FAMILIES OF VALUATIONS AND SEMIGROUPS
OF FRACTIONARY IDEAL CLASSES

BY
ELBERT M. PIRTLE, JR.

Introduction. Let $R$ be an integral domain with quotient field $K$. For any valuation $v$ on $K$ which is nonnegative on $R$, we let $P(v) = \{x \in R \mid v(x) > 0\}$. $P(v)$ is a prime ideal of $R$ and is called the center of $v$ on $R$. In this paper we are concerned mainly with integral domains $R$ which satisfy the following: There exists a family $F$ of valuations on $K$ such that

(i) Each $v \in F$ has rank one.
(ii) $R = \bigcap_{v \in F} R_v$.
(iii) $R_v = R_{P(v)}$, for each $v \in F$.

A family $F$ of valuations on $K$ is said to be of finite character if for $x \in K$, $x \neq 0$, there are only a finite number of $v \in F$ such that $v(x) \neq 0$. $R$ is called a Krull domain if there is a family $F$ of finite character satisfying (i), (ii), (iii), with the additional requirement that each $v \in F$ be discrete. $R$ is called an almost-Krull (AK) domain [7] if $R_P$ is a Krull domain for every proper nonzero prime $P$ of $R$. It follows that $R$ is almost Dedekind (AD) iff $R$ is an AK-domain in which proper prime ideals are maximal [7].

Using the family $F$ of valuations we construct a partially ordered semigroup $\mathcal{A}(R)$ of fractionary ideal classes in §1 and study the relation between $\mathcal{A}(R)$ and $\mathcal{D}(R)$, the divisor group of $R$ (see [1]). Necessary and sufficient conditions for $\mathcal{A}(R)$ and $\mathcal{D}(R)$ to be isomorphic are determined. In §2, condition (S) of [3] is studied. §3 consists of an example.

The notation concerning $\mathcal{D}(R)$ is that of [1]. Otherwise, the notation of [8] is used. Prime ideals are always nonzero and not all of $R$.

1. In order to make this paper as self contained as possible we first list the necessary background results from [1]. $R$ will denote a commutative integral domain with identity and quotient field $K$. $I(R)$ will denote the collection of nonzero fractionary ideals of $R$. A fractionary ideal of the form $Rx$, $x \in K$, $x \neq 0$, is called a principal fractionary ideal.

A relation $<$ is defined on $I(R)$ as follows: $A < B$ iff every principal fractionary ideal of $R$ which contains $A$ also contains $B$. The relation $<$ is a preorder on $I(R)$; i.e., $<$ is a symmetric, transitive relation. If we define $\equiv$ on $I(R)$ by $A \equiv B$ iff $A < B$ and $B < A$, then $\equiv$ is an equivalence relation on $I(R)$. For $A \in I(R)$, $\text{div}_R(A)$
denotes the equivalence class of $A$ with respect to $\equiv$ and is called the divisor of $A$; $\mathfrak{D}(R)$ denotes the set of all such equivalence classes.

For $A \in I(R)$, we put $\overline{A} = \bigcap_{a \in A} Rx$. A fractionary ideal $B$ of $R$ is said to be divisoriel if $B = \overline{B}$. It follows that for $A \in I(R)$, $\text{div}_R(A) = \text{div}_R(\overline{A})$ and that $\overline{A}$ is the unique divisoriel fractionary ideal belonging to $\text{div}_R(A)$. It also follows from the definition that $(\overline{AB})^- = (AB)^-$ for $A, B \in I(R)$ so that $\mathfrak{D}(R)$ together with the operation $+$, defined by $\text{div}_R(A) + \text{div}_R(B) = \text{div}_R(AB)$, is a commutative semigroup with identity $0 = \text{div}_R(R)$. If we define $\leq$ on $\mathfrak{D}(R)$ by $\text{div}_R(A) \leq \text{div}_R(B)$ iff $A \prec B$ then $\mathfrak{D}(R)$ is a lattice ordered semigroup with respect to the partial ordering $\leq$. Furthermore, $\mathfrak{D}(R)$ is a group iff $R$ is completely integrally closed [1, p. 5, Theorem 1].

Let $F$ be a family of valuations on $K$ with the following properties:

(i) Each $v \in F$ has rank one.

(ii) $R = \bigcap_{v \in F} R_v$.

(iii) For each $v \in F$, $R_v = R_{P(v)}$, where $P(v)$ denotes the center of $v$ on $R$.

Occasionally in place of (i) we shall substitute

(i') Each $v \in F$ has rank one and is discrete.

**DEFINITION 1.1.** For $v \in F$, $A \in I(R)$, put $v(A) = \inf \{v(a) | a \in A\}$.

**LEMMA 1.2.** If $A, B \in I(R), v \in F$, then $v(AB) = v(A) + v(B)$.

**Proof.** See [4, p. 712], Theorem 1, part (2).

Now, for $A, B \in I(R)$, define $A \sim B$ iff $v(A) = v(B)$ for all $v \in F$. Then $\sim$ is an equivalence relation on $I(R)$. For $A \in I(R)$ we let $[A]$ denote the equivalence class of $A$ with respect to $\sim$, and we let $\mathfrak{A}(R)$ denote the set of all such equivalence classes.

Define $+$ on $\mathfrak{A}(R)$ by $[A] + [B] = [AB]$. Then $+$ is well defined. Since multiplication of fractionary ideals is commutative and associative, $\mathfrak{A}(R)$ together with $+$ is a commutative semigroup with identity $0 = [R]$.

**LEMMA 1.3.** If $A = Rx$ is a principal fractionary ideal, then $v(A) = v(x)$ for all $v \in F$.

**Proof.** $v(A) = v(Rx) = \inf_{x \in Rx} v(rx) = \inf_{r \in R} v(r) + v(x) = v(1) + v(x) = v(x)$.

If $G$ is a group and $I$ is any nonempty index set, we let $G^I$ denote the direct product of $I$ copies of $G$ and we let $G^{(I)}$ denote the direct sum of $I$ copies of $G$. We shall assume that the value group of each $v \in F$ is a subgroup of the additive group of real numbers. When $v \in F$ is discrete we assume, without loss of generality, that the value group of $v$ is the additive group of integers. $X$ denotes the real numbers and $Z$ denotes the integers.

