ON A COEFFICIENT PROBLEM
IN UNIVALENT FUNCTIONS

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Introduction. Let $S$ denote the family of functions which are regular and
univalent in the unit disc and which possess a power series expansion about the
origin of form
\[ f(z) = z + A_2z^2 + A_3z^3 + \cdots. \]
The coefficient problem for univalent functions proposed by Bieberbach is to
determine the precise region, $V_n$, in $2n-2$ dimensional euclidean space occupied
by points $(A_2, \ldots, A_n)$ where the $A_j$'s appear in (1) for some $f \in S$. Bieberbach [1]
determined $V_3$ in the special case in which $A_2$ and $A_3$ are real, denoting the region
$E^{(0)}$. Using a slight modification of Peschl's notation we determine the region
$E(4, S)$ in this paper. We also adopt the following notation: $D$ denotes the unit
disc centered at 0, $E$ denotes $\{z : |z| > 1\} \cup \{\infty\}$, $R$ denotes $\{f : 1/f(1/z) \in S\}$,
$E(n, S)$ denotes $V_n$ when the $A_j$'s are real for $j = 2, \ldots, n$. The statements $(A_2, \ldots, A_n)$
belongs to $f \in S$ and $f \in S$ belongs to $(A_2, \ldots, A_n)$ will mean that $(A_2, \ldots, A_n)$
e $E(n, S)$ and the $A_j$'s appear in (1) for $f$.

Implicit in results concerning $V_n$ in [2] are the following two propositions about
$E(n, S)$.

**Proposition 1.** $E(n, S)$ is a bounded closed set, the closure of a domain, and is
homeomorphic to the closed $n-1$ dimensional full sphere.

**Proposition 2.** The following statements are equivalent:
(i) $(A_2, \ldots, A_n)$ is an interior point of $E(n, S)$.
(ii) There is a bounded function in $S$ belonging to the point $(A_2, \ldots, A_n)$.

Proofs of these propositions follow directly from the proofs in [2] upon obvious
modifications.

The determination of the functions belonging to boundary points of $E(4, S)$
using the General Coefficient Theorem (GCT) leads to consideration of certain
quadratic differentials on the Riemann sphere. We refer to [4] for definitions and
terminology associated with the GCT and to [5] for the form of the GCT used
here.

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Since the GCT in [4] is phrased in terms of local uniformizing parameters which represent poles of the quadratic differentials as the point at infinity, we consider functions of the family $R$ and their expansions about infinity of form

\[ f(z) = z + C_0 + C_1/z + C_2/z^2 + \cdots. \]

The coefficients in (1) and (2) are related in

\[ C_0 = -A_2, \quad C_1 = A_2^2 - A_3, \quad C_2 = -A_2^2 + 2A_2A_3 - A_4. \]

Clearly $A_2, A_3,$ and $A_4$ are real iff $C_0, C_1,$ and $C_2$ are real. The functions from $R$ to be considered are identified in

**Proposition 3.** Let $t_1$ and $t_2$ be real parameters with $-4 \leq t_1 \leq 4$ and $-\infty < t_2 < \infty$. Then corresponding to the quadratic differentials

\[
Q_1(w, t_1, t_2) \, dw = \left( \frac{(w-t_2)}{w} \right) \, dw, \quad t_2 \geq \max (0, t_1), \\
Q_2(w, t_1, t_2) \, dw = -\left( \frac{(w-t_1)}{w} \right) \, dw, \quad t_2 \leq \min (0, t_1), \\
Q_3(w, t_1) \, dw = \left( \frac{(w-t_1)}{w} \right) \, dw
\]

on the sphere, there are families $F_1(t_1, t_2), F_2(t_1, t_2),$ and $F_3(t_1)$ of functions such that $F_j \subset R$ and each $f \in F_j$ maps $E$ conformally onto a domain $G$ admissible with respect to $Q_j,$ and $G$ is bounded as follows:

(a) If $t_1 \in [-4, 4]$, and $t_2 > \max (0, t_1)$ or $t_2 < \min (0, t_1)$ or $t_2$ does not occur, then $G$ is bounded by the segment from 0 to $t_1$ plus two slits of equal length along trajectories of $Q_j$ having an endpoint at $t_1,$ the slits including $t_1.$

(b) If $t_1 \in [-4, 4] \setminus \{0\}$ and $t_2 = t_1$, then $G$ is bounded by the segment from 0 to $t_1$ plus three slits along trajectories of $Q_j$ with an endpoint at $t_1.$ One of the slits lies along the real axis while the other two slits are of equal length, possibly zero, and all slits include $t_1.$

(c) If $t_1 \in [-4, 4]$ and $t_2 = 0$, then $G$ is bounded by the segment from 0 to $t_1$ plus a slit from 0 to a point on the real axis on the opposite side of the origin from $t_1$, plus two slits of equal length on the trajectories of $Q_j$ with an endpoint at $t_1$, the latter point included.

