similarly for $\mu_2$.

5.4. We now give an application which depends on the actual group structure of $\mathcal{A}^{n,k}$. If $f$ is a self-equivalence of $S^n \times S^k$, we can use $f$ to identify the boundaries of two copies of $D^{n+1} \times S^k$. The resulting space $X_f$ is an object in the category $\mathcal{A}$. Moreover its equivalence class in $\mathcal{A}$ is easily seen to depend only upon the concordance class of $f$. Thus we have defined for $a \in \mathcal{A}^{n,k}$ an equivalence class $X_a$ of objects in $\mathcal{A}$.

Let $\mathcal{A}_2^{n,k} \subset \mathcal{A}^{n,k}$ be the subgroup of self-equivalences which extend to self-equivalences of $D^{n+1} \times S^k$. If $n > k$, we see easily that $\mathcal{A}_2/A_2 \to \mathcal{A}/A \cong \mathbb{Z}_2 + \mathbb{Z}_2$ is an isomorphism.

**Lemma.** If $n > k$ and $a, a' \in \mathcal{A}^{n,k}$, then $X_a = X_a'$ if and only if $a' = \beta_1 a \beta_2$ for some $\beta_i \in \mathcal{A}_2^{n,k}$.

**Proof.** Suppose $f, f'$ represent $a, a'$, respectively, and assume the existence of $\beta_i$, represented by $g_i$. Then $f'$ is concordant to $g_1 \circ f \circ g_2$. Let $\bar{g}_i$ be an extension of $g_i$ to a self-equivalence of $D^{n+1} \times S^k$; by using $\bar{g}_2$ on one copy of $D^{n+1} \times S^k$ and $\bar{g}_1^{-1}$ (or a homotopy inverse, in $\mathcal{A}$) on the other, we can define an equivalence $X_{f'} \to X_f$.

5.5. For the converse, we first deal with the category $\mathcal{A}$. Let $g = p_2 \circ f$, $g' = p_2 \circ f'$, where $p_2: S^n \times S^k \to S^k$ is projection on the second factor. Then $X_a$ (or $X_{a'}$) can be constructed by using $g$ (or $g'$) to attach $D^{n+1} \times S^k$ to $S^k$. It will suffice to construct self-equivalences $h$ of the pair $(D^{n+1} \times S^k, S^n \times S^k)$, and $h'$ of $S^k$, such that $g' \circ h \cong h' \circ g$ on $S^n \times S^k$.

Suppose $h^*: X_a \to X_{a'}$ is a homotopy equivalence. We may assume $h^*(S^k) \subset S^k$; since $n > k$, $h^*$ induces a self-equivalence $h'$ of $S^k$. Let $q: (D^{n+1} \times S^k, S^n \times S^k) \to (X_a, S^k)$ be the identification map—similarly define $q'$ from $X_{a'}$. We want to complete the commutative diagram:

\[(D^{n+1} \times S^k, S^k) \xrightarrow{q} (X_a, S^k) \]
\[\downarrow h \quad \downarrow h^*\]
\[(D^{n+1} \times S^k, S^k) \xrightarrow{q'} (X_{a'}, S^k).\]

Let $E$ be the space of paths in $X_a$, which start in $0 \times S^k$. Let $p: E \to X_{a'}$ be the fibration defined by evaluating at the endpoint and $E_0 = p^{-1}(S^k)$; note that the fiber $F$ is $(n-1)$-connected. Now $h^* \circ q$ and $q'$ both lift to maps $s$ and $s'$, respectively, $(D^{n+1} \times S^k, S^n \times S^k) \to (E, E_0)$, since the only obstruction lies in $\pi_{k-1}(F) = 0$.