**PROPOSITION 1.4.** Let $F = \{v_i | i \in I\}$ where $I$ is an index set. The map $f: \mathfrak{A}(R) \to X^I$, defined by $f([A]) = (v_i(A))_{i \in I}$, is a monomorphism.

**Proof.** The proof is straightforward and is omitted.
It follows from Proposition 1.4 that $\mathcal{A}(R)$ is a semigroup in which the cancellation law holds.

We now introduce a partial ordering for $\mathcal{A}(R)$.

**Definition 1.5.** For $[A], [B] \in \mathcal{A}(R)$, put $[A] \leq [B]$ iff $v(A) \leq v(B)$ for all $v \in F$.

**Proposition 1.6.** $\mathcal{A}(R)$ is partially ordered by $\leq$.

**Proof.** The proof is straightforward and is omitted.

As usual, if $[A], [B] \in \mathcal{A}(R)$ are such that $[A] \leq [B]$ and $[A] \neq [B]$, we write $[A] < [B]$. Since $[R] = 0 \in \mathcal{A}(R)$, $[A] \in \mathcal{A}(R)$ is such that $[A] \geq 0$ iff $A$ is an ideal of $R$. For if $A$ is an ideal of $R$, then $A \subseteq R$ so that $v(A) \geq v(R) = 0$, for all $v \in F$. On the other hand, if $[A] \geq 0$, then $v(A) \geq 0$ for all $v \in F$ so that $A \subseteq \bigcap_{v \in F} R_v = R$. Furthermore, if each $v \in F$ is discrete, then $[A] > 0$ iff $A \subseteq P(v)$ for some $v \in F$. For if $[A] > 0$, then, since each $v$ is discrete, $v(A) \geq 1 > 0$ for some $v \in F$. But then $A \subseteq P(v)$, and conversely. We can use these properties of $\leq$ to characterize the positive elements of $\mathcal{A}(R)$ when the elements of $F$ are discrete.

For $n$ a positive integer and $P$ a minimal prime of $R$, put $P^{(n)} = P^n \cap R$. We shall assume that $F$ satisfies (i'), (ii), (iii) in Propositions 1.7 and 1.8.

**Proposition 1.7.** If $P$ is a minimal prime of $R$ and $P = P(v)$ for some $v \in F$, then $P^{(n)} = \{x \in R \mid v(x) \geq n\}$ for every positive integer $n$.

**Proof.** We have $P^{(n)} = P^n \cap R = (PR_v)^n \cap R$. So if $x \in P^{(n)}$, then $x \in (PR_v)^n$ and so $v(x) \geq n$; i.e., $P^{(n)} = \{x \in R \mid v(x) \geq n\}$. On the other hand, if $x \in R$ is such that $v(x) \geq n$, then $x \in P$ and hence $x \in PR_v$. Since $v(x) \geq n$ we have $x \in (PR_v)^n$, and so $x \in P^{(n)}$; i.e., $\{x \mid x \in R, v(x) \geq n\} \subseteq P^{(n)}$.

As is well known, $P^{(n)}$ is a $P$-primary ideal of $R$.

**Proposition 1.8.** Let $A$ be an ideal of $R$ such that $[A] > 0$, and let

$$J = \{j \in I \mid v_j(A) > 0\}.$$ 

Then $A \subseteq \bigcap_{j \in J} P_j^{(n_j)}$, and $[A] = [\bigcap_{j \in J} P_j^{(n_j)}]$, where for each $j \in J$, $P_j = P(v_j)$ and $n_j = v_j(A)$.

**Proof.** If $x \in A$, $j \in J$, then $v_j(x) \geq v_j(A) = n_j$; i.e., $x \in P_j^{(n_j)}$ by Proposition 1.7. Thus for each $j \in J$, $A \subseteq P_j^{(n_j)}$; i.e., $A \subseteq \bigcap_{j \in J} P_j^{(n_j)}$; which proves the first assertion.

Now let $k \in J$. Since $A \subseteq \bigcap_{j \in J} P_j^{(n_j)}$, we have $v_k(A) \geq v_k(\bigcap_{j \in J} P_j^{(n_j)})$. On the other hand, let $x \in \bigcap_{j \in J} P_j^{(n_j)}$. Then $x \in P_k^{(n_k)}$, and $v_k(x) \geq n_k = v_k(A)$; i.e., $v_k(\bigcap_{j \in J} P_j^{(n_j)}) \geq v_k(A)$. So if $k \in J$, then $v_k(A) = v_k(\bigcap_{j \in J} P_j^{(n_j)})$. If $k \in I - J$, then $0 = v_k(A) \geq v_k(\bigcap_{j \in J} P_j^{(n_j)}) \geq 0$; i.e., $v_k(A) = 0 = v_k(\bigcap_{j \in J} P_j^{(n_j)})$ for all $k \in I - J$, Thus

$$v_i(A) = v_i(\bigcap_{j \in J} P_j^{(n_j)})$$

for all $i \in I$; i.e., $[A] = [\bigcap_{j \in J} P_j^{(n_j)}]$.

It follows that $\bigcap_{j \in J} P_j^{(n_j)}$ is the largest ideal $B$ of $R$ such that $[A] = [B]$. 

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We now drop the assumption that each $v \in F$ is discrete so that $F$ satisfies (i), (ii) and (iii). Property (i) says that $R_v$ is a rank one valuation ring and hence completely integrally closed for each $v \in F$. Property (ii) shows that $R$ is the intersection of completely integrally closed overlings and hence is completely integrally closed. So (i) and (ii) insure that $\mathcal{D}(R)$ is a group. We now study relations between the semigroup $\mathcal{A}(R)$ and the group $\mathcal{D}(R)$.

The next two propositions have been proved in [6] for the case when $F$ is the family of essential valuations of an AD-domain $R$.

**Proposition 1.9.** Let $A \in I(R)$. Then, considering $[A]$ and $\text{div}_R(A)$ as subsets of $I(R)$, $[A] \subseteq \text{div}_R(A)$.