**Proof.** If $t_2 > \max (0, t_1)$ or $t_2 < \min (0, t_1)$ or $t_2$ does not appear, and if $t_1 \neq 0$, then $Q_j$ has a simple zero at $t_1$. Let $G_1$ be the simply connected domain on the sphere bounded by the segment from 0 to $t_1$ and two slits of equal length, $L,$ along the other two trajectories of $Q_j$ with endpoint at $t_1.$ By the Riemann Mapping Theorem there is a conformal mapping, $f$, of $E$ onto $G_1$ with expansion about infinity of form

\[ f(z) = d(L)z + d_0 + d_1/z + d_2/z^2 + \cdots, \quad d(L) > 0. \]

For $L=0$, $G_1$ is bounded by the segment from 0 to $t_1$ hence $f$ reduces to

\[ f(z) = (|t_1|/4)(z + 2 \sgn t_1 + 1/z). \]

(\textsuperscript{c}) Differentials and families are often written briefly as $Q_j$ and $F_j,$ $j=1, 2, 3,$ in what follows.
by uniqueness in the Riemann Mapping Theorem. If $\delta$ denotes the diameter of the complement of $G_1$, then the diameter theorem for functions from the family $\Sigma$ gives $2d(L) \leq \delta \leq 4d(L)$. Schwarz’s Lemma implies that $d(L)$ increases with $L$. By the above bounds on $d(L)$, $\sup_L d(L) \geq 1$. Also $d(L)$ is a continuous function of $L$ as a result of Carathéodory’s theorem on variable regions [7, Theorem 2.1, p. 343]. Thus as $L$ ranges from $0$ to $\infty$, $d(L)$ increases continuously from $|t_1|/4$ through $1$ so for some value of $L$, $d(L) = 1$. The corresponding function is in $R$ and since $d(L)$ increases with $L$, this function is the only member of $F_j(t_1, t_2)$. Analogous reasoning gives the result if $t_1 = 0$ and $t_2 \neq 0$. If $t_1 = t_2 \neq 0$, $Q_1$ and $Q_2$ have double zeros at $t_1$ and four trajectories with limiting endpoints at $t_1$. Let $\Delta$ be a domain on the sphere bounded by the segment from $0$ to $t_1$ plus two slits of equal length $L$ on the two trajectories of $Q_j$, $j = 1, 2$, not lying along the real axis. As above, for some choice of $L$ the function mapping $E$ conformally onto $\Delta$ has $d(L) = 1$ in its expansion of form (4) about $\infty$, and $d(L)$ decreases as $L$ decreases. Fixing $L$ so that $d(L) < 1$, we can increase $d(L)$ by introducing a slit of length $L_1$ along the real axis on the trajectory of $Q_j$ from $t_1$ to $\infty$. Then writing $d(L, L_1)$ instead of $d(L)$, we have that for some choice of $L_1$, $d(L, L_1) = 1$ and the corresponding function is in $R$. Q.E.D.

Note that for each choice of $t_1 \in (-4, 4) - \{0\}$ with $t_2 = t_1$, $F_j(t_1, t_2)$ is a one parameter family of functions in which $L$ can be chosen as the parameter. Note also that if $t_1 = \pm 4$ in any of the above cases, or when $t_1 = t_2$, if $L = 0$, the corresponding function is one of the Koebe functions $k_1(z) = z + 2 + 1/z$ or $k_2(z) = z - 2 + 1/z$.

Construction of the mappings of Proposition 3 and expressions for the coefficients $C_0$, $C_1$, and $C_2$ of expansion (2) proceeds as follows. The upper half $w$-plane is mapped onto the $\zeta$-plane by

$$\zeta_w = \int_0^w [Q_j(w)]^{1/2} dw$$

where the branch of $[Q_j(w)]^{1/2}$ is the one taking large positive values for $w$ large and positive. The domain $E \cap \text{Im } z > 0$ is mapped onto the upper half of the $W$-plane by $W = z + 2 \operatorname{sgn} t_1 + 1/z$ where we make the agreement that $\operatorname{sgn} t_1 = 1$ if $t_1 = 0$. Next the upper half $W$-plane is mapped into the $\zeta$-plane so that the image of the former coincides with the image of the upper half $w$-plane under $\zeta_w$ with the exception that a horizontal segment is appended to the boundary of the latter image at the point $\zeta_w(t_1)$. The mappings are to be conformal on their domains so that the composed mapping from $z$ to the $w$-plane is conformal from $E \cap \text{Im } z > 0$ onto the set $\text{Im } w > 0$ minus a slit on a trajectory of $Q_j$ emanating from $t_1$. Reflection then extends the composed mapping to a conformal mapping of $E$ onto a domain bounded as described in (a), (b), or (c) of Proposition 3.

In the case of $Q_1$ we have

$$\zeta_w = \int_0^w [(w - t_1)(w - t_2)/w]^{1/2} dw$$
and the mapping from the $W$-plane into the $\zeta$-plane is given by

$$
\zeta_w = \int_0^W (W-\alpha)(W-\beta)^{1/2} W^{-1/2} (W-4 \text{ sgn } t_1)^{-1/2} dW + T
$$

where $\alpha$, $\beta$, and $T$ are real parameters with $\alpha$ between 0 and $4 \text{ sgn } t_1$,

$$
\beta \geq \max (0, 4 \text{ sgn } t_1),
$$

$T = -2(-t_1)^{3/2}/3 - 32/3 - 4\alpha$ with $T$ appearing only if $t_2 = 0$. Boundaries in the $\zeta$-plane are matched by the conditions:

$$
\begin{align*}
\zeta_w(t_1) &= \zeta_w(4), & \zeta_w(t_2) &= \zeta_w(\beta) & \text{if } 0 \leq t_1 < t_2, \\
\zeta_w(t_1) &= \zeta_w(-4), & \zeta_w(t_2) &= \zeta_w(\beta) & \text{if } -4 \leq t_1 < t_2,
\end{align*}
$$

