FAMILIES OF VALUATIONS AND SEMIGROUPS
OF FRACTIONARY IDEAL CLASSES

BY
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Introduction. Let $R$ be an integral domain with quotient field $K$. For any
valuation $v$ on $K$ which is nonnegative on $R$, we let $P(v) = \{x \in R \mid v(x) > 0\}$. $P(v)$ is
a prime ideal of $R$ and is called the center of $v$ on $R$. In this paper we are concerned
mainly with integral domains $R$ which satisfy the following: There exists a family
$F$ of valuations on $K$ such that
(i) Each $v \in F$ has rank one.
(ii) $R = \cap_{v \in F} R_v$.
(iii) $R_v = R_{P(v)}$, for each $v \in F$.

A family $F$ of valuations on $K$ is said to be of finite character if for $x \in K$, $x \neq 0$,
there are only a finite number of $v \in F$ such that $v(x) \neq 0$. $R$ is called a Krull domain
if there is a family $F$ of finite character satisfying (i), (ii), (iii), with the additional
requirement that each $v \in F$ be discrete. $R$ is called an almost-Krull (AK) domain
[7] if $R_P$ is a Krull domain for every proper nonzero prime $P$ of $R$. It follows that
$R$ is almost Dedekind (AD) iff $R$ is an AK-domain in which proper prime ideals
are maximal [7].

Using the family $F$ of valuations we construct a partially ordered semigroup
$\mathcal{A}(R)$ of fractionary ideal classes in §1 and study the relation between $\mathcal{A}(R)$ and
$\mathcal{D}(R)$, the divisor group of $R$ (see [1]). Necessary and sufficient conditions for $\mathcal{A}(R)$
and $\mathcal{D}(R)$ to be isomorphic are determined. In §2, condition $(S)$ of [3] is studied.
§3 consists of an example.

The notation concerning $\mathcal{D}(R)$ is that of [1]. Otherwise, the notation of [8] is
used. Prime ideals are always nonzero and not all of $R$.

1. In order to make this paper as self contained as possible we first list the
necessary background results from [1]. $R$ will denote a commutative integral
domain with identity and quotient field $K$. $I(R)$ will denote the collection of non-
zero fractionary ideals of $R$. A fractionary ideal of the form $Rx$, $x \in K$, $x \neq 0$, is
called a principal fractionary ideal.

A relation $<$ is defined on $I(R)$ as follows: $A < B$ iff every principal fractionary
ideal of $R$ which contains $A$ also contains $B$. The relation $<$ is a preorder on $I(R)$;
i.e., $<$ is a symmetric, transitive relation. If we define $\equiv$ on $I(R)$ by $A \equiv B$ iff
$A < B$ and $B < A$, then $\equiv$ is an equivalence relation on $I(R)$. For $A \in I(R)$, $\text{div}_B(A)$

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denotes the equivalence class of \( A \) with respect to \( \equiv \) and is called the divisor of \( A \); \( \mathcal{D}(R) \) denotes the set of all such equivalence classes.

For \( A \in I(R) \), we put \( \overline{A} = \bigcap_{x \in Rx} Rx \). A fractionary ideal \( B \) of \( R \) is said to be divisoriel if \( B = \overline{B} \). It follows that for \( A \in I(R) \), \( \text{div}_R(A) = \text{div}_R(\overline{A}) \) and that \( \overline{A} \) is the unique divisoriel fractionary ideal belonging to \( \text{div}_R(A) \). It also follows from the definition that \( (\overline{AB})_\sim = (AB)_\sim \) for \( A, B \in I(R) \) so that \( \mathcal{D}(R) \) together with the operation \( + \), defined by \( \text{div}_R(A) + \text{div}_R(B) = \text{div}_R(AB) \), is a commutative semigroup with identity \( 0 = \text{div}_R(R) \). If we define \( \leq \) on \( \mathcal{D}(R) \) by \( \text{div}_R(A) \leq \text{div}_R(B) \) iff \( A < B \) then \( \mathcal{D}(R) \) is a lattice ordered semigroup with respect to the partial ordering \( \leq \). Furthermore, \( \mathcal{D}(R) \) is a group iff \( R \) is completely integrally closed [1, p. 5, Theorem 1].

Let \( F \) be a family of valuations on \( K \) with the following properties:

(i) Each \( v \in F \) has rank one.

(ii) \( R = \bigcap_{v \in F} R_v \).

(iii) For each \( v \in F \), \( R_v = R_{P(v)} \), where \( P(v) \) denotes the center of \( v \) on \( R \).

Occasionally in place of (i) we shall substitute

(i') Each \( v \in F \) has rank one and is discrete.