By the homotopy excision theorem [22, p. 484], $q_*$: $\pi_i(D^{n+1} \times S^k, S^n \times S^k) \to \pi_i(X_a, S^k)$, $q'_*: \pi_i(D^{n+1} \times S^k, S^n \times S^k) \to \pi_i(X_{a'}, S^k)$ are isomorphisms for $i \leq 2n-1$ and epimorphisms for $i = 2n$. The same is, therefore, true of $s_*$ and $s'_*$. Since $s, s': D^{n+1} \times S^k \to E$ are homotopy equivalences, the maps $S^n \times S^k \to E_0$, induced by $s$ and $s'$, are $(2n-1)$-connected.
We wish to complete the commutative diagram:

\[(D^{n+1} \times S^k, S^n \times S^k) \xrightarrow{s} (E, E_0) \xleftarrow{h} (D^{n+1} \times S^k, S^n \times S^k)\]

If we replace \(E\) by the mapping cylinder of \(s'\), we may assume \(s'\) is an inclusion. Since \(s'(D^{n+1} \times S^k)\) is, then a (weak) deformation retract of \(E\), it is only necessary to deform \(s\) so that \(s(S^n \times S^k) \subset s'(S^n \times S^k)\). But \((E_0, s'(S^n \times S^k))\) is \((2n-1)\)-connected and \(n+k \leq 2n-1\).

5.6. To complete the proof of Lemma 5.4, we now consider \(\mathcal{A} = \mathcal{P}\) or \(\mathcal{D}\). Suppose \(f, f'\) represent \(\alpha, \alpha' \in \mathcal{A}^{n,k}\), and an equivalence \(h: X_f \to X_{f'}\) exists. Since \(n > k\), \(\pi_n(X_{f'})\) is infinite cyclic and \(h|0 \times S^k\) represents a generator. After an isotopy, we may assume \(h(0 \times S^k) = 0 \times S^k\) (by general position). By the tubular or regular neighborhood theorem, we may assume \(h(D^{n+1} \times S^k) = D^{n+1} \times S^k\). Therefore \(h\) determines two self-equivalences \(h_1, h_2\) of \(D^{n+1} \times S^k\)—one for each copy used in constructing \(X_f\) and \(X_{f'}\). Since \(h_1\) and \(h_2\) determine an equivalence \(X_f \to X_{f'}\), it follows that \(h_2 \circ f = f' \circ h_1\) on \(S^n \times S^k\). We now let \(\beta_i\) be the self-equivalence of \(S^n \times S^k\) determined by restricting \(h_i\).

5.7. Now \(A_2/A_2 \cong Z_2 + \mathbb{Z}_2\) (recall \(n > k\)) and is generated by \(\rho, \rho' \in A_2\), where if \(r'\) and \(r\) are reflections of \(S^n\) and \(S^k\), respectively, we can take \(r' \times 1\) and \(1 \times r\) as representatives of \(\rho'\) and \(\rho\), respectively.

Suppose \(\alpha, \alpha' \in A_0\) and \(X_\alpha = X_{\alpha'}\). Then, by Lemma 5.4, \(\alpha' = \beta_1 \alpha' \beta_2\) for some \(\beta_1, \beta_2 \in A_2\). Projecting on \(A|A\) tells us that \(\beta_1 \beta_2 \in A\). It follows that we may write \(\alpha' = \beta(\beta_1 \alpha' \beta_2)\beta^{-1}\), where \(\beta_1 \in A_2\) and \(\beta = 1, \rho, \rho'\) or \(\rho \rho'\). Now by projecting onto \(A / A_0 \cong A_2\), we find \(\beta_1 = \beta_2^{-1}\), since \(\beta^{-1} \alpha' \beta \in A_0\).

We also point out that, if \(\alpha \in A_0\), then \(\rho \alpha \rho' = -\alpha\). If \(\mathcal{A} = \mathcal{H}\), this follows from the characterization of \(H_0 = H_1\) as \(\pi_4(G_{k+1})\). If \(\mathcal{A} = \mathcal{P}\) or \(\mathcal{D}\), suppose \(\alpha\) is represented by \(f\) satisfying \(f|D^n \times S^k\) = inclusion. We may extend \(f|D^n \times S^k\) to a self-equivalence \(g\) of \(D^{n+1} \times S^k\) in such a way that \(g|D^n \times S^k = (r' \times 1) \circ f\circ (r' \times 1)|D^n \times S^k\). It follows easily that \((r' \times 1) \circ f \circ (r' \times 1) \circ f = g|S^n \times S^k\), by checking on \(D^n \times S^k\) and \(D^n \times S^k\) separately. Thus \(\rho' \alpha \rho' \alpha \in A_2 \cap A_0 = \{1\}\).