Expanding the integrands in the expressions for $\zeta_w$ and $\zeta_w$, choosing the earlier mentioned branches of the root functions, we integrate termwise, insert a trial expansion of form (2) into the resulting expression for $\zeta_w$, express $\zeta_w$ in terms of $z$ and equate coefficients of like powers of $z$ giving

$$
\begin{align*}
C_0 &= g_1 + C + r - \beta - 2\alpha, \\
C_1 &= -g_2 - C_0^2/4 + C_0 g_1/2 + 2/3 + B_1 + (C/3)(C + (3/2)(r - \beta - 2\alpha)), \\
C_2 &= -g_1 g_2/6 + g_2 C_0/2 - C_0 C_1/2 + C_0^3/24 + C_1 g_1/2 - C_0 g_1/8 + B_2 \\
& \quad - C B_1/2 + C/3
\end{align*}
$$

where $g_1 = t_1 + t_2$, $g_2 = (t_1 - t_2)^2/4$, $C = 2 \text{ sgn } t_1$, $r = 2C$,

$$
B_2 = (\beta + r)^3/24 - r\beta^2/12 - r^3/4 + ar^2/4 - a\beta^2/12 - r\alpha/6
$$

and

$$
B_1 = (\beta + r)^2/4 - r^2 + r\alpha - \alpha\beta.
$$

In the case of $Q_2$ the same expressions for $C_0$, $C_1$, and $C_2$ result but parameter ranges are changed so that $t_2 \leq \min (t_1, 0)$, $-4 \leq t_1 \leq 4$, $\beta \leq \min (4 \text{ sgn } t_1, 0)$ and $\alpha$ is between 0 and $4 \text{ sgn } t_1$.

For $Q_3$, $C_0$ and $C_1$ were determined in [6, p. 170] and found there to be

$$
\begin{align*}
C_0 &= (t_1/2)[1 - \ln (|t_1|/4)], \\
C_1 &= (t_1^2/8)[1 - 2 \ln (|t_1|/4)] - 1
\end{align*}
$$

with $C_0 = 0$ and $C_1 = -1$ for $t_1 = 0$. Use of the explicit mapping [6, Equation (8), p. 170](3) gives

$$
C_2 = (t_1^2/32)(1 - 2 \ln (|t_1|/4) - 2 \ln^2 (|t_1|/4)) - t_1/2.
$$

Now define $F$ to be $\bigcup [F_1(t_1, t_2) \cup F_2(t_1, t_2) \cup F_3(t_1)]$ where the outer union is over $t_1 \in [-4, 4]$ and $t_2$ restricted as described in Proposition 3. Then the family $F$

(3) In [6, Equation (8), p. 170] a factor $\tau$ multiplying the log term is missing.
gives the complete collection of extremal functions for $E(4, S)$ in the following sense.

**Proposition 4.** $(A_2, A_3, A_4) \in \partial E(4, S)$ iff $(A_2, A_3, A_4) \in \mathcal{E}$ and $f(z) = 1/g(1/z) \in F$.

**Proof.** Let $f \in F$. From the description in Proposition 3 of the boundary of the image of $E$ under a member of $F$ it follows that the range of $f$ never excludes a neighborhood of the origin. Thus $1/f(1/z) \in S$ is unbounded. If the point $(A_2, A_3, A_4)$ belonging to $1/f(1/z)$ is an interior point of $E(4, S)$, then by Proposition 2 there is a bounded function $g$ belonging to the point. Applying the GCT to the Riemann sphere with the image of $E$ under $f$ as the admissible domain, the function $(1/g)^{-1}$ is unbounded, and the differential $Q_1$ associated with the family $F_j$ to which $f$ belongs, we find the fundamental inequality to be a zero equality. Since $Q_1$ has a pole of order 4 or 5 at $\infty$, the equality statement in the GCT gives $1/g(1/z) = f(z)$, a contradiction since $1/f(1/z)$ is unbounded while $g(z)$ is bounded. Thus $(A_2, A_3, A_4)$ is a boundary point of $E(4, S)$. To complete the proof of Proposition 4 we introduce a topology on the function family $F$ and show that the resulting space is topologically the two sphere. Let the topology on $F$ be given by the metric $d(f, g) = \sup |f(2e^{i\theta}) - g(2e^{i\theta})|$, with the sup taken over $\theta \in [0, 2\pi]$. We map the set $F - \{k_1(z), k_2(z)\}$ into the plane as follows:

(a) If $-4 < t_1 < 4$ and $t_2 > \max(t_1, 0)$ or $t_2 < \min(t_1, 0)$, the single member of $F_1(t_1, t_2)$ is mapped onto the plane point $(t_1, [4 + \arctan |t_2|] \text{ sgn } t_2)$ for $j = 1, 2$.