**Proof.** Let $B \in [A]$. Then $v(B) = v(A)$ for all $v \in F$. If $A \subseteq Rx$, then $v(A) \geq v(Rx) = v(x)$, and so $v(B) \geq v(x)$ for all $v \in F$. If $b \in B$, then $v(b) - v(x) \geq 0$ for all $v \in F$; i.e., $v(b/x) \geq 0$ for all $v \in F$. Thus if $b \in B$ then $b/x \in \bigcap_{v \in F} R_v = R$, and $b \in Rx$; i.e., $B \subseteq Rx$. Similarly, if $B \subseteq Ry$ then $A \subseteq Ry$. In this case, $A = \bigcap_{A \subseteq Ry} Ry = \bigcap_{B \subseteq Rx} Rx = B$, and $\text{div}_R(A) = \text{div}_R(B)$. Hence $B \in \text{div}_R(A)$.

**Proposition 1.10.** The map $g: \mathcal{A}(R) \to \mathcal{D}(R)$ defined by $g([A]) = \text{div}_R(A)$ is an order preserving homomorphism of the partially ordered semigroup $\mathcal{A}(R)$ onto the lattice ordered group $\mathcal{D}(R)$.

**Proof.** Proposition 1.9 shows that $g$ is well defined and onto. It follows directly that $g$ is a homomorphism. To see that $g$ preserves order, suppose $[A], [B] \in \mathcal{A}(R)$ with $[A] \subseteq [B]$. If $A \subseteq Rx$, it follows as in the proof of 1.9 that $B \subseteq Rx$ so that $A \subseteq B$, and hence $\text{div}_R(A) \subseteq \text{div}_R(B)$.

Now let $T$ be a domain such that $R \subseteq T \subseteq K$ and such that there is a subfamily $G$ of $F$ such that $T = \bigcap_{w \in G} R_w$. It is easy to show that $G$ is a family of valuations for $T$ satisfying (i), (ii), (iii).

**Proposition 1.11.** The map $\sigma: \mathcal{A}(R) \to \mathcal{A}(T)$, defined by $\sigma([A]) = [AT]$, is an order preserving homomorphism of $\mathcal{A}(R)$ onto $\mathcal{A}(T)$.

**Proof.** Here, $\mathcal{A}(T)$ denotes the semigroup of fractionary ideal classes of $T$ formed with the family $G$.

It is clear that $\sigma$ is well defined. To see that $\sigma$ is onto, let $\mathcal{U}$ be any nonzero fractionary ideal of $T$. Then $\mathcal{U} = (1/d)\mathcal{B}$, where $\mathcal{B}$ is an ideal of $T$, $d \in R$, $d \neq 0$. Put $A = (1/d)B$, where $B = \mathcal{U} \cap R$. It can be shown that $v(B) = v(\mathcal{B})$ for all $v \in G$, and hence $\sigma([A]) = [\mathcal{U}]$. It is straightforward to show that $\sigma$ is a homomorphism which preserves order.

**Corollary 1.12.** If $T$ is as in 1.11, then $\mathcal{D}(T)$ is a homomorphic image of $\mathcal{A}(R)$.
Let $T$ be as in 1.11, and consider the following diagram:

Diagram 1.13.

\[
\begin{array}{ccc}
\mathcal{A}(R) & \xrightarrow{\sigma} & \mathcal{A}(T) \\
g_1 \downarrow & & \downarrow g_2 \\
\mathcal{D}(R) & \xrightarrow{\rho} & \mathcal{D}(T)
\end{array}
\]

Here $\sigma$ is the homomorphism of 1.11, $g_1$ and $g_2$ are the canonical homomorphisms of 1.10. In general, this diagram may not be completed commutatively by a homomorphism $\rho$. For let $R$ be an AD-domain which is not Dedekind, and let $F$ denote the family of essential valuations of $R$. By a result in [6], $R$ contains at least one proper prime $P$ which is not divisoriel. Then $P < \overline{P}$, and hence $\overline{P} = R$ since $P$ is maximal. Since $R$ is AD, there is $v \in F$ such that $P = P(v)$, for some $v \in F$. Take $T = R_P = R_v$ and assume that $\rho$ completes the following diagram commutatively:

Diagram 1.14.

\[
\begin{array}{ccc}
\mathcal{A}(R) & \xrightarrow{\sigma} & \mathcal{A}(R_P) \\
g_1 \downarrow & & \downarrow g_2 \\
\mathcal{D}(R) & \xrightarrow{\rho} & \mathcal{D}(R_P)
\end{array}
\]

Then we must have $\rho(g_1([P])) = g_2(\sigma([P]))$. However, $g_1([P]) = \text{div}_R(P) = 0$ (since $\overline{P} = R$) so that $\rho(g_1([P])) = 0$; and on the other hand $\sigma([P]) = [PR_P]$. But since $R_P$ is a Dedekind domain with unique maximal ideal $PR_P$, we have that $\text{div}_{R_P}(PR_P) > 0$; i.e., $g_2(\sigma([P])) = \text{div}_{R_P}(PR_P) > 0$. Thus $g_1 \neq g_2 \sigma$, contradicting our assumption on $\rho$. This proves the assertion that, in general, Diagram 1.13 may not be completed commutatively.

Equivalent conditions for an AD-domain $R$ to be Dedekind are given in terms of $\mathcal{A}(R)$ in [6]. If we are to extend these results we need to know something about the inverses of elements of $\mathcal{A}(R)$ whenever they exist.

**Proposition 1.15.** If $[A] \in \mathcal{A}(R)$ has an inverse then $-[A] = [R:A]$.

**Proof.** Suppose $[A] \in \mathcal{A}(R)$ has an inverse $[B]$. Since the canonical map $g$: $\mathcal{A}(R) \rightarrow \mathcal{D}(R)$ is a homomorphism, we must have that $g(-[A]) = -g([A]) = -\text{div}_R(A) = \text{div}_R(R:A)$. Thus $g([B]) = g(-[A]) = \text{div}_R(R:A)$. But by definition of $g$, $g([B]) = \text{div}_R(B)$ so that $\text{div}_R(B) = \text{div}_R(R:A)$. Since $R:A$ is divisoriel we have $B \subseteq B = R:A = R:A$. Then $AB \subseteq A(R:A) \subseteq R$ so that $0 = [AB] \geq [A(R:A)] \geq 0$. Thus $0 = [A] + [B] = [A] + [R:A]$; i.e., $-[A] = [B] = [R:A]$.

