FAMILIES OF VALUATIONS AND SEMIGROUPS
OF FRACTIONARY IDEAL CLASSES

BY

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Introduction. Let $R$ be an integral domain with quotient field $K$. For any valuation $v$ on $K$ which is nonnegative on $R$, we let $P(v) = \{ x \in R \mid v(x) > 0 \}$. $P(v)$ is a prime ideal of $R$ and is called the center of $v$ on $R$. In this paper we are concerned mainly with integral domains $R$ which satisfy the following: There exists a family $F$ of valuations on $K$ such that

(i) Each $v \in F$ has rank one.
(ii) $R = \bigcap_{v \in F} R_v$.
(iii) $R_v = R_{P(v)}$, for each $v \in F$.

A family $F$ of valuations on $K$ is said to be of finite character if for $x \in K$, $x \neq 0$, there are only a finite number of $v \in F$ such that $v(x) \neq 0$. $R$ is called a Krull domain if there is a family $F$ of finite character satisfying (i), (ii), (iii), with the additional requirement that each $v \in F$ be discrete. $R$ is called an almost-Krull (AK) domain [7] if $R_P$ is a Krull domain for every proper nonzero prime $P$ of $R$. It follows that $R$ is almost Dedekind (AD) iff $R$ is an AK-domain in which proper prime ideals are maximal [7].

Using the family $F$ of valuations we construct a partially ordered semigroup $\mathcal{A}(R)$ of fractionary ideal classes in §1 and study the relation between $\mathcal{A}(R)$ and $\mathcal{D}(R)$, the divisor group of $R$ (see [1]). Necessary and sufficient conditions for $\mathcal{A}(R)$ and $\mathcal{D}(R)$ to be isomorphic are determined. In §2, condition $(S)$ of [3] is studied.

§3 consists of an example.

The notation concerning $\mathcal{D}(R)$ is that of [1]. Otherwise, the notation of [8] is used. Prime ideals are always nonzero and not all of $R$.

1. In order to make this paper as self contained as possible we first list the necessary background results from [1]. $R$ will denote a commutative integral domain with identity and quotient field $K$. $I(R)$ will denote the collection of nonzero fractionary ideals of $R$. A fractionary ideal of the form $Rx$, $x \in K$, $x \neq 0$, is called a principal fractionary ideal.

A relation $<$ is defined on $I(R)$ as follows: $A < B$ iff every principal fractionary ideal of $R$ which contains $A$ also contains $B$. The relation $<$ is a preorder on $I(R)$; i.e., $<$ is a symmetric, transitive relation. If we define $\equiv$ on $I(R)$ by $A \equiv B$ iff $A < B$ and $B < A$, then $\equiv$ is an equivalence relation on $I(R)$. For $A \in I(R)$, $\text{div}_R(A)$

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denotes the equivalence class of \( A \) with respect to \( \equiv \) and is called the divisor of \( A \); \( \mathcal{D}(R) \) denotes the set of all such equivalence classes.

For \( A \in I(R) \), we put \( \overline{A} = \bigcap_{A \leq Rx} Rx \). A fractionary ideal \( B \) of \( R \) is said to be divisoriel if \( B = \overline{B} \). It follows that for \( A \in I(R) \), \( \text{div}_{R}(A) = \text{div}_{R}(\overline{A}) \) and that \( \overline{A} \) is the unique divisoriel fractionary ideal belonging to \( \text{div}_{R}(A) \). It also follows from the definition that \( (\overline{A} \overline{B})^{-} = (AB)^{-} \) for \( A, B \in I(R) \) so that \( \mathcal{D}(R) \) together with the operation \( + \), defined by \( \text{div}_{R}(A) + \text{div}_{R}(B) = \text{div}_{R}(AB) \), is a commutative semigroup with identity \( 0 = \text{div}_{R}(R) \). If we define \( \leq \) on \( \mathcal{D}(R) \) by \( \text{div}_{R}(A) \leq \text{div}_{R}(B) \) iff \( A < B \) then \( \mathcal{D}(R) \) is a lattice ordered semigroup with respect to the partial ordering \( \leq \). Furthermore, \( \mathcal{D}(R) \) is a group iff \( R \) is completely integrally closed \cite[1, p. 5, Theorem 1]{1}.

Let \( F \) be a family of valuations on \( K \) with the following properties:

(i) Each \( v \in F \) has rank one.

(ii) \( R = \bigcap_{v \in F} R_{v} \).

(iii) For each \( v \in F \), \( R_{v} = R_{P(v)} \), where \( P(v) \) denotes the center of \( v \) on \( R \).

Occasionally in place of (i) we shall substitute

(i') Each \( v \in F \) has rank one and is discrete.

**Definition 1.1.** For \( v \in F \), \( A \in I(R) \), put \( v(A) = \inf \{v(a) \mid a \in A\} \).

**Lemma 1.2.** If \( A, B \in I(R) \), \( v \in F \), then \( v(AB) = v(A) + v(B) \).

**Proof.** See \[4, p. 712\], Theorem 1, part (2).

Now, for \( A, B \in I(R) \), define \( A \sim B \) iff \( v(A) = v(B) \) for all \( v \in F \). Then \( \sim \) is an equivalence relation on \( I(R) \). For \( A \in I(R) \) we let \( [A] \) denote the equivalence class of \( A \) with respect to \( \sim \), and we let \( \mathcal{A}(R) \) denote the set of all such equivalence classes.

Define \( + \) on \( \mathcal{A}(R) \) by \( [A] + [B] = [AB] \). Then \( + \) is well defined. Since multiplication of fractionary ideals is commutative and associative, \( \mathcal{A}(R) \) together with \( + \) is a commutative semigroup with identity \( 0 = [R] \).

**Lemma 1.3.** If \( A = Rx \) is a principal fractionary ideal, then \( v(A) = v(x) \) for all \( v \in F \).

**Proof.** \( v(A) = v(Rx) = \inf_{r \in Rx} v(rx) = \inf_{r \in R} v(r) + v(x) = v(1) + v(x) = v(x) \).

If \( G \) is a group and \( I \) is any nonempty index set, we let \( G^{I} \) denote the direct product of \( I \) copies of \( G \) and we let \( G^{(I)} \) denote the direct sum of \( I \) copies of \( G \). We shall assume that the value group of each \( v \in F \) is a subgroup of the additive group of real numbers. When \( v \in F \) is discrete we assume, without loss of generality, that the value group of \( v \) is the additive group of integers. \( X \) denotes the real numbers and \( Z \) denotes the integers.