(b) If $0 \leq t_1 < 4$ and $t_1 = t_2$, then the proper one of conditions (5) gives $t_1 \leq \alpha \leq 4/3 + t_1^{3/2}/3$ and as noted earlier $F_1(t_1, t_1)$ is a one parameter family of functions with parameter $L$ defined in Proposition 3. As $\alpha$ increases from $t_1$ to $4/3 + t_1^{3/2}/3$, $L$ increases from 0 to its maximum. Then using $\alpha$ instead of $L$ as parameter and calling the corresponding member of $F_1(t_1, t_1)$ $f_{a}$, we map $f_{a}$ onto $(t_1 + 2, \alpha)$ for $t_1 < \alpha \leq 4/3 + t_1^{3/2}/3$.

(c) If $-4 < t_1 < 0$ and $t_2 = 0$, then from conditions (5), $-4 \leq \alpha \leq -8/3 - (t_1)^{3/2}/6$. With $\alpha$ as parameter map $f_{a}$ in $F_1(t_1, t_1)$ onto the point $(t_1 + 2, \alpha + 4)$.

(d) If $0 \leq t_1 < 4$ and $t_2$ and $\alpha - 4$ are not simultaneously zero, then the boundary matching conditions for $Q_2$ analogous to (5) give $8/3 + t_1^{3/2}/6 \leq \alpha \leq 4$. Map $f_{a}$ in $F_2(t_1, 0)$ onto the point $(t_1 - 2, \alpha - 4)$ for $\alpha$ in the above range, $\alpha \neq 4$.

(e) If $-4 < t_1 < 0$ and $t_1 = t_2$, the conditions analogous to (5) give $-4/3 - (t_1)^{3/2}/3 \leq \alpha \leq t_1$. Map $f_{a}$ in $F_2(t_1, t_1)$ onto the point $(t_1 - 2, \alpha)$.

(f) Map the single member of $F_3(t_1)$ onto the point $(t_1, 4 + \pi/2)$ for $-4 < t_1 < 4$.

Denote the two disjoint plane sets described in (a) by $D_1$ and $D_2$ where $t_2 > 0$ in $D_1$, and sets described in (b) through (f) by $D_3$ through $D_7$. The mapping just described is one-to-one from $F - \{k_1(z), k_2(z)\}$ into the plane. The functions $k_1(z)$ and $k_2(z)$ correspond to those points on the boundaries of the $D_i$'s for which $t_1 = \pm 4$ in any of the $D_i$'s; $\alpha = t_1$ in $D_3$ and $D_6$; $\alpha = -4$ and $t_1 = 0$ in $D_4$; and $\alpha = 4$, $t_1 = 0$ in $D_5$. This statement is easily verified by substituting the appropriate values.
for the parameters in equations (6) and in later expressions for the \( C_j \)'s in the case of \( F_3 \). In each case it will be seen that \( C_0 = \pm 2 \).

Denote the set of boundary points of the \( D_j \)'s corresponding to \( k_1 \) and \( k_2 \) by \( K \) and \( \bigcup (D_j) \cup K, j = 1, \ldots, 7 \), by \( H \). We now introduce a topology on \( H \) under which the correspondence just described from \( F \) into \( H \) becomes a homeomorphism. Using \( \text{Cl} \) for plane closure we form the free union \( \text{Cl} (D_2) + \text{Cl} (D_1 \cup D_\gamma) + \text{Cl} (D_3 \cup D_4 \cup D_5 \cup D_6) \) and denote it \( \mathcal{P} \). Certain points in this space are identified.

\[
(t_1, t_2) \text{ in } \partial D_1 \text{ is identified with } (t_1 + 2, 4/3 + t_2^{3/2}/3) \text{ in } \partial D_3 \text{ for } 0 \leq t_1 \leq 4.
\]

\[
(t_1, 4) \text{ in } \partial D_1 \text{ is identified with } (t_1 + 2, +4/3 - (-t_1)^{3/2}/6) \text{ in } \partial D_4 \text{ for } -4 \leq t_1 \leq 0.
\]

\[
(t_1, -4) \text{ in } \partial D_2 \text{ is identified with } (t_1 - 2, -4/3 + t_1^{3/2}/6) \text{ in } \partial D_3 \text{ for } 0 \leq t_1 \leq 4.
\]

\[
(t_1, -4 - \arctan (-t_1)) \text{ in } \partial D_2 \text{ is identified with } (t_1 - 2, -4/3 - (-t_1)^{3/2}/3) \text{ in } \partial D_6 \text{ for } -4 \leq t_1 \leq 0.
\]

These identifications are homeomorphisms between certain boundary continua on the \( D_j \)'s. Further each of \( \text{Cl} (D_1 \cup D_\gamma), \text{Cl} (D_2), \) and \( \text{Cl} (D_3 \cup D_4 \cup D_5 \cup D_6) \) is homeomorphic to the closed two disc, \( \text{Cl} (D) \). Calling the equivalence relation given by the above identifications \( R_1 \), we have that the space \( P/R_1 \) is also homeomorphic to the closed two disc since identifications were made along boundary continua in a manner preserving simple connectivity. Using \([x, y]\) to denote the equivalence class containing the plane point \((x, y)\), we now map \( P/R_1 \) onto a rectangle in the plane with sides parallel to the coordinate axes so that images of the points \([t_1, 4 + \pi/2]\) form the upper boundary and images of the points \([t_1, -4 - \pi/2]\) form the lower boundary in such a way that images of points having the same first coordinate also have the same first coordinate. Note that the vertical sides of the rectangle correspond to \( K \) while the lower horizontal side does not correspond to any member of \( F \). If \( g \) denotes the mapping of \( P/R_1 \) onto the rectangle, we remove the additional boundary points by identifying the upper and lower horizontal sides of the rectangle under the equivalence relation \( R_2 \) defined by

\[
g([t_1, 4 + \pi/2]) \sim g([t_1, -4 - \pi/2]), \quad -4 < t_1 < 4.
\]

