HOMOTOPY PROPERTIES OF DECOMPOSITION SPACES(1)

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1. **Introduction.** The purpose of this paper is to apply results of [3] and [4] to the study of cellular decompositions of manifolds, decompositions of retracts into retracts, and decompositions of a more general nature than either of the two previously mentioned types.

UV properties of compact sets are studied in [3], and in [4], a number of basic lemmas are established concerning decompositions of spaces into compact sets with UV properties. Some applications of the results of [4] were made in §5 of [4]. We give additional applications here.

In §2, we study decompositions of LC^n spaces. Under suitable hypotheses, decomposition spaces of LC^n spaces are shown to be LC^n. The results of §2 provide a partial generalization of Smale's mapping theorem for homotopy [12]. We also study decompositions of retracts in §2.

Homotopy type is studied in §3. We are able to show that in a number of cases, the original space and the decomposition space have the same homotopy type. In particular, if G is a cellular decomposition of a compact manifold M such that M/G is a manifold N, then M and N have the same homotopy type. This may be compared with the results of [2] for dimension 3 that if M is a 3-manifold and G is a cellular decomposition of M such that M/G is a 3-manifold N, then M and N are homeomorphic. One interesting consequence of the theorem mentioned above concerning homotopy type deals with cellular decompositions of S^n, for $n \geqq 5$. We show that if $n \geqq 5$ and G is a cellular decomposition of S^n such that S^n/G is an n-manifold, then S^n/G is homeomorphic to S^n. Hence the results of [2] extend to cellular decompositions of n-spheres that yield n-manifolds provided $n \geqq 5$.

In §4, we study ULC properties of open sets. We give conditions under which it can be concluded that if U is an open set in a decomposition space such that U is n-ULC, then $P^{-1}[U]$ is n-ULC, where P is the projection map.

In §5, we apply the results of the preceding sections to the study of cellularity of subsets of manifolds. We consider, in particular, the problem of determining whether the elements of a given decomposition of a manifold are cellular subsets of that manifold.

Presented to the Society, January 23, 1968; received by the editors April 18, 1968.

(1) Research supported in part by the National Science Foundation.

There is a close connection between upper semicontinuous decompositions and compact mappings; see §6 of [4]. For the convenience of the reader, we have reformulated some of the results of this paper in terms of compact mappings.

The author thanks R. H. Bing, D. R. McMillan, Jr., Joseph Martin, and E. R. Fadell for helpful discussions on various matters related to this paper. Some of the results of this paper, as well as other closely related results, have been obtained by R. C. Lacher [10].

Definitions of the UV properties of compact sets are given in [3]. Definitions of LC properties of spaces also occur in [3]. We shall use the notation and terminology concerning decompositions given in [4].

2. **Decompositions of** LC **spaces.**

In this section we apply results from [3] and [4] to the study of decompositions of LC spaces and of absolute retracts.

LEMMA 2.1. *If n is a nonnegative integer, X is a metric* LC^n *space, and G is an upper semicontinuous decomposition of X into compact sets, each of which has property* UV^n, *then X/G is n-*LC.

Proof. Suppose x is a point of X/G and U is an open set in X/G containing x. Then $P^{-1}[U]$ is an open set in X containing the element x of G. Since x has property UV^n, there is an open set W in X such that $x\subset W\subset P^{-1}[U]$, each singular n-sphere in W is homotopic to 0 in $P^{-1}[U]$, and W is a union of elements of G.

Let V be $P[W]$, and suppose f is a continuous function from S^n into V. By Lemma 4.2 of [4], there is a continuous function h from S^n into W such that $Ph\sim f$ in V. Now $h\sim 0$ in $P^{-1}[U]$ and hence $Ph\sim 0$ in U. Therefore $f\sim 0$ in U, and it follows that X/G is n-LC.

COROLLARY 2.2. *Under the hypothesis of Lemma* 9.1, *X/G is* LC^n.

COROLLARY 2.3. *Suppose n is a positive integer, X is a locally compact metric* LC^n *space, and G is an upper semicontinuous decomposition of X such that each element of G is compact,* LC^{n-1}, *and n-connected. Then X/G is* LC^n.

Proof. By Lemma 5.5 of [3] each set of G has property UV^n. Corollary 2.3 then follows from Corollary 2.2.