**Definition 1.1.** For \( v \in F \), \( A \in I(R) \), put \( v(A) = \inf \{ v(a) \mid a \in A \} \).

**Lemma 1.2.** If \( A, B \in I(R), v \in F, \) then \( v(AB) = v(A) + v(B) \).

**Proof.** See [4, p. 712], Theorem 1, part (2).

Now, for \( A, B \in I(R) \), define \( A \sim B \) iff \( v(A) = v(B) \) for all \( v \in F \). Then \( \sim \) is an equivalence relation on \( I(R) \). For \( A \in I(R) \) we let \( [A] \) denote the equivalence class of \( A \) with respect to \( \sim \), and we let \( \mathcal{A}(R) \) denote the set of all such equivalence classes.

Define \( + \) on \( \mathcal{A}(R) \) by \( [A] + [B] = [AB] \). Then \( + \) is well defined. Since multiplication of fractionary ideals is commutative and associative, \( \mathcal{A}(R) \) together with \( + \) is a commutative semigroup with identity \( 0 = [R] \).

**Lemma 1.3.** If \( A = Rx \) is a principal fractionary ideal, then \( v(A) = v(x) \) for all \( v \in F \).

**Proof.** \( v(A) = v(Rx) = \inf_{x \in Rx} v(rx) = \inf_{r \in R} v(r) + v(x) = v(1) + v(x) = v(x) \).

If \( G \) is a group and \( I \) is any nonempty index set, we let \( G^I \) denote the direct product of \( I \) copies of \( G \) and we let \( G^{[I]} \) denote the direct sum of \( I \) copies of \( G \). We shall assume that the value group of each \( v \in F \) is a subgroup of the additive group of real numbers. When \( v \in F \) is discrete we assume, without loss of generality, that the value group of \( v \) is the additive group of integers. \( X \) denotes the real numbers and \( Z \) denotes the integers.

**Proposition 1.4.** Let \( F = \{ v_i \mid i \in I \} \) where \( I \) is an index set. The map \( f_\sim : \mathcal{A}(R) \to X^I \), defined by \( f([A]) = (v_i(A))_{i \in I} \), is a monomorphism.

**Proof.** The proof is straightforward and is omitted.
It follows from Proposition 1.4 that $\mathfrak{A}(R)$ is a semigroup in which the cancellation law holds.

We now introduce a partial ordering for $\mathfrak{A}(R)$.

**Definition 1.5.** For $[A], [B] \in \mathfrak{A}(R)$, put $[A] \leq [B]$ iff $v(A) \leq v(B)$ for all $v \in F$.

**Proposition 1.6.** $\mathfrak{A}(R)$ is partially ordered by $\leq$.

**Proof.** The proof is straightforward and is omitted.

As usual, if $[A], [B] \in \mathfrak{A}(R)$ are such that $[A] \leq [B]$ and $[A] \neq [B]$, we write $[A] < [B]$. Since $[R] = 0 \in \mathfrak{A}(R)$, $[A] \in \mathfrak{A}(R)$ is such that $[A] \geq 0$ iff $A$ is an ideal of $R$. For if $A$ is an ideal of $R$; then $A \subseteq R$ so that $v(A) \geq v(R) = 0$, for all $v \in F$. On the other hand, if $[A] \geq 0$, then $v(A) \geq 0$ for all $v \in F$ so that $A \subseteq \bigcap_{v \in F} R_v = R$. Furthermore, if each $v \in F$ is discrete, then $[A] > 0$ iff $A \subseteq P(v)$ for some $v \in F$. For if $[A] > 0$, then, since each $v$ is discrete, $v(A) \geq 1 > 0$ for some $v \in F$. But then $A \subseteq P(v)$, and conversely. We can use these properties of $\leq$ to characterize the positive elements of $\mathfrak{A}(R)$ when the elements of $F$ are discrete.

For $n$ a positive integer and $P$ a minimal prime of $R$, put $P^{(n)} = P^n R_P \cap R$. We shall assume that $F$ satisfies (i'), (ii), (iii) in Propositions 1.7 and 1.8.

**Proposition 1.7.** If $P$ is a minimal prime of $R$ and $P = P(v)$ for some $v \in F$, then $P^{(n)} = \{ x \in R \mid v(x) \geq n \}$ for every positive integer $n$.

**Proof.** We have $P^{(n)} = P^n R_P \cap R = (P R_P)^n \cap R$. So if $x \in P^{(n)}$, then $x \in (P R_P)^n$ and so $v(x) \geq n$; i.e., $P^{(n)} = \{ x \in R \mid v(x) \geq n \}$. On the other hand, if $x \in R$ is such that $v(x) \geq n$, then $x \in P$ and hence $x \in P R_P$. Since $v(x) \geq n$ we have $x \in (P R_P)^n$, and so $x \in P^{(n)}$; i.e., $\{ x \mid x \in R, v(x) \geq n \} = P^{(n)}$.

As is well known, $P^{(n)}$ is a $P$-primary ideal of $R$.

**Proposition 1.8.** Let $A$ be an ideal of $R$ such that $[A] > 0$, and let

$$J = \{ j \in I \mid v_j(A) > 0 \}.$$ 

Then $A \subseteq \bigcap_{j \in J} P_j^{(n_j)}$, and $[A] = [\bigcap_{j \in J} P_j^{(n_j)}]$ , where for each $j \in J$, $P_j = P(v_j)$ and $n_j = v_j(A)$.