Since identified points have the same first coordinate in the rectangle, the quotient space \((P/R_3)/R_2\) is just \( S^1 \times I \) where \( S^1 \) is the circle and \( I \) is a nondegenerate closed interval, say \([a, b]\). The sets \( A = \{(x, a) : a \in S^1\} \) and \( B = \{(x, b) : x \in S^1\} \) represent the functions \( k_1(z) \) and \( k_2(z) \) while every other point \((x, y)\) in \( S^1 \times I \) is the unique representative of a point of the set \( F - \{k_1(z), k_2(z)\} \). We identify all points in \( A \), and identify all points in \( B \) calling the equivalence relation \( R_3 \). Then \((S^1 \times I)/R_3 \) is the suspension of \( S^1 \) and hence is homeomorphic to \( S^2 \). Thus the topology given to \( H \) is the quotient of plane topology under the equivalence relations \( R_1 \) followed by \( R_2 \) and \( R_3 \), and the resulting topological space is homeomorphic to \( S^2 \). To show that the one-to-one correspondence from \( F \) onto \( H \) is a homeomorphism, we consider its inverse, call it \( h \). Points of \( H \) are equivalence classes of plane points with the following structures: (i) classes with a single member, (ii) classes containing pairs of identified boundary points, (iii) two classes each containing a continuum.
of points representing $k_1(z)$ and $k_2(z)$ respectively. Because of the topologies on $H$ and $F$ it suffices to deal with sequences in any discussion of continuity. Let $\{[x_n, y_n]\}$ be a sequence of points of $H$ which converges to $[x, y]$, the latter being in $H$ by compactness. The class $[x, y]$ contains plane points as enumerated in (i) through (iii) above. Suppose first that $(x, y)$ is the only member of $[x, y]$. Then for all but finitely many $n$, $[x_n, y_n]$ has only one member since $(x, y)$ is an interior point of $D_j$, $j=1, \ldots, 6$, and the topology there is essentially plane topology. Thus $x_n \to x$ and $y_n \to y$. If $(x, y) \in D_1 \cup D_2$ we can conclude that $t_1^{(n)} \to t_1$ and $t_2^{(n)} \to t_2$ where $t_1^{(n)}$ and $t_2^{(n)}$ are the distinct zeros of the quadratic differentials

$$Q_j^{(n)}(w) \, dw^2 = \pm \frac{\left[(w-t_1^{(n)})(w-t_2^{(n)})/w\right]}{w} \, dw^2.$$

The corresponding sequence of functions in $F$ is $\{f_n\}$ where $f_n$ maps $E$ conformally onto a domain $G_n$ admissible with respect to $Q_j^{(n)}$ as described in Proposition 3. We show that $h([x_n, y_n]) \to h([x, y])$ as $n \to \infty$ by considering cases. Consider first the case in which $[x, y] \in D_1$. Let $h([x, y])=f$. To show that $f_n \to f$ in the topology of $F$ it is enough to show that $f_n \to f$ uniformly on compact subsets of $E$. The latter will follow by Carathéodory’s theorem on variable regions [7, Theorem 2.1, p. 343] if it is first shown that $\{G_n\}$ converges to its kernel with respect to $\infty$, $G_\infty$, and that $G_\infty = G$, the image of $E$ under $f$. The boundary of $G$ as described in Proposition 3 is the segment from 0 to $t_1$, plus two symmetric slits of equal length along the other two trajectories of $[(w-t_1)(w-t_2)/w] \, dw^2$ with endpoint at $t_1$. Extend these slits along the trajectories until they meet the circle $|w|=4$ and denote the so augmented boundary of $G$ by $B$. Then it is asserted that the boundary of $G_\infty$ is contained in $B$. First note that the complement of $G_\infty$ contains the segment from 0 to $t_1$ along the real axis and is symmetric in the real axis. If $\partial G_\infty$ is not contained in $B$, take $w_0 \in \partial G_\infty$ at distance $\delta > 0$ from $B$. With no loss of generality we may assume that $w_0$ is in the half plane $\text{Im } w > 0$. Suppose first that $w_0 \neq t_2$. Let $N(w_0)$ be a neighborhood of $w_0$ of radius less than $\delta/2$ and chosen so that 0, $t_1$, and $t_2$ are not in $N(w_0)$. Since $w_0 \in \partial G_\infty$ there is a sequence $\{n(k)\}$ of integers such that $\partial G_{n(k)}$ intersects $\text{Cl } (N(w_0))$ and the zeros of $Q_1^{(n(k))}$ are exterior to $\text{Cl } (N(w_0))$ for $k=1, 2, \ldots$. If we restrict consideration to $\text{Im } w > 0$ where the functions