The following statement of Corollary 2.3 is closely related to a result of [14].

COROLLARY 2.4. *Suppose n is a positive integer, X is a locally compact metric* LC^n *space, f is a compact mapping from X onto a metric space Y, and if $y\in Y$, $f^{-1}[Y]$ is compact,* LC^{n-1}, *and n-connected. Then Y is* LC^n.

COROLLARY 2.5. *Suppose n is a nonnegative integer, X is a locally compact* LC^n *metric space, and G is an upper semicontinuous decomposition of X into compact absolute retracts. Then X/G is* LC^n.

Proof. By Corollary 5.6 of [3], each element of G has property UV^n. Corollary 2.5 then follows from Corollary 2.2.

We now consider decompositions of compact absolute retracts. Theorems 2.6 and 2.7 have been established by Borsuk [6].

THEOREM 2.6. *Suppose that X is a compact metric absolute retract and G is an upper semicontinuous decomposition of X into compact absolute retracts. If X/G has finite dimension, then X/G is an absolute retract.*

Proof. Suppose X/G has dimension m. We shall show that X/G is LC^m and m-connected. Since X is LC^m [6, p. 289], then by Corollary 2.5, X/G is LC^m. Now suppose f is a continuous function from S^m into X/G. By Lemma 4.2 of [4], there is a continuous function h from S^m into X such that $Ph \sim f$ in X/G. Since X is m-connected [6, p. 289], then $h \sim 0$ in X. Hence $Ph \sim 0$ in X/G, and therefore $f \sim 0$ in X/G. Therefore X/G is m-connected and since X/G has dimension m, it follows [6, p. 289] that X/G is an absolute retract.

THEOREM 2.7. *Suppose X is a compact metric absolute neighborhood retract and G is an upper semicontinuous decomposition of X into compact absolute retracts. If X/G has finite dimension, then X/G is an absolute neighborhood retract.*

In [9], Hyman studies ANR divisors. If g is an ANR divisor in an ANR X, and Y is the space obtained by collapsing g to a point, then Y has nice homotopy properties. However, even ANR divisors which are metric continua need not have UV properties; see [9, Theorem 5.4], for instance.

3. **Applications to homotopy type.** Our objective in this section is to show that under certain conditions, given a space X and an upper semicontinuous decomposition of X, then X and the associated decomposition space have the same homotopy type. Joseph Martin pointed out to the author that Theorem 5.1 of [4] has interesting applications to homotopy type. The main lemma that we shall use is a result due to J. H. C. Whitehead [15] that we shall state now.

Suppose X is a metric continuum and an absolute neighborhood retract. By ΔX is meant the minimum dimension of all finite simplicial complexes that dominate X. In particular, ΔX does not exceed the dimension of X.

If X and Y are spaces as above and f is a continuous function from X into Y, then for each positive integer n, f_n denotes the homomorphism from $\pi_n(X)$ into $\pi_n(Y)$ induced by f. Whitehead's result is as follows.

LEMMA 3.1. *Suppose X and Y are metric continua which are absolute neighborhood retracts, $N = \max\{\Delta X, \Delta Y\}$, f is a continuous function from X into Y, and if $n = 1, 2, \ldots,$ or N, f_n is an isomorphism from $\pi_n(X)$ onto $\pi_n(Y)$. Then f is a homotopy equivalence.*

THEOREM 3.2. *Suppose X is a compact connected metric absolute neighborhood retract of finite dimension n, G is an upper semicontinuous decomposition of X into compact sets, X/G is of finite dimension n, $N = \max\{m, n\}$, and each set of G has property UV^N. Then P is a homotopy equivalence and hence X and X/G are of the same homotopy type.*

Proof. By Corollary 2.2, X/G is LC^m and hence [6, p. 289] X/G is a compact connected absolute neighborhood retract. By Theorem 5.1 of [4], if $k=1, 2, \ldots,$ or N, then the homomorphism P_n from $\pi_k(X)$ into $\pi_k(X/G)$ induced by P is an isomorphism onto. By Lemma 3.1, P is a homotopy equivalence.