**Proposition 1.4.** Let \( F = \{v_{i} \mid i \in I\} \) where \( I \) is an index set. The map \( f : \mathcal{A}(R) \rightarrow X^{I} \), defined by \( f([A]) = (v_{i}(A))_{i \in I} \), is a monomorphism.

**Proof.** The proof is straightforward and is omitted.
It follows from Proposition 1.4 that \( \mathcal{A}(R) \) is a semigroup in which the cancellation law holds.

We now introduce a partial ordering for \( \mathcal{A}(R) \).

**Definition 1.5.** For \([A], [B] \in \mathcal{A}(R)\), put \([A] \leq [B]\) iff \(v(A) \leq v(B)\) for all \(v \in F\).

**Proposition 1.6.** \( \mathcal{A}(R) \) is partially ordered by \( \leq \).

**Proof.** The proof is straightforward and is omitted.

As usual, if \([A], [B] \in \mathcal{A}(R)\) are such that \([A] \leq [B]\) and \([A] \neq [B]\), we write \([A] < [B]\). Since \([R] = 0 \in \mathcal{A}(R)\), \([A] \in \mathcal{A}(R)\) is such that \([A] \geq 0\) iff \(A\) is an ideal of \(R\). For if \(A\) is an ideal of \(R\), then \(A \subseteq R\) so that \(v(A) \geq v(R) = 0\), for all \(v \in F\). On the other hand, if \([A] \geq 0\), then \(v(A) \geq 0\) for all \(v \in F\) so that \(A \subseteq \bigcap_{v \in F} R_v = R\). Furthermore, if each \(v \in F\) is discrete, then \([A] > 0\) iff \(A \subseteq P(v)\) for some \(v \in F\). For if \([A] > 0\), then, since each \(v\) is discrete, \(v(A) \geq 1 > 0\) for some \(v \in F\). But then \(A \subseteq P(v)\), and conversely. We can use these properties of \(\leq\) to characterize the positive elements of \(\mathcal{A}(R)\) when the elements of \(F\) are discrete.

For \(n\) a positive integer and \(P\) a minimal prime of \(R\), put \(P^{(n)} = P^n R_P \cap R\).

**Proposition 1.7.** If \(P\) is a minimal prime of \(R\) and \(P = P(v)\) for some \(v \in F\), then \(P^{(n)} = \{x \in R \mid v(x) \geq n\}\) for every positive integer \(n\).

**Proof.** We have \(P^{(n)} = P^n R_P \cap R = (PR_P)^n \cap R\). So if \(x \in P^{(n)}\), then \(x \in (PR_P)^n\) and \(v(x) \geq n\); i.e., \(P^{(n)} = \{x \in R \mid v(x) \geq n\}\). On the other hand, if \(x \in R\) is such that \(v(x) \geq n\), then \(x \in P\) and hence \(x \in PR_P\). Since \(v(x) \geq n\) we have \(x \in (PR_P)^n\), and so \(x \in P^{(n)}\); i.e., \(\{x \mid x \in R, v(x) \geq n\} \subseteq P^{(n)}\).

As is well known, \(P^{(n)}\) is a \(P\)-primary ideal of \(R\).

**Proposition 1.8.** Let \(A\) be an ideal of \(R\) such that \([A] > 0\), and let

\[
J = \{j \in I \mid v_j(A) > 0\}.
\]

Then \(A \subseteq \bigcap_{j \in J} P_j^{(n_j)}\), and \([A] = [\bigcap_{j \in J} P_j^{(n_j)}]\), where for each \(j \in J\), \(P_j = P(v_j)\) and \(n_j = v_j(A)\).

**Proof.** If \(x \in A\), \(j \in J\), then \(v_j(x) \geq v_j(A) = n_j\); i.e., \(x \in P_j^{(n_j)}\) by Proposition 1.7. Thus for each \(j \in J\), \(A \subseteq P_j^{(n_j)}\); i.e., \(A \subseteq \bigcap_{j \in J} P_j^{(n_j)}\); which proves the first assertion.

Now let \(k \in J\). Since \(A \subseteq \bigcap_{j \in J} P_j^{(n_j)}\), we have \(v_k(A) \geq \bigcap_{j \in J} P_j^{(n_j)}\). On the other hand, let \(x \in \bigcap_{j \in J} P_j^{(n_j)}\). Then \(x \in P_k^{(n_k)}\), and \(v_k(x) \geq n_k = v_k(A)\); i.e., \(v_k(\bigcap_{j \in J} P_j^{(n_j)}) \geq v_k(A)\). So if \(k \in J\), then \(v_k(A) = v_k(\bigcap_{j \in J} P_j^{(n_j)})\). If \(k \in I - J\), then \(0 = v_k(A) \geq v_k(\bigcap_{j \in J} P_j^{(n_j)}) \geq 0\); i.e., \(v_k(\bigcap_{j \in J} P_j^{(n_j)}) = 0\) for all \(k \in I - J\). Thus

\[
v_k(A) = v_k(\bigcap_{j \in J} P_j^{(n_j)})
\]

for all \(i \in I\); i.e., \([A] = [\bigcap_{j \in J} P_j^{(n_j)}]\).

It follows that \(\bigcap_{j \in J} P_j^{(n_j)}\) is the largest ideal \(B\) of \(R\) such that \([A] = [B]\).
We now drop the assumption that each \( v \in F \) is discrete so that \( F \) satisfies (i), (ii) and (iii). Property (i) says that \( R_v \) is a rank one valuation ring and hence completely integrally closed for each \( v \in F \). Property (ii) shows that \( R \) is the intersection of completely integrally closed overrings and hence is completely integrally closed. So (i) and (ii) insure that \( \mathcal{D}(R) \) is a group. We now study relations between the semigroup \( \mathcal{A}(R) \) and the group \( \mathcal{D}(R) \).

The next two propositions have been proved in [6] for the case when \( F \) is the family of essential valuations of an AD-domain \( R \).

**Proposition 1.9.** Let \( A \in I(R) \). Then, considering \( [A] \) and \( \text{div}_R(A) \) as subsets of \( I(R) \), \( [A] \subseteq \text{div}_R(A) \).