Now consider the following diagram.
Diagram 1.16.

\[ \mathcal{A}(R) \xrightarrow{f} X' \]
\[ \mathcal{D}(R) \]
\[ g \downarrow \]
\[ \lambda \]

Here \( g \) is the canonical homomorphism and \( f \) is the homomorphism of 1.4. \( g \) is surjective and \( f \) is injective.

**Proposition 1.17.** Diagram 1.16 may be completed commutatively by a homomorphism \( \lambda \) iff \( g \) is an isomorphism.

We can now prove the following theorem.

**Theorem 1.18.** Let \( R \) be an integral domain with quotient field \( K \), and let \( F \) be a family of valuations satisfying (i), (ii), (iii). The following statements are equivalent.

1. \( \mathcal{A}(R) \) is a group.
2. \( R: A = R: B \Rightarrow [A] = [B] \), for all \( A, B \in I(R) \).
3. \( v(A) = v(\widehat{A}) \) for all \( A \in I(R) \) and \( v \in F \).
4. The map \( g: \mathcal{A}(R) \to \mathcal{D}(R) \) is an isomorphism.

**Proof.** (1) \( \Rightarrow \) (2) Suppose \( \mathcal{A}(R) \) is a group. Let \( A, B \in I(R) \) be such that \( R: A = R: B \). By 1.15 we have \( -[A] = [R: A] = [R: B] = -[B] \) and hence \( [A] = [B] \).

(2) \( \Rightarrow \) (3) We have \( R: A = R: \widehat{A} \) for all \( A \in I(R) \). If (2) holds then \( [A] = [\widehat{A}] \) for all \( A \in I(R) \); i.e., \( v(A) = v(\widehat{A}) \) for all \( v \in F \), \( A \in I(R) \).

(3) \( \Rightarrow \) (4) Consider Diagram 1.16. If \( v(A) = v(\widehat{A}) \) for all \( v \in F \) and \( A \in I(R) \), we can define \( \lambda: \mathcal{D}(R) \to X' \) by \( \lambda(\text{div}_R(A)) = (v_i(A))_{i \in I} \). It follows that \( \lambda \) is a homomorphism and that \( \lambda \circ g = f \). By 1.17, \( g \) is an isomorphism.

(4) \( \Rightarrow \) (1) obvious.

We observe that the converse of statement (2) in 1.18 is always true in \( R \). For if \( [A] = [B] \), then \( A \in [B] \subseteq \text{div}_R(B) \), and \( B \in \text{div}_R(B) \) so that \( \text{div}_R(A) = \text{div}_R(B) \) and hence \( R: A = R: B \).

When the valuations in \( F \) are discrete we obtain a partial generalization of a result in [6] with the aid of the following lemma.

**Lemma 1.19.** Assume that each \( v \in F \) is discrete. Then for each \( v \in F \), if \( \text{div}_R(P(v)) \neq 0 \) then \( P(v) = (P(v))^- \).

**Proof.** We have \( P(v) \subseteq (P(v))^- \subseteq R \). If \( P(v) < (P(v))^- \), there is \( x \in (P(v))^- \), \( x \notin P(v) \). Then \( v(x) = 0 = v(P(v))^- \). Also, for \( w \in F \), \( w \neq v \), we have \( 0 = w(P(v)) \geq w(P(v))^- \geq 0 \). Thus \( [(P(v))^-] = 0 \). Since the canonical map \( g \) from \( \mathcal{A}(R) \) onto \( \mathcal{D}(R) \) is a homomorphism, we should have \( 0 = [(P(v))^-] = \text{div}_R(P(v))^- = \text{div}_R(P(v)) = 0 \). But \( \text{div}_R(P(v)) \neq 0 \) by assumption. Thus we must have \( (P(v))^-=P(v) \) if \( \text{div}_R(P(v)) \neq 0 \).

**Theorem 1.20.** Assume each \( v \in F \) is discrete. Then the canonical map \( g \) from \( \mathcal{A}(R) \) onto \( \mathcal{D}(R) \) is an isomorphism iff \( P(v) \) is divisoriel for each \( v \in F \).
Proof. (⇒) Suppose \( g \) is an isomorphism. If \( P = P(v) \) for some \( v \in F \), then \( [P] > 0 \) since \( v(P) = 1 \). If \( P \) is not divisoriel then \( g([P]) = \text{div}_R (P) = 0 \), by Lemma 1.19. But then \( g \) is not 1-1 and hence not an isomorphism. For \( [R] = 0 \neq [P] \).

(⇐) Suppose \( P(v) \) is divisoriel for each \( v \in F \). Then if \( A \in I(R) \) is such that \( \text{div}_R (A) = 0 \) we must have \( A \subseteq R \) (for this result see [1, bottom of p. 4]). Moreover, \( A \nsubseteq P(v) \) for any \( v \in F \). If \( A \subseteq P = P(v) \) for some \( v \in F \), then \( \text{div}_R (A) \geq \text{div}_R (P) > 0 \), a contradiction. Thus \( g([A]) = 0 \) iff \( [A] = 0 \). Now suppose \( [A], [B] \in \mathcal{A}(R) \) are such that \( g([A]) = g([B]) \). Then \( \text{div}_R (A) = \text{div}_R (B) \) so that \( \text{div}_R (A) - \text{div}_R (B) = 0 \). Since \( g([A : B]) = g([B : A]) = 0 \), we must have \( [A : B] = 0 = [B : A] \). Since each \( v \in F \) is discrete, for each \( v \in F \) there is \( x \in A : B \) such that \( v(x) = v(A : B) = 0 \). Now \( xB \subseteq A \) (by definition of \( A : B \)) so that \( v(x) + v(B) = v(xB) \geq v(A) \); i.e., \( v(B) \geq v(A) \). Thus \( v(B) \geq v(A) \) for all \( v \in F \). Similarly \( v(A) \geq v(B) \) for all \( v \in F \), and \( [A] = [B] \). This shows that \( g \) is 1-1 and hence an isomorphism.

When \( R \) is AD, the author has shown in [6] that \( P(v) \) is divisoriel for each \( v \in F \) iff \( R \) is Dedekind. To date, however, the author has been unable to prove the following conjecture: If \( R \) is AK and \( P(v) \) is divisoriel for each \( v \in F \), then \( R \) is a Krull domain.

When \( R \) is AK, we do have the following theorem.