We can summarize the above observations in:

**Proposition.** If \(n > k\) and \(\alpha, \alpha' \in A_0^{n,k}\), then \(X_\alpha = X_{\alpha'}\) if and only if there exists \(\beta \in A_2^{n,k}\) such that:

\[\beta \alpha \beta^{-1} = \pm \alpha' \quad \text{or} \quad \pm \rho \alpha' \rho^{-1}.\]

5.8. The objects \(\{X_\alpha\}\), for \(\alpha \in A_0\), include the \(k\)-sphere bundles over \(S^{n+1}\). In the category \(\mathcal{H}\), Proposition 5.7, in light of Proposition 4.2, agrees with the classification given by James and Whitehead [9].
5.9. As one application of Proposition 5.7 we show:

**Corollary.** There exists a smooth closed manifold $M$, tangentially homotopy equivalent but not homeomorphic to $S^n \times S^k$, for $n = 6$ and $k = 2, 3$ or $n = 14$ and $2 \leq k \leq 6$.

**Proof.** In these dimensions $P_n \cong \mathbb{Z}_2$, $\partial : P_n \to P(n-1, k)$ is nonzero and $\theta \circ \partial : P_n \to \Gamma_{n-1}$ is zero (see (3.1) and (5.1)). Let $\alpha \in P(n-1, k) \cong P_1^{n-1, k}$ be the nonzero element of $\partial(P_n)$; since $\theta(\alpha) = 0$, let $\alpha = \mu_1(\alpha')$ (see 3.1(a)). Then $M = X_\alpha$ is a smooth $(n+k)$-manifold. Because $\mu_2 \circ \mu_1(\alpha') = \mu_2(\alpha) = 0$, $M$ is homotopy equivalent to $S^n \times S^k$ (Proposition 5.7). On the other hand $M$ is not piecewise-linearly homeomorphic to $S^n \times S^k$, by Proposition 5.7, since $\alpha$ is not conjugate to 0. Therefore, by Sullivan [34], $M$ is not homeomorphic to $S^n \times S^k$.

To complete the proof of the Corollary, we show that any homotopy equivalence $M \to S^n \times S^k$ is tangential. Since $\alpha'$ represents a diffeomorphism of $S^{n-1} \times S^k$ which extends to one of $S^{n-1} \times D^k+1$, we may use such an extension to attach two copies of $D^n \times D^k+1$ along $S^{n-1} \times D^k+1$, thereby obtained a smooth homology $n$-sphere bounded by $M$. This manifold, and, therefore, $M$, is stably parallelizable, since $\pi_{n-1}(SO) = 0$. But $S^n \times S^k$ is also stably parallelizable and the Euler class of a manifold is a homotopy invariant. Therefore, any homotopy equivalence $M \to S^n \times S^k$ is tangential.

5.10. Proposition 5.8 can also be used to study the inertia group of a smooth manifold (see [14]).

**Corollary.** Suppose $\alpha \in D_1^{n,k}$, $\beta \in D_2^{n,k}$ satisfy $\phi(\beta) \cdot \alpha = \alpha$. Then $\tau(\beta) \cdot \alpha \in \Gamma^{n+k+1}$ belongs to the inertia group of $X_\alpha$.

**Proof.** We have $\beta \alpha \beta^{-1} = \phi(\beta) \cdot \alpha + \tau(\beta) \cdot \alpha = \alpha + \tau(\beta) \cdot \alpha$. If $\alpha' = \beta \alpha \beta^{-1}$, then $X_{\alpha'} = X_\alpha$. But it is easy to see (compare [1, Lemma 1]) that $X_{\alpha'} = X_\alpha \# \Sigma$, where $\Sigma$ represents $\tau(\beta) \cdot \alpha$.

For example, if $\alpha$ is the suspension of an element of $D_1^{n,k-1}$, the hypothesis is satisfied for any $\beta$, by Proposition 4.5 (compare [14, Theorem 1]).

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BRANDEIS UNIVERSITY, 
WALTHAM, MASSACHUSETTS