**Proof.** Let \( B \in [A] \). Then \( v(B) = v(A) \) for all \( v \in F \). If \( A \subseteq Rx \), then \( v(A) \geq v(Rx) = v(x) \), and so \( v(B) \geq v(x) \) for all \( v \in F \). If \( b \in B \), then \( v(b - x) \geq 0 \) for all \( v \in F \); i.e., \( v(b/x) \geq 0 \) for all \( v \in F \). Thus if \( b \in B \) then \( b/x \in \bigcap_{v \in F} R_v = R \), and \( b \in Rx \); i.e., \( B \subseteq Rx \). Similarly, if \( B \subseteq Ry \) then \( A \subseteq Ry \). In this case, \( \bar{A} = \bigcap_{A \subseteq Ry} Ry = \bigcap_{B \subseteq Rx} Rx = B \), and \( \text{div}_R(A) = \text{div}_R(B) \). Hence \( B \in \text{div}_R(A) \).

**Proposition 1.10.** The map \( g : \mathcal{A}(R) \rightarrow \mathcal{D}(R) \) defined by \( g([A]) = \text{div}_R(A) \) is an order preserving homomorphism of the partially ordered semigroup \( \mathcal{A}(R) \) onto the lattice ordered group \( \mathcal{D}(R) \).

**Proof.** Proposition 1.9 shows that \( g \) is well defined and onto. It follows directly that \( g \) is a homomorphism. To see that \( g \) preserves order, suppose \( [A], [B] \in \mathcal{A}(R) \) with \( [A] \subseteq [B] \). If \( A \subseteq Rx \), it follows as in the proof of 1.9 that \( B \subseteq Rx \) so that \( A \subseteq B \), and hence \( \text{div}_R(A) \subseteq \text{div}_R(B) \).

Now let \( T \) be a domain such that \( R \subseteq T \subseteq K \) and such that there is a subfamily \( G \) of \( F \) such that \( T = \bigcap_{w \in G} R_w \). It is easy to show that \( G \) is a family of valuations for \( T \) satisfying (i), (ii), (iii).

**Proposition 1.11.** The map \( \sigma : \mathcal{A}(R) \rightarrow \mathcal{A}(T) \), defined by \( \sigma([A]) = [AT] \), is an order preserving homomorphism of \( \mathcal{A}(R) \) onto \( \mathcal{A}(T) \).

**Proof.** Here, \( \mathcal{A}(T) \) denotes the semigroup of fractionary ideal classes of \( T \) formed with the family \( G \).

It is clear that \( \sigma \) is well defined. To see that \( \sigma \) is onto, let \( \mathcal{U} \) be any nonzero fractionary ideal of \( T \). Then \( \mathcal{U} = (1/d)\mathcal{B} \), where \( \mathcal{B} \) is an ideal of \( T \), \( d \in R \), \( d \neq 0 \). Put \( A = (1/d)B \), where \( B = \mathcal{U} \cap R \). It can be shown that \( v(B) = v(\mathcal{B}) \) for all \( v \in G \), and hence \( \sigma([A]) = [\mathcal{U}] \). It is straightforward to show that \( \sigma \) is a homomorphism which preserves order.

**Corollary 1.12.** If \( T \) is as in 1.11, then \( \mathcal{D}(T) \) is a homomorphic image of \( \mathcal{A}(R) \).
Let $T$ be as in 1.11, and consider the following diagram:

Diagram 1.13.

$$
\begin{array}{ccc}
\mathcal{A}(R) & \xrightarrow{\sigma} & \mathcal{A}(T) \\
\downarrow g_1 & & \downarrow g_2 \\
\mathcal{D}(R) & \xrightarrow{\rho} & \mathcal{D}(T)
\end{array}
$$

Here $\sigma$ is the homomorphism of 1.11, $g_1$ and $g_2$ are the canonical homomorphisms of 1.10. In general, this diagram may not be completed commutatively by a homomorphism $\rho$. For let $R$ be an AD-domain which is not Dedekind, and let $F$ denote the family of essential valuations of $R$. By a result in [6], $R$ contains at least one proper prime $P$ which is not divisoriel. Then $P < \bar{P}$, and hence $\bar{P} = R$ since $P$ is maximal. Since $R$ is AD, there is $v \in F$ such that $P = P(v)$, for some $v \in F$. Take $T = R_P = R_v$ and assume that $\rho$ completes the following diagram commutatively:

Diagram 1.14.

$$
\begin{array}{ccc}
\mathcal{A}(R) & \xrightarrow{\sigma} & \mathcal{A}(R_P) \\
\downarrow g_1 & & \downarrow g_2 \\
\mathcal{D}(R) & \xrightarrow{\rho} & \mathcal{D}(R_P)
\end{array}
$$

Then we must have $\rho(g_1([P])) = g_2(\sigma([P]))$. However, $g_1([P]) = \text{div}_R(P) = 0$ (since $\bar{P} = R$) so that $\rho(g_1([P])) = 0$; and on the other hand $\sigma([P]) = [PR_P]$. But since $R_P$ is a Dedekind domain with unique maximal ideal $PR_P$, we have that $\text{div}_{R_P}(PR_P) > 0$; i.e., $g_2(\sigma([P])) = \text{div}_{R_P}(PR_P) > 0$. Thus $g_2 \neq g_2 \sigma$, contradicting our assumption on $\rho$. This proves the assertion that, in general, Diagram 1.13 may not be completed commutatively.

Equivalent conditions for an AD-domain $R$ to be Dedekind are given in terms of $\mathcal{A}(R)$ in [6]. If we are to extend these results we need to know something about the inverses of elements of $\mathcal{A}(R)$ whenever they exist.

**Proposition 1.15.** If $[A] \in \mathcal{A}(R)$ has an inverse then $-[A] = [R:A]$.

**Proof.** Suppose $[A] \in \mathcal{A}(R)$ has an inverse $[B]$. Since the canonical map $g: \mathcal{A}(R) \to \mathcal{D}(R)$ is a homomorphism, we must have that $g(-[A]) = -g([A]) = -\text{div}_R(A) = \text{div}_R(R:A)$. Thus $g([B]) = g(-[A]) = \text{div}_R(R:A)$. But by definition of $g$, $g([B]) = \text{div}_R(B)$ so that $\text{div}_R(B) = \text{div}_R(R:A)$. Since $R:A$ is divisoriel we have $B \subseteq \bar{B} = R:A = R: \bar{A}$. Then $AB \subseteq A(R:A) \subseteq R$ so that $0 = [AB] \succeq [A(R:A)] \geq 0$. Thus $0 = [A] + [B] = [A] + [R:A]$; i.e., $-[A] = [B] = [R:A]$.