**Theorem 1.21.** Let \( R \) be an AK-domain with family \( F \) of essential valuations and let \( \Delta \) denote the collection of maximal ideals of \( R \). Every minimal prime of \( R \) is divisoriel iff \( \tilde{\Delta} = \bigcap_{M \in \mathcal{M}} (AR_M)^\sim \) for every ideal \( A \) of \( R \).

Proof. Here \( \tilde{\Delta} = \bigcap_{M \in \mathcal{M}} AR_M \) and \( (AR_M)^\sim = \bigcap_{M \in \mathcal{M}} \mathcal{M}_M R_M \). For any maximal ideal \( M \) of \( R \), \( F_M \) denotes the family of essential valuations of the Krull domain \( R_M \). Recall that \( F_M \subseteq F \).

(⇒) Let \( A \) be an ideal of \( R \). If \( M \) is any maximal ideal of \( R \) then \( v(A) = v(AR_M) \) for all \( v \in F_M \). Since \( R_M \) is a Krull domain, \( v(AR_M) = v(AR_M)^\sim \) for all \( v \in F_M \) so that \( v(A) = V(AR_M)^\sim \) for all \( v \in F_M \).

Case 1. \( v(A) = 0 \) for all \( v \in F \).

Then \( P < A \) for every minimal prime \( P \) of \( R \). In this case \( \tilde{\Delta} = R = \bigcap_{M \in \mathcal{M}} R_M = \bigcap_{M \in \mathcal{M}} (AR_M)^\sim \).

Case 2. \( v(A) > 0 \) for some \( v \in F \).

For each maximal ideal \( M \) of \( R \), if there is \( v \in F_M \) such that \( 0 < v(A) = v(AR_M)^\sim \), then we can write

\[
(AR_M)^\sim = \bigcap_{v \in F_M : v(A) > 0} Q_v(n_v),
\]

where \( n_v = v_i(AR_M)^\sim \) and \( Q_v \) is the center of \( v_i \) on \( R_M \). Then for each \( i \) such that \( v_i(A) > 0 \) we have \( Q_v = P_i R_i \) where \( P_i = P(v_i) \) in \( R \). Thus \( (AR_M)^\sim = \bigcap_i (P_i R_i)^{n_v} = \bigcap_i (P_i R_i) \cap R_M = \bigcap_i (P_i R_i) \cap R_M \) where \( i \) runs over all indices such that \( v_i \in F_M \) and \( v_0(A) > 0 \), and \( n_i = v_i(A) \) for each such \( i \). It can then be shown that,

\[
C = \bigcap_{M \in \mathcal{M}} (AR_M)^\sim = \bigcap \{P_i^{n_v} : v_i \in F, v_i(A) > 0\}.
\]

Then \( [C] = [A] \). Since \( [A] = \text{div}_R (A) \), it follows that \( \tilde{\Delta} = C \).
Suppose $P$ is a minimal prime of $R$. If $M \in \Delta$, then either $P \subseteq M$ or $P \not\subseteq M$. If $P \not\subseteq M$ then $PR_M = M$. If $P \subseteq M$, then $PR_M$ is a minimal prime of the Krull domain $R_M$ and thus $(PR_M)^{-1} = PR_M$. Since $P$ is contained in some maximal ideal $M$ we have $P = \bigcap_{M \in \Delta} PR_M = P$.

We now drop the assumption that $R$ is an AK-domain and assume only that $F$ satisfies axioms (i), (ii), (iii) at the beginning of this section. The next lemma tells us more about the elements of $A(R)$ which have inverses and enables us to partially describe $D(R)$ in certain cases where $A(R)$ may not be a group.

**Lemma 1.22.** If $[A] \in A(R)$ is such that $[A]$ has an inverse then $[A] = [\bar{A}]$.

**Proof.** If $[A] \in A(R)$ has an inverse then $-[A] = [R : A]$ by Proposition 1.15. Now $A \subseteq \bar{A}$ and $R : A = R : \bar{A}$. Thus $A(R : A) \subseteq \bar{A}(R : A) = \bar{A}(R : \bar{A}) \subseteq R$. These containment relations yield the following: $0 = [A(R : A)] \geq [\bar{A}(R : \bar{A})] \geq 0$. Thus

$$0 = [A] + [R : A] = [\bar{A}] + [R : \bar{A}]$$

and $[A] = [\bar{A}]$.

**Corollary 1.23.** If $[A], [B]$ have inverses in $A(R)$, then $[A] + [B] = [\bar{A}B] = [AB]^{-1}$; i.e., $\sigma(AB) = \sigma(AB)^{-1}$ for all $v \in F$.

**Proof.** By 1.22 above, if $[A], [B]$ have inverses then $[A] = [\bar{A}]$ and $[B] = [\bar{B}]$. Moreover, $[A] + [B]$ has an inverse. Thus $[A] + [B] = [AB] = [AB]^{-1}$ by 1.22; i.e., $[\bar{A}] + [\bar{B}] = [\bar{A}\bar{B}] = [AB]^{-1}$.

Now consider the map $\rho : D(R) \to X^I$ defined by $\rho(\sigma(A)) = (v_1(\bar{A}))_{i \in I}$. $\rho$ is well defined, for if $\sigma(A) = \sigma(B)$ then $\bar{A} = \bar{B}$. Conversely, if $(v_1(\bar{A}))_{i \in I} = (v_1(\bar{B}))_{i \in I}$ then $[\bar{B}] = [\bar{A}]$ and so $\sigma(A) = \sigma(\bar{A}) = \sigma(\bar{B}) = \sigma(B)$ by the remark following the proof of 1.18. Thus $\rho$ is 1-1. We can now give a description of $D(R)$ when $R$ is fairly well behaved.

**Theorem 1.24.** The map $\rho : D(R) \to X^I$ defined by $\rho(\sigma(A)) = (v_1(\bar{A}))_{i \in I}$ is 1-1. Furthermore $\rho$ is a homomorphism iff $[\bar{A}] \in A(R)$ has an inverse for all $A \in I(R)$.