$$\zeta_{n(k)}(w) = \int_0^w [Q_1^{(n(k))}(w)]^{1/2} \, dw \quad \text{and} \quad \zeta(w) = \int_0^w [Q_1(w)]^{1/2} \, dw$$

are continuous and one-to-one, we find that the trajectories of $Q_1^{(n(k))}$ from 0 to $t_1^{(n(k))}$ and from $t_1^{(n(k))}$ to $\infty$ are mapped by $\zeta_{n(k)}$ onto the negative real axis (recall the choice of root determination specified earlier). Thus there is a sequence $\{p_{n(k)}\}$ of points in $\text{Cl } (N(w_0))$ such that $\text{Im } \zeta_{n(k)}(p_{n(k)}) = 0$, $k=1, 2, \ldots$. Choose a convergent subsequence $\{q_j\}$ of $\{p_{n(k)}\}$, and let $\{\zeta_j\}$ denote the corresponding subsequence of $\{\zeta_{n(k)}\}$. Then $\zeta_j \to \zeta$ uniformly on $\text{Cl } (N(w_0))$. Thus if $q=\lim q_j$, we have $q \in \text{Cl } (N(w_0))$ and $\zeta$ maps $q$ onto the real axis which contradicts the one-to-one nature of $\zeta$ since $\text{Cl } (N(w_0))$ is $\delta/2$ distance from the trajectories of $Q_1$ with limiting
endpoint at $t_1$. If $w_0 = t_2$, the above proof is valid upon replacing $\text{Cl}(N(w_0))$ by a half disc centered at $w_0$. Thus $\partial G_\infty \subseteq B$.

Note here that any subsequence of $\{G_n\}$ has kernel with respect to $\infty$ containing $G_\infty$. Let $\{G_{nk}\}$ be a subsequence of domains, and $\{G_{mk}\}$ a sub-subsequence such that $\{f_{m(k)}\}$ converges uniformly inside $E$ to the limit function $f^\#$. Then $f^\#$ maps $E$ conformally onto the kernel of $\{G_{m(k)}\}$ with respect to $\infty$ which we denote by $J$. Then $J$ is simply connected and $f^\#$ has expansion about $\infty$ of form (2). It now follows from Schwarz’s Lemma that $f^\#$ and $f$ are identical since $f$ had expansion about $\infty$ of form (2) also and either $\partial G \subseteq \partial J$ or $\partial J \subseteq \partial G$. Similarly the kernel of any convergent subsequence $\{G_{nk}\}$ must be $G$, hence $G_\infty \subseteq G$ which implies $\partial G \subseteq \partial G_\infty$.

If the last containment is proper, then for some point $p \in \partial G_\infty - \partial G$ there is a sequence $\{G_{nk}\}$ and a sequence of points $\{p_{nk}\}$ such that $p_{nk} \in \partial G_{nk}$ and $p_{nk} \to p$. Choosing a convergent subsequence $\{G_{mk}\}$ of $\{G_{nk}\}$ we have that $p_{mk} \to p$. But the kernel of $\{G_{mk}\}$ is $G$ by above while $\partial[\text{Ker} \{G_{mk}\}]$ contains $p$, and this is a contradiction since $p \in G$. Thus $\partial G = \partial G_\infty$, so that $G = G_\infty$ since they are open, connected and nondisjoint. As remarked above $G_\infty \subseteq \text{Ker} \{G_{mk}\}$ and $\text{Ker} \{G_{nk}\} \subseteq G$ for any subsequence of $\{G_n\}$. Hence $\{G_n\}$ converges to $G$ and so by Carathéodory’s theorem $f_n \to f$ uniformly on compact subsets of $E$. Similarly if $(x, y) \in D_3$ we have $f_n \to f$ inside $E$. If $(x, y)$ is an interior point of $D_3$, then $(x_n, y_n) = (t_1^{(n)} + 2, \alpha^{(n)})$ and the coefficients $C_0$, $C_1$, and $C_2$ are given by (6), including proper superscripts, with $t_1^{(n)} = t_2^{(n)} = \beta^{(n)} = 4$. The coefficients are continuous functions of $t_1$ and $\alpha$ so that $t_1^{(n)} \to t_1$ and $\alpha^{(n)} \to \alpha$ imply that $C_j^{(n)} \to C_j$, $j = 0, 1, 2$, $n \to \infty$. Taking any convergent subsequence of $\{f_n\}$, the sequence of functions associated with $(x_n, y_n)$, we have that the limit function, $f^*$, has $C_0$, $C_1$, and $C_2$ as constant term and coefficients of $1/z$ and $1/z^2$ in its expansion of form (2). This is also true of the function associated with $(x, y)$, the limit of $(x_n, y_n)$, since $(x, y)$ is just $(t_1 + 2, \alpha)$. The fundamental inequality of GCT is a zero equality when applied to $f$ and $f^*$. $Q_1(w, t_1, t_2) \, dw^2$ has a pole of order five at $\infty$ so $f = f^*$. Thus every convergent subsequence of the normal family $\{f_n\}$ has limit $f$, so $f_n \to f$. If $(x, y)$ is an interior point of $D_j$, $j = 4, 5, 6$; then $C_0^{(n)}$, $C_1^{(n)}$, and $C_2^{(n)}$ are also given by (6) with proper values of the parameters and proper superscripts and the same argument as the one above for $(x, y)$ in the interior of $D_3$ gives $f_n \to f$. Suppose now that $\{(x_n, y_n)\}$ has limit on the boundaries of the $D_j$’s. First let the limit point correspond to the Koebe function $k_1$. Then it is possible that $\{(x_n, y_n)\}$ has subsequences in each $D_j$, $j \neq 6$, simultaneously. The subsequence in $D_1$ has terms of the form $(t_1^{(n)}, 4 + \arctan t_2^{(n)})$ with $t_1^{(n)} \to 4$ but no requirement on $t_2^{(n)}$. To prove that the limit of the associated sequence of functions, $\{f_n\}$, is $k_1$ consider the sequence $\{G_n\}$ of domains which are images of $E$ under the $f_n$. Since $t_1^{(n)} \to 4$, the boundary of the kernel of $\{G_n\}$ with respect to $\infty$ contains the segment of the real axis from 0 to 4. Taking any convergent subsequence $\{f_{nk}\}$ of $\{f_n\}$, we have that the boundary of the kernel of $\{G_{nk}\}$ also contains the segment from 0 to 4 because $t_1^{(nk)} \to 4$. Thus the limit function of $\{f_{nk}\}$ must omit the value $w = 4$. Hence by the Koebe
Theorem, this limit function is \( k_1 \). Since \( \{ f_n \} \) is a normal family it follows that \( f_n \to k_1 \). Similarly the subsequence of points in \( D_2 \) is such that the corresponding sequence of functions converges uniformly inside \( E \) to \( k_1 \). For the sequences in each of \( D_3, D_4, D_5, \) and \( D_7 \), the coefficient \( C_0 \) is a continuous function of \( t_1 \) and \( \alpha \), and an examination of (6) with proper values for the parameters shows that \( C_0^{(n)} \to 2 \) as \( (x_n, y_n) \) converges to a point representing \( k_1 \). Thus \( k_1 \) is the only function to which a subsequence of functions can converge, and since each subsequence is a normal family, \( k_1 \) is the limit of each subsequence. The situation is analogous for a sequence of points of \( H \) with limit point corresponding to the Koebe function \( k_2 \).