Now consider the following diagram.
Diagram 1.16.

\[
\begin{array}{ccc}
\mathcal{A}(R) & \xrightarrow{f} & X' \\
g \downarrow & & \downarrow \lambda \\
\mathcal{D}(R) & & \\
\end{array}
\]

Here \(g\) is the canonical homomorphism and \(f\) is the homomorphism of 1.4. \(g\) is surjective and \(f\) is injective.

**Proposition 1.17.** Diagram 1.16 may be completed commutatively by a homomorphism \(\lambda\) iff \(g\) is an isomorphism.

We can now prove the following theorem.

**Theorem 1.18.** Let \(R\) be an integral domain with quotient field \(K\), and let \(F\) be a family of valuations satisfying (i), (ii), (iii). The following statements are equivalent.

1. \(\mathcal{A}(R)\) is a group.
2. \(R:A = R:B \Rightarrow [A] = [B]\), for all \(A, B \in I(R)\).
3. \(v(A) = v(\bar{A})\) for all \(A \in I(R)\) and \(v \in F\).
4. The map \(g: \mathcal{A}(R) \to \mathcal{D}(R)\) is an isomorphism.

**Proof.** (1) \(\Rightarrow\) (2) Suppose \(\mathcal{A}(R)\) is a group. Let \(A, B \in I(R)\) be such that \(R:A = R:B\). By 1.15 we have \(-[A] = [R:A] = [R:B] = -[B]\) and hence \([A] = [B]\).

(2) \(\Rightarrow\) (3) We have \(R:A = R:\bar{A}\) for all \(A \in I(R)\). If (2) holds then \([A] = [\bar{A}]\) for all \(A \in I(R)\); i.e., \(v(A) = v(\bar{A})\) for all \(v \in F, A \in I(R)\).

(3) \(\Rightarrow\) (4) Consider Diagram 1.16. If \(v(A) = v(\bar{A})\) for all \(v \in F\) and \(A \in I(R)\), we can define \(\lambda: \mathcal{D}(R) \to X'\) by \(\lambda(\text{div}_R(A)) = (v_i(A))_{i \in I}\). It follows that \(\lambda\) is a homomorphism and that \(\lambda \circ g = f\). By 1.17, \(g\) is an isomorphism.

(4) \(\Rightarrow\) (1) obvious.

We observe that the converse of statement (2) in 1.18 is always true in \(R\). For if \([A] = [B]\), then \(A \in [B] \subseteq \text{div}_R(B)\), and \(B \in \text{div}_R(B)\) so that \(\text{div}_R(A) = \text{div}_R(B)\) and hence \(R:A = R:B\).

When the valuations in \(F\) are discrete we obtain a partial generalization of a result in [6] with the aid of the following lemma.

**Lemma 1.19.** Assume that each \(v \in F\) is discrete. Then for each \(v \in F\), if \(\text{div}_R(P(v)) \neq 0\) then \(P(v) = (P(v))^{-}\).

**Proof.** We have \(P(v) \subseteq (P(v))^{-} \subseteq R\). If \(P(v) < (P(v))^{-}\), there is \(x \in (P(v))^{-}\), \(x \notin P(v)\). Then \(v(x) = 0 = v(P(v))^{-}\). Also, for \(w \in F, w \neq v\), we have \(0 = w(P(v)) \supseteq w(P(v))^{-} \geq 0\). Thus \([(P(v))^{-}] = 0\). Since the canonical map \(g\) from \(\mathcal{A}(R)\) onto \(\mathcal{D}(R)\) is a homomorphism, we should have \(0 = [(P(v))^{-}] \rightarrow \text{div}_R(P(v))^{-} = \text{div}_R(P(v)) = 0\). But \(\text{div}_R(P(v)) \neq 0\) by assumption. Thus we must have \((P(v))^{-} = P(v)\) if \(\text{div}_R(P(v)) \neq 0\).

**Theorem 1.20.** Assume each \(v \in F\) is discrete. Then the canonical map \(g\) from \(\mathcal{A}(R)\) onto \(\mathcal{D}(R)\) is an isomorphism iff \(P(v)\) is divisoriel for each \(v \in F\).
Proof. (\(\Rightarrow\)) Suppose \(g\) is an isomorphism. If \(P = P(v)\) for some \(v \in F\), then \([P] > 0\) since \(v(P) = 1\). If \(P\) is not divisoriel then \(g([P]) = \text{div}_R(P) = 0\), by Lemma 1.19. But then \(g\) is not 1-1 and hence not an isomorphism. For \([R] = 0 \neq [P]\).

(\(\Leftarrow\)) Suppose \(P(v)\) is divisoriel for each \(v \in F\). Then if \(A \in I(R)\) is such that \(\text{div}_R(A) = 0\) we must have \(A \subseteq R\) (for this result see [1, bottom of p. 4]). Moreover, \(A \trianglelefteq P(v)\) for any \(v \in F\). If \(A \subseteq P = P(v)\) for some \(v \in F\), then \(\text{div}_R(A) \geq \text{div}_R(P) > 0\), a contradiction. Thus \(g([A]) = 0\) iff \([A] = 0\). Now suppose \([A], [B] \in \mathcal{A}(R)\) are such that \(g([A]) = g([B])\). Then \(\text{div}_R(A) = \text{div}_R(B)\) so that \(\text{div}_R(A) - \text{div}_R(B) = 0 = \text{div}_R(B) - \text{div}_R(A)\); i.e., \(\text{div}_R(A : B) = 0 = \text{div}_R(B : A)\). Since \(g([A : B]) = g([B : A]) = 0\), we must have \([A : B] = 0 = [B : A]\). Since each \(v \in F\) is discrete, for each \(v \in F\) there is \(x \in A : B\) such that \(v(x) = v(A : B) = 0\). Now \(xB \subseteq A\) (by definition of \(A : B\)) so that \(v(x) + v(B) = v(xB) \geq v(A)\); i.e., \(v(B) \geq v(A)\). Thus \(v(B) \geq v(A)\) for all \(v \in F\). Similarly \(v(A) \geq v(B)\) for all \(v \in F\), and \([A] = [B]\). This shows that \(g\) is 1-1 and hence an isomorphism.