**Proof.** The first assertion is proved in the immediately preceding remarks. We now prove the second assertion.

$(\Rightarrow)$ Suppose $\rho$ is a homomorphism. Then since $D(R)$ is a group, for $\sigma(A) \in D(R)$, $-\sigma(A) = \sigma(A : A)$. Thus $\rho(\sigma(A) + \sigma(A : A)) = 0 = \rho(\sigma(A)) + \rho(\sigma(A : A)) = (v_1(\bar{A}))_{i \in I} + (v_1(\bar{A}))_{i \in I}$. It follows that $[\bar{A}]$ has an inverse in $A(R)$.

$(\Leftarrow)$ Suppose that $[\bar{A}]$ has an inverse for all $A \in I(R)$. By Corollary 1.23, we have that $[\bar{A}B] = [AB]^{-1}$ for all $A, B \in I(R)$. Thus for $A, B \in I(R)$ we have $\rho(\sigma(AB)) = (v_1(\bar{A}) v_1(\bar{B}))_{i \in I} = (v_1(\bar{A}))_{i \in I} + (v_1(\bar{B}))_{i \in I} = \rho(\sigma(A)) + \rho(\sigma(B))$ and $\rho$ is a homomorphism.

Now, let $R$ be an AK-domain. Then $R_P$ is a Krull domain for any prime ideal $P$ of $R$. However, these are not the only Krull domains $T$ such that $R \subseteq T \subseteq K$. For if $A = \{P_1, \ldots, P_n\}$ is any finite collection of prime ideals of $R$ then $T = \bigcap_{P_i \in A} R_{P_i}$ is also a Krull domain. Thus there is a large class of Krull domains $T$ such that
R \subseteq T \subseteq K$. When $R$ is an AK-domain in which every minimal prime is divisoriel we always have that $\mathcal{D}(T)$ is a homomorphic image of $\mathcal{D}(R)$, where $T$ is an AK-domain such that $R \subseteq T \subseteq K$. For, $\mathcal{A}(T)$ is a homomorphic image of the group $\mathcal{A}(R)$ and so is a group. Then $\mathcal{A}(R) \cong \mathcal{D}(R)$ and $\mathcal{A}(T) \cong \mathcal{D}(T)$. When $T$ is a Krull domain and $R$ is an AK-domain for which the map $\rho$ of Theorem 1.24 is a homomorphism we also get that $\mathcal{D}(T)$ is a homomorphic image of $\mathcal{D}(R)$ as follows.

**Proposition 1.25.** Let $R$ be an AK-domain and let $T$ be a Krull domain such that $R \subseteq T \subseteq K$. If $[\tilde{A}]$ has an inverse for every $[\tilde{A}] \in \mathcal{A}(R)$ then the map $\tau: \mathcal{D}(R) \to \mathcal{D}(T)$, defined by $\tau(\text{div}_R(A)) = \text{div}_T(\tilde{A}T)$, is a homomorphism of $\mathcal{D}(R)$ onto $\mathcal{D}(T)$.

**Proof.** $\tau$ is well defined, for if $\text{div}_R(A) = \text{div}_R(B)$ then $\tilde{A} = \tilde{B}$ so that $\tilde{A}T = \tilde{B}T$. Now consider the following diagram.

\[
\begin{array}{ccc}
\mathcal{D}(R) & \xrightarrow{\rho} & \mathbb{Z}^I \\
\tau \downarrow & & \downarrow \pi \\
\mathcal{D}(T) & \xrightarrow{\gamma} & \mathbb{Z}^J
\end{array}
\]

Here, $I$ is the index set for the family of essential valuations of $R$; $J$ is the index set for the family of essential valuations of $T$; $\pi$ is the projection of $\mathbb{Z}^I$ onto $\mathbb{Z}^J$; $\rho$ is the (injective) homomorphism of 1.24; $\gamma$ is the injection of 1.4. It is well known that $\gamma$ is also surjective; i.e., $\gamma$ is an isomorphism. Consider the map $\gamma^{-1} \circ \pi \circ \rho: \mathcal{D}(R) \to \mathcal{D}(T)$. We have, for $\text{div}_R(A) \in \mathcal{D}(R)$, $(\gamma^{-1} \circ \pi \circ \rho)(\text{div}_R(A)) = (\gamma^{-1} \circ \pi) \times (v_i(\tilde{A}))_{i \in I} = (v_i(\tilde{A}))_{i \in I}$. Since $T$ is a Krull domain we have that $v(B) = v(\tilde{B})$ for all fractionary ideals $B$ and all essential valuations $v$. Thus $(v_i(\tilde{A}))_{i \in J} = (v_i(\tilde{A}T))_{i \in J} = (v_i(\tilde{AT}))_{i \in J}$ so that $\gamma^{-1}(v_i(\tilde{A}))_{i \in J} = \text{div}_R(\tilde{A}T)$. Then $\gamma^{-1} \circ \pi \circ \rho = \tau$ and $\tau$ is a homomorphism since $\tau$ is a composition of homomorphisms. To see that $\tau$ is surjective it is sufficient to show that for every divisoriel fractionary ideal $\mathcal{W}$ of $T$ there is a divisoriel fractionary ideal $A$ of $R$ such that $\tau(\text{div}_R(A)) = \text{div}_T(\mathcal{W})$. So let $\mathcal{W}$ be a fractionary ideal of $T$. There are elements $x, y \in K$ such that $\mathcal{W} = Tx \cap Ty$ [1, p. 13]. Let $A = Rx \cap Ry$. Then $A$ is divisoriel and $v_j(A) = v_j(\mathcal{W})$ for all $j \in J$ and $\tau(\text{div}_R(A)) = \text{div}_T(\mathcal{W})$.

2. Let $R$ be an integral domain with quotient field $K$. Suppose that $F$ is a family of valuations on $K$ satisfying the following:

1. $R = \bigcap_{v \in F} R_v$,
2. $R_v = R_{\text{Pic}(v)}$, for each $v \in F$.

Following Gilmer in [3], we make the following definition.

**Definition 2.1.** We say that $R$ satisfies property $(\star)$ with respect to $F$ iff for distinct subsets $F_1, F_2$ of $F$ we have that $\bigcap_{v \in F_1} R_v \neq \bigcap_{v \in F_2} R_v$.

When $R$ is a Prüfer domain and $F$ is the family of valuations induced by the collection of maximal ideals, then property $(\star)$ is the same as property $(\star \star)$ in [2].
For $v \in F$, we let $F_v = F - \{v\}$.