COROLLARY 3.3. *Suppose X is a compact finite-dimensional metric absolute neighborhood retract and G is an upper semicontinuous decomposition of X into compact absolute retracts. If X/G has finite dimension, then P is a homotopy equivalence, and X and X/G are of the same homotopy type.*

Theorem 2 of [10] is a result closely related to Corollary 3.3.

We consider now cellular decompositions of manifolds that yield manifolds. It was proved in [2] that if M is a 3-manifold and G is a cellular decomposition of M such that M/G is a 3-manifold N, then M and N are homeomorphic. The proof of this result makes strong use of results from the topology of 3-manifolds. We have the following results for manifolds of other dimensions.

THEOREM 3.4. *Suppose M is a compact manifold and G is a cellular decomposition of M such that M/G is a manifold N. Then P is a homotopy equivalence and hence M and N have the same homotopy type.*

One interesting corollary of Theorem 3.4 deals with cellular decompositions of n-spheres that yield n-manifolds. In case $n \geqq 5$, we are able to extend the results of [2] to this situation.

THEOREM 3.5. *Suppose n is a positive integer, G is a cellular decomposition of S^n such that S^n/G is an n-manifold. Then S^n/G is homeomorphic to S^n.*

Proof. By Theorem 3.4, S^n/G has the homotopy type of S^n. By [8], S^n/G is homeomorphic to S^n.

It is clear that one need not require, in the hypothesis of Theorem 3.5, that the elements of G be cellular in S^n. It would be sufficient to know that each element of G is compact and has property UV^n (and hence property UV^∞). However, it follows from Corollary 5.5 that if $n \geqq 5$ and G is an upper semicontinuous decomposition of S^n into compact sets, each with property UV^∞, such that S^n/G is an n-manifold, then each element of G is cellular in S^n.

4. ULC **properties.** It is frequently useful to know that an open set has a certain ULC property. In this section we shall establish a lemma which shows that under appropriate conditions, if U is an open set in a decomposition space and U is ULC in a certain dimension n, then $P^{-1}[U]$ is ULC in dimension n. We then apply this lemma to obtain a result on decompositions of E^3 and to establish a theorem related to Theorem 1 of [1].

Suppose X is a metric space. If ε is a positive number, by an *ε-subset* of X is meant a subset of X of diameter less than ε. Suppose U is an open subset of X and n is a nonnegative integer. Then U is n-ULC (*uniformly locally connected* in dimension n) if and only if for each positive number ε, there is a positive number δ such that if f is any continuous function from Bd I^{n+1} into a δ-subset of U, f is homotopic to 0 on an ε-subset of U.

LEMMA 4.1. *Suppose X is a metric space with metric d, n is a nonnegative integer, X is n-*ULC, *and G is an upper semicontinuous decomposition of X into compact sets, each with property* UV^n. *Suppose U is an open set in X/G such that* (1)$\overline{U}$ *is compact*, (2) *U is n-*ULC, *and* (3) Bd U *contains the image of no nondegenerate element of G. Then $P^{-1}[U]$ is n-*ULC.

Proof. Suppose ε is a positive number. We show first that there exist a compact set M in U and a positive number α such that if A is any subset of $U-M$ of diameter less than α, then (diam $P^{-1}[A])<\varepsilon$.

Suppose no such M and α exist. For each positive integer i, let M_i be

$$U-V(1/i, \text{Bd } U).$$

Then $M_1, M_2, M_3, \ldots$ are compact subsets of M, $M_1 \subset M_2 \subset M_3 \cdots$, and $U=\bigcup_{i=1}^{\infty} M_i$. For each positive integer i, let α_i be $1/i$.

By supposition, for each positive integer i there is a subset A_i of $U-M_i$ such that (diam $A)<1/i$ but (diam $P^{-1}[A_i])\geqq\varepsilon$. Since $\overline{U}$ is compact, there is a convergent subsequence $A_{i_1}, A_{i_2}, A_{i_3}, \ldots$ of $A_1, A_2, A_3, \ldots$. Suppose $A_{i_1}, A_{i_2}, A_{i_3}, \ldots$ converges to B. Clearly $B\subset$ Bd U and (diam $B)=0$, so there is a point p of Bd U such that $B=\{p\}$. Since no nondegenerate element of G has its image lying on Bd U, it follows that $P^{-1}[p]$ is a singleton $\{q\}$. Let W be an open neighborhood of q such that $W\subset V(\varepsilon/2, q)$ and W is a union of elements of G. Then $P[W]$ is open and contains p, and hence, for some positive integer m, contains A_{i_m}. Therefore $P^{-1}[A_{i_m}]\subset W$, but this contradicts the fact that (diam $P^{-1}[A_{i_m}])\geqq\varepsilon$. There is, accordingly, an M and α as described above.