When \(R\) is AD, the author has shown in [6] that \(P(v)\) is divisoriel for each \(v \in F\) iff \(R\) is Dedekind. To date, however, the author has been unable to prove the following conjecture: If \(R\) is AK and \(P(v)\) is divisoriel for each \(v \in F\), then \(R\) is a Krull domain.

When \(R\) is AK, we do have the following theorem.

**Theorem 1.21.** Let \(R\) be an AK-domain with family \(F\) of essential valuations and let \(\Delta\) denote the collection of maximal ideals of \(R\). Every minimal prime of \(R\) is divisoriel iff \(\bar{A} = \bigcap_{M \in \Delta} (AR_M)^{\sim}\) for every ideal \(A\) of \(R\).

Proof. Here \(\bar{A} = \bigcap_{A \subseteq R^x} Rx\) and \((AR_M)^{\sim} = \bigcap_{A \subseteq M \subseteq R_M} R_Mv\). For any maximal ideal \(M\) of \(R\), \(F_M\) denotes the family of essential valuations of the Krull domain \(R_M\). Recall that \(F_M \subseteq F\).

(\(\Rightarrow\)) Let \(A\) be an ideal of \(R\). If \(M\) is any maximal ideal of \(R\) then \(v(A) = v(AR_M)\) for all \(v \in F_M\). Since \(R_M\) is a Krull domain, \(v(AR_M) = v(AR_M)^{\sim}\) for all \(v \in F_M\) so that \(v(A) = V(AR_M)^{\sim}\) for all \(v \in F_M\).

Case 1. \(v(A) = 0\) for all \(v \in F\).

Then \(P < A\) for every minimal prime \(P\) of \(R\). In this case \(\bar{A} = R = \bigcap_{M \in \Delta} R_M = \bigcap_{M \in \Delta} (AR_M)^{\sim}\).

Case 2. \(v(A) > 0\) for some \(v \in F\).

For each maximal ideal \(M\) of \(R\), if there is \(v \in F_M\) such that \(0 < v(A) = v(AR_M)^{\sim}\), then we can write
\[(AR_M)^{\sim} = \bigcap_{v \in F_M : v(A) > 0} Q_i^{(n_i)},\]

where \(n_i = v_i(AR_M)^{\sim}\) and \(Q_i\) is the center of \(v_i\) on \(R_M\). Then for each \(i\) such that \(v_i(A) > 0\) we have \(Q_i = P_iR_i\) where \(P_i = P(v_i)\) in \(R\). Thus \((AR_M)^{\sim} = \bigcap_i (P_iR_M)^{(n_i)} = \bigcap_i ((P_iR_M)^{n_i} : R_M) \cap R_M = \bigcap_i (P_i^{n_i} : R_i \cap R_M)\) where \(i\) runs over all indices such that \(v_i \in F_M\) and \(v_i(A) > 0\), and \(n_i = v_i(A)\) for each such \(i\). It can then be shown that, \(C = \bigcap_{M \in \Delta} (AR_M)^{\sim} = \bigcap \{P_i^{n_i} \mid v_i \in F, v_i(A) > 0\}\). Then \([C] = [A]\). Since \([A] = \text{div}_R(A)\), it follows that \(\bar{A} = C\).
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\[(\Leftarrow)\] Suppose \(P\) is a minimal prime of \(R\). If \(M \in \Delta\), then either \(P \subseteq M\) or \(P \nsubseteq M\). If \(P \nsubseteq M\) then \(PR_M = M\). If \(P \subseteq M\), then \(PR_M\) is a minimal prime of the Krull domain \(R_M\) and thus \((PR_M)^{-1} = PR_M\). Since \(P\) is contained in some maximal ideal \(M\) we have \(P = \bigcap_{M \in \Delta} PR_M\).

We now drop the assumption that \(R\) is an AK-domain and assume only that \(F\) satisfies axioms (i), (ii), (iii) at the beginning of this section. The next lemma tells us more about the elements of \(\mathcal{A}(R)\) which have inverses and enables us to partially describe \(\mathcal{D}(R)\) in certain cases where \(\mathcal{A}(R)\) may not be a group.

**Lemma 1.22.** If \([A] \in \mathcal{A}(R)\) is such that \([A]\) has an inverse then \([A] = [\tilde{A}]\).

**Proof.** If \([A] \in \mathcal{A}(R)\) has an inverse then \(-[A] = [R : A]\) by Proposition 1.15. Now \(A \subseteq \tilde{A}\) and \(R : A = R : \tilde{A}\). Thus \(A(R : A) \subseteq \tilde{A}(R : A) = \tilde{A}(R : \tilde{A}) \subseteq R\). These containment relations yield the following: \(0 \leq [A(R : A)] \leq [\tilde{A}(R : \tilde{A})] \geq 0\). Thus

\[0 = [A] + [R : A] = [\tilde{A}] + [R : A] \quad \text{and} \quad [A] = [\tilde{A}].\]

**Corollary 1.23.** If \([A], [B]\) have inverses in \(\mathcal{A}(R)\), then \([\tilde{A} + [B] = [\tilde{A}B] = [AB]^{-1}\); i.e., \(v(\tilde{AB}) = v(AB)^{-1}\) for all \(v \in F\).

**Proof.** By 1.22 above, if \([A], [B]\) have inverses then \([A] = [\tilde{A}]\) and \([B] = [\tilde{B}]\). Moreover, \([A] + [B]\) has an inverse. Thus \([A] + [B] = [AB] = [AB]^{-1}\) by 1.22; i.e., \([\tilde{A}] + [\tilde{B}] = [\tilde{AB}] = [AB]^{-1}\).

Now consider the map \(\rho: \mathcal{D}(R) \to X^I\) defined by \(\rho(div_R (A)) = (v_i(\tilde{A}))_{i \in I}\). \(\rho\) is well defined, for if \(div_R (A) = div_R (B)\) then \(\tilde{A} = \tilde{B}\). Conversely, if \((v_i(\tilde{A}))_{i \in I} = (v_i(\tilde{B}))_{i \in I}\) then \([\tilde{B} = [\tilde{A}]\) and so \(div_R (A) = div_R (\tilde{A}) = div_R (\tilde{B}) = div_R (B)\) by the remark following the proof of 1.18. Thus \(\rho\) is 1-1. We can now give a description of \(\mathcal{D}(R)\) when \(R\) is fairly well behaved.