**Proposition 2.2.** $R$ has property $(\ast)$ with respect to $F$ iff for each $v \in F$, $\bigcap_{u \in F_v} R_u \subseteq R_v$.

**Proof.** The proof is substantially the same as that of Lemma 1 in [2] and is omitted.

**Corollary 2.3.** If $R$ satisfies $(\ast)$ with respect to $F$ and if $G$ is any nonempty subset of $F$, then $T = \bigcap_{u \in G} R_u$ satisfies $(\ast)$ with respect to $G$.

We note that if $R$ satisfies $(\ast)$ with respect to $F$ then $P(v) \nsubseteq P(w)$ for $v \neq w$. For if $P(v) \subseteq P(w)$ for some $w \neq v$, then $R_{P(w)} \subseteq R_{P(v)}$; i.e., $R_w \subseteq R_v$. Then we have the following: $(\bigcap_{u \in F - \{v, w\}} R_u) \cap (R_v \cap R_w) = (\bigcap_{u \in F - \{v, w\}} R_u) \cap R_w = \bigcap_{u \in F_v} R_u$, and $F \neq F_v$, a contradiction.

**Proposition 2.4.** If $F$ is of finite character and is such that $P(u) \nsubseteq P(v)$ if $u \neq v$, then $R$ satisfies $(\ast)$ with respect to $F$.

**Proof.** Let $v \in F$ and let $x \in R$, $x \neq 0$, be such that $v(x) > 0$. Let $v_1, \ldots, v_n$ be the distinct (from $v$ and each other) valuations such that $v_i(x) \neq 0$, $i = 1, \ldots, n$. There exists $y \in (\bigcap_{i=1}^n P(v_i)) - P(v)$. For if $\bigcap_{i=1}^n P(v_i) \subseteq P(v)$, then $\prod_{i=1}^n P(v_i) \subseteq \bigcap_{i=1}^n P(v_i) \subseteq P(v)$ and so $P(v_i) \subseteq P(v)$ for some $j$, $1 \leq j \leq n$, contradicting our hypothesis. Choose $n$ large enough so that $w(y^n/x) \geq 0$ for $w \in F_v$. This is possible since $F$ is of finite character and $w(y^n/x) \geq 0$ for all $w \in F_v$. Then $w(y^n/x) \geq 0$ for all $w \in F_v$ and $v(y^n/x) = -v(x) < 0$. Thus $y^n/x \in (\bigcap_{u \in F_v} R_u) - R_v$; i.e., $\bigcap_{u \in F_v} R_u \not\subseteq R_v$. So $R$ satisfies $(\ast)$ with respect to $F$ by 2.2.

Let $R$ be an integral domain with family $F$ of valuations satisfying (1) and (2) listed at the beginning of this section. $R$ is called a generalized Krull domain if $F$ satisfies the following two additional properties (see [5]).

(3) Each $v \in F$ has rank one.

(4) $F$ is of finite character.

**Corollary 2.5.** If $R$ is a Krull domain, or a generalized Krull domain with family $F$ of valuations, then $R$ satisfies $(\ast)$ with respect to $F$.

**Proof.** In this case $F$ is a family of rank one valuations of finite character, so that if $u, v \in F$, $u \neq v$, then $P(v) \nsubseteq P(u)$.

**Proposition 2.6.** Let $R$ be an AD-domain. The following conditions on $R$ are equivalent.

(1) $R$ satisfies $(\ast)$ with respect to $F$, the family of essential valuations of $R$.

(2) $R$ is Dedekind.

(3) Every minimal prime of $R$ is divisoriel.

**Proof.** (1) $\Rightarrow$ (2) is Theorem 3 of [3].

(2) $\Rightarrow$ (3) is found in [6].

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Thus we see that in the case of almost-Dedekind domains, the divisoriel property of the minimal prime ideals completely determines whether or not $R$ satisfies property $(\ast)$. We shall see that the divisoriel property of the minimal primes is always sufficient for $R$ to satisfy $(\ast)$.

**Proposition 2.7.** Let $R$ be an integral domain with family $F$ of valuations such that

(i) Each $v \in F$ has rank one.

(ii) $R = \bigcap_{v \in F} R_v$.

(iii) $R_v = R_{P(v)}$ for each $v \in F$.

If $P(v)$ is divisoriel for each $v \in F$, then $R$ satisfies $(\ast)$ with respect to $F$.

**Proof.** We note that since $R$ is the intersection of rank one valuation rings, $R$ is completely integrally closed and hence $\mathcal{D}(R)$ is a group. If each $P(v)$ is divisoriel, then each $v \in F$ is discrete. For if $P = P(v)$ is divisoriel we must have $P^2 < P$. For if $P^2 = P$, then $\text{div}_R(P^2) = \text{div}_R(P)$; i.e., $2 \text{ div}(P) = \text{ div}(P)$. Thus $\text{ div}(P) = 0$ and $\bar{P} = R \neq P$, contradicting $\bar{P} = P$.

Since $P^2 < P$, we have $P^2 R_\varphi < P R_\varphi$ and so $R_\varphi$ is a discrete valuation ring. We now show that $\{P(v) \mid v \in F\}$ is the set of all minimal divisoriel primes of $R$. Clearly, $\{P(v) \mid v \in F\}$ is contained in the set of all divisoriel minimal primes. Now let $P$ be a minimal, divisoriel prime of $R$. If $P \neq P(v)$ for any $v \in F$, then $P \notin P(v)$ for any $v \in F$ and so $\nu(P) = 0$ for all $v \in F$; i.e., $[P] = 0$. But then we would have $g([P]) = 0$; i.e., $\text{ div}(P) = 0$; i.e., $\bar{P} = R$, contradicting $\bar{P} = P < R$. So we must have that

$$\{P(v) \mid v \in F\}$$

is the set of all divisoriel minimal primes of $R$. Now let $G$ be any subset of $F$ such that $R = \bigcap_{u \in G} R_u$. $P(u)$ is divisoriel for each $u \in G$ since $G \subseteq F$. By what we have just shown, $\{P(u) \mid u \in G\}$ is the collection of all minimal divisoriel primes of $R$; i.e., $G = F$. Thus for any $v \in F$, $\bigcap_{u \in F} R_u \neq R_v$ and so $R$ satisfies $(\ast)$ with respect to $F$.