Suppose now that \( \{ [x_n, y_n] \} \) converges to \( [x, y] \) and that this equivalence class contains a point on the boundary of \( D_1 \) and a point on the boundary of \( D_3 \), neither corresponding to \( k_1 \). Assume also that \( \{ [x_n, y_n] \} \) consists of two subsequences of plane points \( \{ (x^{(n)}_1, y^{(n)}_1) \} \) from \( D_1 \) and \( \{ (x^{(n)}_3, y^{(n)}_3) \} \) from \( D_3 \). Then by the same arguments as above, the sequences of functions associated with the subsequences of plane points converge to functions \( f \) and \( f^* \). The points of \( D_1 \) are of form \( (t_1^{(n)} + 2, \alpha^{(n)}) \) and their subsequence has limit \( (t_1 + 2, 4/3 + t_1^{(n)}) \). \( f \) maps \( E \) onto a domain slit along trajectories of \( Q_1(w, t_1, t_1) \, dw^2 \) as described in part (b) of Proposition 3 with the slit on the trajectory from \( t_1 \) to \( \infty \) having length zero. The last comment follows since any point to the right of \( t_1 \) on the real axis is in the kernel of \( \{ G_n \} \) because \( t_1^{(n)} - t_1 \). Further \( f^* \) maps \( E \) onto the domain just described since generally the limit of a sequence of functions associated with points of \( D_3 \) maps \( E \) onto a domain bounded as described in part (b) of Proposition 3. In this case the length of the slit on the trajectory from \( t_1 \) to \( \infty \) is \( -4t_1^{(n)} + 4t_1 - 16/3 \), and here \( \alpha = 4/3 + t_1^{(n)}/3 \), so the slit has length zero.

The quadratic differential \( Q_1(w, t_1, t_1) \, dw^2 \) enters here as above and hence we can use Schwarz's Lemma to assert that \( f=f^* \). Suppose \( \{ [x_n, y_n] \} \) converges to a point \( [x, y] \) which contains plane points \( (t_1, \pi/2) \) on the boundary of \( D_1 \) and \( (t_1, -\pi/2) \) on the boundary of \( D_2 \). Then \( t_1^{(n)} - t_1 \) and \( \{ t_2^{(n)} \} \) consists of two subsequences, one with limit \( +\infty \) and the other with limit \( -\infty \). The quadratic differentials

\[
[(w-t_1^{(n)})(w/t_2^{(n)}-1)/w] \, dw^2
\]