There is a positive number β_0 such that $\beta<\alpha$ and if $x\in$ Bd U and $y\in M$, $d(x, y)>\beta_0$. Let β be $\beta_0/2$. Since U is n-ULC, there is a positive number r such that if f is a continuous function from Bd I^{n+1} into r-subset of U, then $f\sim 0$ on a β-subset of U.

Let W be $U \cap V(\beta$, Bd $U)$. Observe that if $A\subset U$, (diam $A)<\beta$, and A intersects W, then $A\subset U-M$. Now P is uniformly continuous on the closure of $P^{-1}[U]$, and hence there is a positive number γ such that if $A\subset P^{-1}[U]$ and (diam $A)<\gamma$, then (diam $P[A])<r$.

Let f be a continuous function from Bd I^{n+1} into a γ-subset of $P^{-1}[W]$. Then Pf is into an r-subset of W. Therefore $Pf\sim 0$ on a β-subset of U. Let h be a continuous extension of Pf to I^{n+1}, such that $h[I^{n+1}]\subset U$ and (diam $h[I^{n+1}])<\beta$. Let Q be an open subset of U of diameter less than β and containing $h[I^{n+1}]$. Now Q intersects W and has diameter less than β, and hence lies in $U-M$. Since $\beta<\beta_0<\alpha$, it follows that (diam $P^{-1}[Q])<\varepsilon$. By Lemma 6.1, $f\sim 0$ in $P^{-1}[Q]$, and hence $f\sim 0$ on an ε-subset of $P^{-1}[U]$.

Since M is compact, it follows that $P^{-1}[M]$ is compact. Since X is n-ULC, it can be proved that $P^{-1}[M]$ has a compact neighborhood N contained in $P^{-1}[U]$ and such that there is a positive number σ such that if ϕ is a continuous function from Bd I^{n+1} into a σ-subset of N, then $\phi \sim 0$ on an ε-subset of $P^{-1}[U]$. We may assume also that any σ-subset of $P^{-1}[U]$ that intersects $P^{-1}[M]$ lies in N. Let δ be $\min\{\gamma, \sigma\}$. It follows that if f is any continuous function from Bd I^{n+1} into a δ-subset of $P^{-1}[U]$, $f \sim 0$ on an ε-subset of $P^{-1}[U]$. Therefore $P^{-1}[U]$ is n-ULC.

With the aid of Lemma 4.1, we can establish a result closely related to Theorem 1 of [1].

THEOREM 4.2. *Suppose G is an upper semicontinuous decomposition of E^3 into compact sets, each with property UV^1, and such that E^3/G is homeomorphic to E^3. Suppose M is a tame compact 2-manifold in E^3/G and M contains the image, under P, of no nondegenerate element of G. Then $P^{-1}[M]$ is tame in E^3.*

Proof. By [5], a compact 2-manifold K in E^3 is tame if and only if each complementary domain of K is 1-ULC. This fact and Lemma 4.1 yield Theorem 4.2.

Now we shall establish a result related to Theorem 5.5 of [4].

THEOREM 4.3. *Suppose n is a positive integer, $n \neq 4$, G is an upper semicontinuous decomposition of S^n into compact sets, each with property* UV^1, *and S is an $(n-1)$-sphere in S^n such that S intersects no nondegenerate element of G. Suppose W is a complementary domain of S in S^n and* $P[W]$ *is* 1-ULC. *Then W is an open n-cell.*

Proof. By Lemma 4.1, W is 1-ULC. By [12, Theorem 4], W is an open n-cell.

5. **Applications to cellularity.** One of the questions concerning decompositions of manifolds that is of considerable interest is the following: When can it be shown that the elements of a decomposition of a manifold are cellular subsets of that manifold? In this section we give some applications of the results of previous sections to this question. Our methods are similar to those of Martin in [11], and we depend heavily on [12].