The first part of the proof of Proposition 2.7 shows that if $P$ is the center of a rank one valuation $v$, then $v$ is discrete if $P$ is divisoriel. This enables us to characterize Krull domains in the class of generalized Krull domains as follows.

**Corollary 2.8.** Let $R$ be a generalized Krull domain with family $F$ of valuations. $R$ is a Krull domain iff $P(v)$ is divisoriel for each $v \in F$.

Let $R$ be an integral domain with quotient field $K$ and let $F$ be a family of valuations on $K$ satisfying conditions (1) and (2) stated at the beginning of this section. Let $x$ be an indeterminate and let $F'$ denote the family of valuations on $K(x)$ which are canonical extensions of elements of $F$. Let $G$ denote the family of $p(x)$-adic valuations on $K(x)$, where $p(x)$ is a nonconstant irreducible polynomial in $K[x]$. Then $F' \cup G$ is a family of valuations on $K(x)$ satisfying (1) and (2) with $R[x]$ in place of $R$. 

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Proposition 2.9. If \( R \) satisfies (\(*\)) with respect to \( F \), then \( R[x] \) satisfies (\(*\)) with respect to \( F' \cup G \).

Proof. Let \( w \in F' \cup G \). If \( w \in G \), then \( w \) is a \( p(x) \)-adic valuation for some non-constant irreducible polynomial \( p(x) \in K[x] \). Without loss of generality we may assume that \( p(x) \in R[x] \). Suppose \( p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \), \( a_i \in R \).

Let \( b = \prod_{a_k \neq 0} a_k \). Then \( b \neq 0 \) since \( a_n \neq 0 \), and \( v(b) = \sum_{a_k \neq 0} v(a_k) \geq \min_{0 \leq j \leq n} v(a_j) \) for all \( v \in F \) since \( b \in R \) and \( a_k \in R \) for all \( k = 0, 1, \ldots, n \), and every \( v \in F \) is non-negative on \( R \). Then for \( v' \in F' \), \( v'(b/p(x)) = v'(b) - v'(p(x)) = v(b) - \min_{0 \leq j \leq n} v(a_j) \geq 0 \). If \( u \in G \) and \( u \neq w \), then \( u \) is a \( q(x) \)-adic valuation for some nonconstant irreducible polynomial \( q(x) \) such that \( q(x) \mid p(x) \). Then \( u(b/p(x)) = 0 \). Thus \( b/p(x) \in \left( \bigcap_{u \in F' \cup G} (R[x])_u \right) \setminus (R[x])_w \) since \( w(b/p(x)) = -1 < 0 \). Thus if \( w \in G \cap \left( \bigcap_{u \in F' \cup G} (R[x])_u \right) \setminus (R[x])_w \), then \( w(v(b/p(x))) = -1 < 0 \). Thus if \( w \in F' \), then \( w = v' \) for some \( v \in F \). Since \( \bigcap_{u \in F' \cup G} (R[x])_u \neq \emptyset \), there is a \( a \in (\bigcap_{u \in F'} R_u) - R_v \subseteq (\bigcap_{u \in F'} R_u) \cap \{ x \in K \} \) such that \( v'(a) = v(a) < 0 \). Thus for every \( w \in F' \cup G \) we have \( \bigcap_{u \in F' \cup G} (R[x])_u \neq (R[x])_w \) and thus \( R \) satisfies (\(*\)) with respect to \( F' \cup G \) by 2.2.

3. In [6] it was shown that if \( R \) is an almost-Dedekind domain with family \( F \) of essential valuations, then \( R \) is Dedekind if every minimal prime of \( R \) is divisoriel. Thus in an AD-domain \( R \), every minimal prime is divisoriel iff \( F \) is of finite character. In §1 it was conjectured that if \( R \) is an AK-domain with family \( F \) of essential valuations, then \( R \) is Krull if \( P(v) \) is divisoriel for each \( v \in F \); i.e., \( F \) is of finite character if \( P(v) \) is divisoriel for each \( v \in F \). In this section we give an example to show that this conjecture is false if the AK requirement is dropped. We also give an example of an AK-domain which is neither a Krull domain nor an AD-domain.

Let \( R \) denote the set of entire functions, \( C \) denote the set of complex numbers, \( Z \) denote the additive group of integers. It is well known that \( R \) is an integral domain under the usual pointwise definitions of addition and multiplication. For \( a \in C \) we define \( v_a : R - \{ 0 \} \rightarrow Z \) by \( v_a(f(z)) = n \) if \( a \) is a zero of \( f(z) \) of order \( n \). If \( a \) is not a zero of \( f(z) \) then \( v_a(f(z)) = 0 \). If \( f(z) \equiv 0 \) we put \( v_a(f(z)) = +\infty \) for each \( z \in C \). It is easy to show that each \( v_a \) can be extended to a valuation on the quotient field of \( R \). We let \( F \) denote this family of valuations. \( F \) has the following properties:

(i) Each \( v \in F \) has rank one and is discrete; (ii) \( R = \bigcap \{ R_v \mid v \in F \} \); (iii) \( R_v = R_{P(v)} \) for each \( v \in F \); (iv) For \( a \in C \), \( P(v_a) = \langle z - a \rangle R \), and hence is divisoriel; (v) \( F \) is not of finite character. Furthermore, \( P(v) \) is maximal for each \( v \in F \). However, these are not all the maximal ideals of \( R \). For let \( \{ z_n \}_{n=1}^\infty \) be a sequence of complex numbers such that \( \lim z_n = \infty \). For each positive integer \( m \), let \( f_m(z) \) be an entire function whose zeros are exactly \( \{ z_m, z_{m+1}, \ldots \} \). The ideal generated by \( \{ f_1(z), f_2(z), \ldots \} \) is proper and is contained in a maximal ideal \( M \). However, \( R_M \) is not a Krull domain. It follows that \( R \) is not AK.

It was shown in [7] that if \( R \) is AK and \( X_1, \ldots, X_n \) are indeterminates, then \( R[X_1, \ldots, X_n] \) is AK. Let \( R \) be an AD-domain which is not a Dedekind domain.
Such a domain is given in example 2 of [2]. Then $R[X_1, \ldots, X_n]$ is an AK-domain which is neither a Krull domain nor an AD-domain. We observe that example 1 of [2] is a generalized Krull domain which is not a Krull domain.

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**References**