**Theorem 1.24.** The map \(\rho: \mathcal{D}(R) \to X^I\) defined by \(\rho(div_R (A)) = (v_i(\tilde{A}))_{i \in I}\) is 1-1. Furthermore \(\rho\) is a homomorphism iff \([\tilde{A}] \in \mathcal{A}(R)\) has an inverse for all \(A \in I(R)\).

**Proof.** The first assertion is proved in the immediately preceding remarks. We now prove the second assertion.

\((\Rightarrow)\) Suppose \(\rho\) is a homomorphism. Then since \(\mathcal{D}(R)\) is a group, for \(div_R (A) \in \mathcal{D}(R), \ -div_R (A) = div_R (R : A)\). Thus \(\rho(div_R (A)) + div_R (R : A) = 0 = \rho(div_R (A)) + \rho(div_R (R : A)) = (v_i(\tilde{A}))_{i \in I} + (v_i(R : A))_{i \in I}\). It follows that \([\tilde{A}]\) has an inverse in \(\mathcal{A}(R)\).

\((\Leftarrow)\) Suppose that \([\tilde{A}]\) has an inverse for all \(A \in I(R)\). By Corollary 1.23, we have that \([\tilde{A}B] = [AB]^{-1}\) for all \(A, B \in I(R)\). Thus for \(A, B \in I(R)\) we have \(\rho(div_R (AB)) = (v_i(AB))_{i \in I} = (v_i(\tilde{A}B))_{i \in I} = (v_i(\tilde{A}))_{i \in I} + (v_i(B))_{i \in I} = \rho(div_R (A)) + \rho(div_R (B))\) and \(\rho\) is a homomorphism.

Now, let \(R\) be an AK-domain. Then \(R_P\) is a Krull domain for any prime ideal \(P\) of \(R\). However, these are not the only Krull domains \(T\) such that \(R \subseteq T \subseteq K\). For if \(A = \{P_1, \ldots, P_n\}\) is any finite collection of prime ideals of \(R\) then \(T = \bigcap_{i \in A} R_{P_i}\) is also a Krull domain. Thus there is a large class of Krull domains \(T\) such that
When $R$ is an AK-domain in which every minimal prime is divisoriel we always have that $D(T)$ is a homomorphic image of $D(R)$, where $T$ is an AK-domain such that $R \subseteq T \subseteq K$. For, $A(T)$ is a homomorphic image of the group $A(R)$ and so is a group. Then $A(R) \cong D(R)$ and $A(T) \cong D(T)$. When $T$ is a Krull domain and $R$ is an AK-domain for which the map $\rho$ of Theorem 1.24 is a homomorphism we also get that $D(T)$ is a homomorphic image of $D(R)$ as follows.

**Proposition 1.25.** Let $R$ be an AK-domain and let $T$ be a Krull domain such that $R \subseteq T \subseteq K$. If $[A]$ has an inverse for every $[A] \in A(R)$ then the map $\tau: D(R) \to D(T)$, defined by $\tau(\text{div}_R([A])) = \text{div}_T([AT])$, is a homomorphism of $D(R)$ onto $D(T)$.

**Proof.** $\tau$ is well defined, for if $\text{div}_R([A]) = \text{div}_R([B])$ then $[A] = [B]$ so that $[AT] = [BT]$. Now consider the following diagram.

**Diagram 1.26.**

\[
\begin{array}{ccc}
D(R) & \xrightarrow{\rho} & Z^I \\
\tau \downarrow & & \downarrow \pi \\
D(T) & \xrightarrow{\gamma} & Z^{(J)}
\end{array}
\]

Here, $I$ is the index set for the family of essential valuations of $R$; $J$ is the index set for the family of essential valuations of $T$; $\pi$ is the projection of $Z^I$ onto $Z^{(J)}$; $\rho$ is the (injective) homomorphism of 1.24; $\gamma$ is the injection of 1.4. It is well known that $\gamma$ is also surjective; i.e., $\gamma$ is an isomorphism. Consider the map $\gamma^{-1} \circ \pi \circ \rho: D(R) \to D(T)$. We have for $\text{div}_R([A]) \in D(R)$, $(\gamma^{-1} \circ \pi \circ \rho)(\text{div}_R([A])) = (\gamma^{-1} \circ \pi) \times (v_j([A]))_{j \in J} = \gamma^{-1}((v_j([A]))_{j \in J})$. Since $T$ is a Krull domain we have that $v([B]) = v([B])$ for all fractionary ideals $B$ and all essential valuations $v$. Thus $(v_j([AT]))_{j \in J} = (v_j([AT]))_{j \in J}$ so that $\gamma^{-1}((v_j([A]))_{j \in J}) = \text{div}_R([AT])$. Then $\gamma^{-1} \circ \pi \circ \rho = \tau$ and $\tau$ is a homomorphism since $\tau$ is a composition of homomorphisms. To see that $\tau$ is surjective it is sufficient to show that for every divisoriel fractionary ideal $\mathcal{U}$ of $T$ there is a divisoriel fractionary ideal $A$ of $R$ such that $\tau(\text{div}_R([A])) = \text{div}_T(\mathcal{U})$. So let $\mathcal{U}$ be a fractionary ideal of $T$. There are elements $x, y \in K$ such that $\mathcal{U} = Tx \cap Ty$ [1, p. 13]. Let $A = Rx \cap Ry$. Then $A$ is divisoriel and $v_j(A) = v_j(\mathcal{U})$ for all $j \in J$ and $\tau(\text{div}_R([A])) = \text{div}_T(\mathcal{U})$.

2. Let $R$ be an integral domain with quotient field $K$. Suppose that $F$ is a family of valuations on $K$ satisfying the following:

1. $R = \bigcap_{v \in F} R_v$,
2. $R_v = R_{F(\mathcal{U})}$ for each $v \in F$.

Following Gilmer in [3], we make the following definition.

**Definition 2.1.** We say that $R$ satisfies property $(\heartsuit)$ with respect to $F$ iff for distinct subsets $F_1, F_2$ of $F$ we have that $\bigcap_{v \in F_1} R_v \neq \bigcap_{v \in F_2} R_v$.

When $R$ is a Prüfer domain and $F$ is the family of valuations induced by the collection of maximal ideals, then property $(\heartsuit)$ is the same as property $(\diamondsuit)$ in [2].
For \( v \in F \), we let \( F_v = F - \{v\} \).

**Proposition 2.2.** \( R \) has property \((*)\) with respect to \( F \) iff for each \( v \in F \), \( \cap_{w \in F_v} R_w \nsubseteq R_v \).