Suppose X is a topological space and M is a subset of X. Then M has *property* CC in X if and only if for each open set U containing M, there is an open set V containing M such that $V \subset U$ and each singular 1-sphere in $V-M$ is homotopic to 0 in $U-M$. The significance of this concept is indicated by the following lemma.

LEMMA 5.1. *Suppose n is a positive integer, $n>2$, $n \neq 4$, M^n is a piecewise linear manifold, and X is a compact set in M^n with property* UV^∞. *Suppose in addition that if $n=3$, some neighborhood of X can be embedded in E^3. Then if X has property* CC *in M^n, X is cellular in M^n.*

Proof. This is a restatement of Theorems 1 and 1′ of [12], taking into account the remarks in the introduction of [12] concerning replacing "compact absolute retract" in the statements of Theorems 1 and 1′ of [12] by "compact sets having property UV^∞".

We first establish some results concerning property CC.

LEMMA 5.2. *Suppose that X is a metric space and G is an upper semicontinuous decomposition of X into compact sets, each with property* UV^1. *If A is a compact subset of X/G with property* CC, *then $P^{-1}[A]$ has property* CC.

Proof. Let U be an open set in X containing $P^{-1}[A]$. There is an open set W such that $P^{-1}[A]\subset W$, $W\subset U$, and W is a union of elements of G. Let R be $P[W]$; R is open in X/G and contains A. Since A has property CC in X/G, there is an open set Q in X/G such that $A\subset Q\subset R$ and each loop in $Q-A$ is homotopic to 0 in $R-A$. Let V be $P^{-1}[Q]$. Then V is open in X and $P^{-1}[A]\subset V\subset U$.

We shall show that each singular 1-sphere in $V-P^{-1}[A]$ is homotopic to 0 in $U-P^{-1}[A]$. Let γ be a singular 1-sphere in $V-P^{-1}[A]$. Then $P\gamma$ is a singular 1-sphere in $Q-A$ and since A has property CC, $P\gamma\sim 0$ in $R-A$. By Lemma 3.3 of [4], $\gamma\sim 0$ in $P^{-1}[R-A]$, or $\gamma\sim 0$ in $W-P^{-1}[A]$. Therefore $\gamma\sim 0$ in $U-P^{-1}[A]$ and it follows that $P^{-1}[A]$ has property CC.

COROLLARY 5.3. *Suppose n is a positive integer, $n>4$, M^n is a piecewise linear manifold, and G is an upper semicontinuous decomposition of M^n into compact sets, each having property* UV^n. *If M^n/G is a manifold of dimension at least* 3, *then each element of G is cellular.*

For a related result concerning topological manifolds, see [10].

Proof. Suppose $g\in G$. By Proposition 3.4 of [3], g has property UV^∞. Also, g is a point of M^n/G, and $P^{-1}[\{g\}]=g$. Hence by Lemma 5.2, it is only necessary to show that $\{g\}$ has property CC in M^n/G. Let U be an open set in M^n/G containing the point g. Since M^n/G is a manifold, there is an open cell W such that $g\in W\subset U$. Let V be $W-\{g\}$; since M^n/G has dimension at least 3, V is simply connected. Hence $\{g\}$ has property CC. The corollary then follows from Lemmas 5.2 and 5.1.

COROLLARY 5.4. *Suppose M^3 is a* 3-*manifold and G is an upper semicontinuous decomposition of M^3 into compact sets such that each has property* UV^3 *and each has a neighborhood that can be embedded in E^3. If M^3/G is a manifold of dimension at least* 3, *then each element of G is cellular.*

COROLLARY 5.5. *Suppose n is a positive integer, $n>2$, $n\neq 4$, G is an upper semicontinuous decomposition of S^n into compact sets, each having property* UV^∞. *If S^n/G is a manifold of dimension at least* 3, *then each element of G is cellular.*

In [13], Price proved the following theorem: 'If n is a positive integer, $n\neq 4$, G is a cellular decomposition of E^n, and U is an open n-cell open in E^n/G, then $P^{-1}[U]$ is an open n-cell in E^n. In [7], Černavskiĭ and Kompaniec have established essentially the same result. In [7], this result is formulated in terms of cellular mappings. (A notion of *cellular* function was defined in §6 of [4]; the usage in [4] and that of [7] do not agree.)