have the same trajectory structure as the differentials \( Q_1(w, t_1^{(n)}, t_2^{(n)}) \, dw^2 \) when \( t_2^{(n)} \) is positive and \( Q_1(w, t_1^{(n)}, t_2^{(n)}) \, dw^2 \) when \( t_2^{(n)} \) is negative. As a subsequence of the sequence \( \{ t_2^{(n)} \} \) converges either to \( +\infty \) or \( -\infty \), the sequence of rational functions with terms \( [(w-t_1^{(n)})(w/t_2^{(n)}-1)/w] \) converges uniformly on any compact neighborhood of \( w_0 \) not including 0 in the \( w \)-plane to \( -[(w-t_1^{(n)})/w] \). We use this fact as before to prove that the sequence of domains, the images of \( E \) under the associated sequence of functions, converges to its kernel with respect to \( \infty \), and that this kernel is bounded as described in part (c), Proposition 3. Then as before the associated sequence of functions in \( F \) converges to the function in \( F \) mapping \( E \) conformally onto the kernel of the above sequence of domains. The proofs that
the convergence in $H$ of sequences having limits at boundaries of other adjacent or attached sets $D_i$ are carried out exactly as the proofs just completed. Thus the mapping $h$ from $H$ to $F$ is one-to-one and onto, and continuous. $H$ is compact hence $h$ is a homeomorphism.

From (3) relating $A_2$, $A_3$, and $A_4$ to $C_0$, $C_1$, and $C_2$ we see that there is a natural correspondence from $F$ to $\partial E(4, S)$ given by

$$f \rightarrow (-C_0, C_0^2 - C_1, 2C_0C_1 - C_2 - C_0^2).$$

The GCT can be used to show that this correspondence is one-to-one, and continuity follows because convergence of a sequence of functions in the topology of $F$ implies convergence of their series coefficients in (2) to the corresponding coefficients of the limit function. Hence the mapping from $F$ into $\partial E(4, S)$ is a homeomorphism. Finally, the image of $F$ is all of $\partial E(4, S)$ for if this were not so, then because $\partial E(4, S)$ is homeomorphic to $S^2$, it would be possible to construct a homeomorphism of $S^2$ properly into itself in contradiction to the Jordan Separation Theorem. Thus the functions of $F$ determine all of the boundary points of $E(4, S)$ and this completes the proof of Proposition 4.

It is interesting to note here that the coefficient domain $E(3, S)$ is readily found by considering the family $F$. Clearly $f \in F$ implies that $1/f(1/z)$ belongs to a point of $E(3, S)$. The functions belonging to boundary points of $E(3, S)$ are identified in

**Proposition 5.** Let $F^#$ be the subset of $F$ consisting of $F_3 = \bigcup F_3(t_1)$ with the union taken over $t_1 \in [-4, 4]$, and $F^#_4 = \bigcup F_4(t_1, 0)$ with the union taken over those $t_1$ in $[-4, 4]$ for which $-t_1 + T = 4$. Then $(A_2, A_3) \in \partial E(3, S)$ iff $(A_2, A_3)$ belongs to $1/f(1/z)$ for some $f \in F^#$.

**Proof.** The proof that functions in $F^#$ correspond only to boundary points of $E(3, S)$ is the same as the proof of the analogous assertion about $F$ and $E(4, S)$ in Proposition 4. To see that $F^#$ yields all of $\partial E(3, S)$ we recall that for $f \in F_3$ we have

$$A_2 = -C_0 = -t_1[1 - \ln(|t_1|/4)]/2,$$

$$A_3 = C_0^2 - C_1 = t_1^2[1 - \ln(|t_1|/4)]^2/4 - t_1[1 - 2 \ln(|t_1|/4)]/8 + 1$$

for $t_1 \in (-4, 4)$ with $C_0 = 0$, $C_1 = -1$ for $t_1 = 0$. For $f \in F^# - F_3$ we have

$$A_3 = A_2^2 - 1, \quad -2 \leq A_2 \leq 2.$$

The pairs $(A_2, A_3)$ satisfying (7) and (8) trace a simple closed curve hence the boundary of $E(3, S)$. Q.E.D.

For purposes of computation note that in those cases where $t_1 = t_2$ or $t_2 = 0$, equations (6) give the $C_i$'s, and hence the $A_i$'s, in terms of $t_1$ and $\alpha$ since $\beta$ assumes fixed values with $t_2$ as above. Computations can then be made by choosing $t_1 \in [-4, 4]$ and $\alpha$ limited by the proper one of conditions (5) or its analog for $Q_2$. If $t_2 > \max(t_1, 0)$ or $t_2 < \min(t_1, 0)$ conditions (5) or their analog for $Q_2$ relate
$t_1$, $t_2$, $\beta$, and $\alpha$ in a way which can be put in terms of hypergeometric functions and computations appear to be very difficult to carry out. It is interesting to note however that only a relatively small portion of $\partial E(4, S)$ is associated with those cases where $t_2 \neq t_1$ or $t_2 \neq 0$. As a brief illustration of this the points $(A_2, A_3, A_4) = (-0.50563, 1.19559, -0.93709)$ and $(-0.50009, 1.07244, 1.08179)$ are on $\partial E(4, S)$ in the cross section where $A_2$ is approximately $-\frac{1}{2}$. These points lie on curves bounding the portion of $\partial E(4, S)$ associated with the conditions $t_2 > t_1 \geq 0$. The point with smallest $A_3$ in this section is $(-0.50000, -0.75000, 0.87500)$ while the first mentioned point has the largest $A_3$ in this section. A table with points in representative cross sections of the coefficient body has been compiled.

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