**Proof.** The proof is substantially the same as that of Lemma 1 in [2] and is omitted.

**Corollary 2.3.** If \( R \) satisfies \((*)\) with respect to \( F \) and if \( G \) is any nonempty subset of \( F \), then \( T = \cap_{u \in G} R_u \) satisfies \((*)\) with respect to \( G \).

We note that if \( R \) satisfies \((*)\) with respect to \( F \) then \( P(v) \nsubseteq P(w) \) for \( v \neq w \). For if \( P(v) \subseteq P(w) \) for some \( w \neq v \), then \( R_{P(w)} \subseteq R_{P(v)} \); i.e., \( R_w \subseteq R_v \). Then we have the following: \( (\cap_{u \in F - \{v, w\}} R_u) \cap (R_v \cap R_w) = (\cap_{u \in F - \{v, w\}} R_u) \cap R_w = \cap_{u \in F_v} R_u \), and \( F \neq F_v \), a contradiction.

**Proposition 2.4.** If \( F \) is of finite character and is such that \( P(u) \nsubseteq P(v) \) if \( u \neq v \), then \( R \) satisfies \((*)\) with respect to \( F \).

**Proof.** Let \( v \in F \) and let \( x \in R \), \( x \neq 0 \), be such that \( v(x) > 0 \). Let \( v_1, \ldots, v_n \) be the distinct (from \( v \) and each other) valuations such that \( v_i(x) \neq 0 \), \( i = 1, \ldots, n \). There exists \( y \in (\cap_{i=1}^n P(v_i)) - P(v) \). For if \( \cap_{i=1}^n P(v_i) \subseteq P(v) \), then \( \prod_{i=1}^n P(v_i) \subseteq \cap_{i=1}^n P(v_i) \) and so \( P(v_i) \subseteq P(v) \) for some \( j, 1 \leq j \leq n \), contradicting our hypothesis. Choose \( n \) large enough so that \( w(v^n/x) \geq 0 \) for \( w \in F_v \). This is possible since \( F \) is of finite character and \( w(y^n/x) \geq 0 \) for all \( w \in F_v \). Then \( w(v^n/x) \geq 0 \) for all \( w \in F_v \) and \( v(y^n/x) = -v(x) < 0 \). Thus \( y^n/x \in \cap_{u \in F_v} R_u - R_v \); i.e., \( \cap_{u \in F_v} R_u \nsubseteq R_v \). So \( R \) satisfies \((*)\) with respect to \( F \) by 2.2.

Let \( R \) be an integral domain with family \( F \) of valuations satisfying (1) and (2) listed at the beginning of this section. \( R \) is called a generalized Krull domain if \( F \) satisfies the following two additional properties (see [5]).

(3) Each \( v \in F \) has rank one.

(4) \( F \) is of finite character.

**Corollary 2.5.** If \( R \) is a Krull domain, or a generalized Krull domain with family \( F \) of valuations, then \( R \) satisfies \((*)\) with respect to \( F \).

**Proof.** In this case \( F \) is a family of rank one valuations of finite character, so that if \( u, v \in F \), \( u \neq v \), then \( P(v) \nsubseteq P(u) \).

**Proposition 2.6.** Let \( R \) be an AD-domain. The following conditions on \( R \) are equivalent.

(1) \( R \) satisfies \((*)\) with respect to \( F \), the family of essential valuations of \( R \).

(2) \( R \) is Dedekind.

(3) Every minimal prime of \( R \) is divisoriel.

**Proof.** (1) \( \iff \) (2) is Theorem 3 of [3].

(2) \( \iff \) (3) is found in [6].
Thus we see that in the case of almost-Dedekind domains, the divisoriel property of the minimal prime ideals completely determines whether or not \( R \) satisfies property \( (\ast) \). We shall see that the divisoriel property of the minimal primes is always sufficient for \( R \) to satisfy \( (\ast) \).

**Proposition 2.7.** Let \( R \) be an integral domain with family \( F \) of valuations such that

(i) Each \( v \in F \) has rank one.

(ii) \( R = \bigcap_{v \in F} R_v \).

(iii) \( R_v = R_{P(v)} \) for each \( v \in F \).

If \( P(v) \) is divisoriel for each \( v \in F \), then \( R \) satisfies \( (\ast) \) with respect to \( F \).

**Proof.** We note that since \( R \) is the intersection of rank one valuation rings, \( R \) is completely integrally closed and hence \( \mathfrak{D}(R) \) is a group. If each \( P(v) \) is divisoriel, then each \( v \in F \) is discrete. For if \( P = P(v) \) is divisoriel we must have \( P^2 < P \). For if \( P^2 = P \), then \( \text{div}_R(P^2) = \text{div}_R(P) \); i.e., \( 2 \text{ div}(P) = \text{ div}(P) \). Thus \( \text{ div}(P) = 0 \) and \( \mathfrak{p} = R \neq P \), contradicting \( \mathfrak{p} = P \).

Since \( P^2 < P \), we have \( P^2 R_P < PR_P \) and so \( R_P \) is a discrete valuation ring. We now show that \( \{ P(v) \mid v \in F \} \) is the set of all minimal divisoriel primes of \( R \). Clearly, \( \{ P(v) \mid v \in F \} \) is contained in the set of all divisoriel minimal primes. Now let \( P \) be a minimal, divisoriel prime of \( R \). If \( P \neq P(v) \) for any \( v \in F \), then \( P \neq P(v) \) for any \( v \in F \) and so \( v(P) = 0 \) for all \( v \in F \); i.e., \( [P] = 0 \). But then we would have \( g([P]) = 0 \); i.e., \( \text{ div}(P) = 0 \); i.e., \( \mathfrak{p} = R \), contradicting \( \mathfrak{p} = P < R \). So we must have that \( \{ P(v) \mid v \in F \} \) is the set of all divisoriel minimal primes of \( R \). Now let \( G \) be any subset of \( F \) such that \( R = \bigcap_{u \in G} R_u \). \( P(u) \) is divisoriel for each \( u \in G \) since \( G \subseteq F \). By what we have just shown, \( \{ P(u) \mid u \in G \} \) is the collection of all minimal divisoriel primes of \( R \); i.e., \( G = F \). Thus for any \( v \in F \), \( \bigcap_{u \in F} R_u \neq R_v \) and so \( R \) satisfies \( (\ast) \) with respect to \( F \).