A function f from a manifold M into a space Y is a *cellular* function if and only if for each point y of Y, $f^{-1}[y]$ is cellular in M. In this terminology, the result of [7] can be stated as follows: If f is a cellular mapping from S^n onto S^n and A is a cellular subset of S^n, then $f^{-1}[A]$ is cellular. McMillan pointed out to the author that these two results are essentially the same. We shall extend the result of [7] to piecewise linear manifolds of any dimension greater than 4. The proof of the following theorem was pointed out by D. R. McMillan, Jr., and is simpler than the author's original proof.

THEOREM 5.6. *Suppose that each of n and k is a positive integer, $n>4$, $k>2$, M^n is a piecewise linear n-manifold, G is an upper semicontinuous decomposition of M^n into compact sets, each with property* UV^n, *and M^n/G is a k-manifold N^k. If A is a cellular subset of N^k, then $P^{-1}[A]$ is cellular in M^n.*

Proof. Suppose U is an open set in M^n containing $P^{-1}[A]$. Let W be open in M^n such that $P^{-1}[A]\subset W$ and W is a union of elements of G. There is a k-cell C in N^k such that $P^{-1}[A]\subset \operatorname{Int} C$ and $C\subset P[W]$. By Proposition 3.4 of [3], each set of G has property UV$^\infty$. Thus, by Theorem 5.5 of [4], $P^{-1}[\operatorname{Int} C]$ is an open n-cell. Hence there is an n-cell D such that $P^{-1}[A]\subset \operatorname{Int} C$ and $D\subset U$. It follows that $P^{-1}[A]$ is cellular.

An analogous result for topological manifolds is given in [10].

COROLLARY 5.7. *Suppose the same hypothesis as in Theorem* 5.6 *except that G is a cellular decomposition of M^n. Then the conclusion of Theorem* 5.6 *holds.*

We may state Corollary 5.7 in terms of compact mappings.

COROLLARY 5.8. *Suppose that each of n and k is a positive integer, $n>4$, $k>2$, M^n is a piecewise linear n-manifold, and f is a compact cellular mapping from M^n onto a k-manifold N^k. If A is a cellular subset of N^k, then $f^{-1}[A]$ is a cellular subset of M^n.*

For our last result, we consider a generalization of Theorem 5.6. We show that under certain conditions, if a compact set A is a decomposition space has property UV^n, then so does $P^{-1}[A]$.

LEMMA 5.9. *If X is a metric space, n is a nonnegative integer, G is an upper semicontinuous decomposition of X into compact sets, each with property* UV^n, *and A is a compact subset of X/G having property* UV^n, *then $P^{-1}[A]$ has property* UV^n.

Proof. Suppose k is an integer such that $0\leqq k\leqq n$. Suppose U is an open set in X such that $P^{-1}[A]\subset U$. There is an open set W in X such that $P^{-1}[A]\subset W\subset U$ and W is a union of elements of G. There is an open set V_0 in X/G such that $A\subset V_0\subset P[W]$ and each singular k-sphere in V_0 is homotopic to 0 in $P[W]$. Let V be $P^{-1}[V_0]$.

Let γ be a singular k-sphere in V. Then $P\gamma$ is a singular k-sphere in V_0 and consequently $P\gamma \sim 0$ in $P[W]$. By Lemma 3.3 of [4], $\gamma \sim 0$ in W, and so $\gamma \sim 0$ in U. Hence $P^{-1}[A]$ has property k-UV, and it follows that $P^{-1}[A]$ has property UV^n.

COROLLARY 5.10. *Suppose that M^n is a triangulated n-manifold, G is an upper semicontinuous decomposition of M^n into compact sets, each having property* UV^∞, *and M^n/G is a k-manifold N^k. If A is a compact subset of N^k with property* UV^∞, *then $P^{-1}[A]$ has property* UV^∞.

Proof. By Proposition 3.4 of [3], it suffices to show that $P^{-1}[A]$ has property UV^n. Since A has property UV^∞, it has property UV^n. Each element of G has property UV^∞ and hence has property UV^n. By Lemma 5.9, $P^{-1}[A]$ has property UV^n. This establishes Corollary 5.10.

We remark that by combining Corollary 5.10 and Lemmas 5.1 and 5.2, one may give another proof of Theorem 5.6.

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