The first part of the proof of Proposition 2.7 shows that if \( P \) is the center of a rank one valuation \( v \), then \( v \) is discrete if \( P \) is divisoriel. This enables us to characterize Krull domains in the class of generalized Krull domains as follows.

**Corollary 2.8.** Let \( R \) be a generalized Krull domain with family \( F \) of valuations. \( R \) is a Krull domain iff \( P(v) \) is divisoriel for each \( v \in F \).

Let \( R \) be an integral domain with quotient field \( K \) and let \( F \) be a family of valuations on \( K \) satisfying conditions (1) and (2) stated at the beginning of this section. Let \( x \) be an indeterminate and let \( F' \) denote the family of valuations on \( K(x) \) which are canonical extensions of elements of \( F \). Let \( G \) denote the family of \( p(x) \)-adic valuations on \( K(x) \), where \( p(x) \) is a nonconstant irreducible polynomial in \( K[x] \). Then \( F' \cup G \) is a family of valuations on \( K(x) \) satisfying (1) and (2) with \( R[x] \) in place of \( R \).
Proposition 2.9. If \( R \) satisfies \( (*) \) with respect to \( F \), then \( R[x] \) satisfies \( (*) \) with respect to \( F' \cup G \).

Proof. Let \( w \in F' \cup G \). If \( w \in G \), then \( w \) is a \( (x) \)-adic valuation for some nonconstant irreducible polynomial \( p(x) \in K[x] \). Without loss of generality we may assume that \( p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \), \( a_i \in R \). Let \( b = \prod a_{\nu} \). Then \( b \neq 0 \) since \( a_n \neq 0 \), and \( v(b) = \sum_{\nu} v(a_{\nu}) \geq \min_{0 \leq i \leq n} v(a_i) \) for all \( v \in F \) since \( b \in R \) and \( a_{\nu} \in R \) for all \( k = 0, 1, \ldots, n \), and every \( v \in F \) is nonnegative on \( R \). Then for \( v' \in F' \), \( v'(b/p(x)) = v'(b) - v'(p(x)) = v(b) - \min_{0 \leq i \leq n} v(a_i) \geq 0 \). If \( u \in G \) and \( u \neq w \), then \( u \) is a \( (x) \)-adic valuation for some nonconstant irreducible polynomial \( q(x) \) such that \( q(x) \mid p(x) \). Then \( u(b/p(x)) = 0 \). Thus \( b/p(x) \in \bigcap_{v \in (F' \cup G) \setminus u} (R[x])_v \). \( b/p(x) \notin (R[x])_w \) since \( w(b/p(x)) = -1 < 0 \). Thus if \( w \in G \cap \bigcap_{v \in (F' \cup G) \setminus v'} (R[x])_w \), there is \( a \in \left( \bigcap_{v \in F \setminus u} R_v \right) - R_u \subseteq \left( \bigcap_{v' \in F' \setminus v'} R[x]_{v'} \right) \cap [x] = (\bigcap_{v' \in F' \cup G} R[x]_{v'}) \cap (\bigcap_{v \in G} R[x]_v) = \bigcap_{v \in (F' \cup G) \setminus u} R[x]_v \), and \( a \notin R_v \). Then \( a \notin R[x]_v \), for \( v'(a) = v(a) < 0 \). Thus for every \( w \in F' \cup G \) we have \( \bigcap_{v \in (F' \cup G) \setminus u} R[x]_w \neq (R[x])_w \) and thus \( R \) satisfies \( (*) \) with respect to \( F' \cup G \) by 2.2.

3. In [6] it was shown that if \( R \) is an almost-Dedekind domain with family \( F \) of essential valuations, then \( R \) is Dedekind iff every minimal prime of \( R \) is divisoriel. Thus in an AD-domain \( R \), every minimal prime is divisoriel iff \( F \) is of finite character. In §1 it was conjectured that if \( R \) is an AK-domain with family \( F \) of essential valuations, then \( R \) is Krull if \( P(v) \) is divisoriel for each \( v \in F \); i.e., \( F \) is of finite character if \( P(v) \) is divisoriel for each \( v \in F \). In this section we give an example to show that this conjecture is false if the AK requirement is dropped. We also give an example of an AK-domain which is neither a Krull domain nor an AD-domain.

Let \( R \) denote the set of entire functions, \( C \) denote the set of complex numbers, \( Z \) denote the additive group of integers. It is well known that \( R \) is an integral domain under the usual pointwise definitions of addition and multiplication. For \( a \in C \) we define \( v_a: R - \{0\} \to Z \) by \( v_a(f(z)) = n \) if \( a \) is a zero of \( f(z) \) of order \( n \). If \( a \) is not a zero of \( f(z) \) then \( v_a(f(z)) = 0 \). If \( f(z) \equiv 0 \) we put \( v_a(f(z)) = +\infty \) for each \( z \in C \). It is easy to show that each \( v_a \) can be extended to a valuation on the quotient field of \( R \). We let \( F \) denote this family of valuations. \( F \) has the following properties: (i) Each \( v \in F \) has rank one and is discrete; (ii) \( R = \bigcap \{ R_v \mid v \in F \} \); (iii) \( R_v = R_{P(v)} \) for each \( v \in F \); (iv) For \( a \in C \), \( P(v_a) = (z - a)R \), and hence is divisoriel; (v) \( F \) is not of finite character. Furthermore, \( P(v) \) is maximal for each \( v \in F \). However, these are not all the maximal ideals of \( R \). For let \( \{z_n\}_{n=1}^{\infty} \) be a sequence of complex numbers such that \( \lim z_n = \infty \). For each positive integer \( m \), let \( f_m(z) \) be an entire function whose zeros are exactly \( \{z_m, z_m + 1, \ldots\} \). The ideal generated by \( \{f_1(z), f_2(z), \ldots\} \) is proper and is contained in a maximal ideal \( M \). However, \( R_M \) is not a Krull domain. It follows that \( R \) is not AK.

It was shown in [7] that if \( R \) is AK and \( X_1, \ldots, X_n \) are indeterminates, then \( R[X_1, \ldots, X_n] \) is AK. Let \( R \) be an AD-domain which is not a Dedekind domain.
Such a domain is given in example 2 of [2]. Then $\mathbb{R}[X_1, \ldots, X_n]$ is an AK-domain which is neither a Krull domain nor an AD-domain. We observe that example 1 of [2] is a generalized Krull domain which is not a Krull domain.

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**References**


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