**Proof.** If $x \in A$, $j \in J$, then $v_j(x) \geq v_j(A) = n_j$; i.e., $x \in P_j^{(n_j)}$ by Proposition 1.7. Thus for each $j \in J$, $A \subseteq P_j^{(n_j)}$; i.e., $A \subseteq \bigcap_{j \in J} P_j^{(n_j)}$; which proves the first assertion.

Now let $k \in J$. Since $A \subseteq \bigcap_{j \in J} P_j^{(n_j)}$, we have $v_k(A) \geq v_k(\bigcap_{j \in J} P_j^{(n_j)})$. On the other hand, let $x \in \bigcap_{j \in J} P_j^{(n_j)}$. Then $x \in P_k^{(n_k)}$, and $v_k(x) \geq n_k = v_k(A)$; i.e., $v_k(\bigcap_{j \in J} P_j^{(n_j)}) \geq v_k(A)$. So if $k \in J$, then $v_k(A) = v_k(\bigcap_{j \in J} P_j^{(n_j)})$. If $k \in I - J$, then $0 = v_k(A) \geq v_k(\bigcap_{j \in J} P_j^{(n_j)}) \geq 0$; i.e., $v_k(A) = 0 = v_k(\bigcap_{j \in J} P_j^{(n_j)})$ for all $k \in I - J$. Thus

$$v_i(A) = v_i(\bigcap_{j \in J} P_j^{(n_j)})$$

for all $i \in I$; i.e., $[A] = [\bigcap_{j \in J} P_j^{(n_j)}]$.

It follows that $\bigcap_{j \in J} P_j^{(n_j)}$ is the largest ideal $B$ of $R$ such that $[A] = [B]$. 
We now drop the assumption that each \( v \in F \) is discrete so that \( F \) satisfies (i), (ii) and (iii). Property (i) says that \( R_v \) is a rank one valuation ring and hence completely integrally closed for each \( v \in F \). Property (ii) shows that \( R \) is the intersection of completely integrally closed overrings and hence is completely integrally closed. So (i) and (ii) insure that \( \mathcal{D}(R) \) is a group. We now study relations between the semigroup \( \mathcal{A}(R) \) and the group \( \mathcal{D}(R) \).

The next two propositions have been proved in [6] for the case when \( F \) is the family of essential valuations of an AD-domain \( R \).

**Proposition 1.9.** Let \( A \in I(R) \). Then, considering \([A]\) and \( \operatorname{div}_R(A) \) as subsets of \( I(R) \), \([A] \subseteq \operatorname{div}_R(A)\).

**Proof.** Let \( B \in [A] \). Then \( v(B) = v(A) \) for all \( v \in F \). If \( A \subseteq Rx \), then \( v(A) \geq v(Rx) = v(x) \), and so \( v(B) \geq v(x) \) for all \( v \in F \). If \( b \in B \), then \( v(b) - v(x) \geq 0 \) for all \( v \in F \); i.e., \( v(b/x) \geq 0 \) for all \( v \in F \). Thus if \( b \in B \) then \( b/x \in \bigcap_{v \in F} R_v = R \), and \( b \in Rx \); i.e., \( B \subseteq Rx \). Similarly, if \( B \subseteq Ry \) then \( A \subseteq Ry \). In this case, \( \mathcal{A} = \bigcap_{A \subseteq Rx} Ry = \bigcap_{B \subseteq Rx} Rx = B \), and \( \operatorname{div}_R(A) = \operatorname{div}_R(B) \). Hence \( B \in \operatorname{div}_R(A) \).

**Proposition 1.10.** The map \( g : \mathcal{A}(R) \to \mathcal{D}(R) \) defined by \( g([A]) = \operatorname{div}_R(A) \) is an order preserving homomorphism of the partially ordered semigroup \( \mathcal{A}(R) \) onto the lattice ordered group \( \mathcal{D}(R) \).

**Proof.** Proposition 1.9 shows that \( g \) is well defined and onto. It follows directly that \( g \) is a homomorphism. To see that \( g \) preserves order, suppose \([A], [B] \in \mathcal{A}(R)\) with \([A] \leq [B]\). If \( A \subseteq Rx \), it follows as in the proof of 1.9 that \( B \subseteq Rx \) so that \( A \subseteq B \), and hence \( \operatorname{div}_R(A) \leq \operatorname{div}_R(B) \).

Now let \( T \) be a domain such that \( R \subseteq T \subseteq K \) and such that there is a subfamily \( G \) of \( F \) such that \( T = \bigcap_{v \in G} R_v \). It is easy to show that \( G \) is a family of valuations for \( T \) satisfying (i), (ii), (iii).

**Proposition 1.11.** The map \( \sigma : \mathcal{A}(R) \to \mathcal{A}(T) \), defined by \( \sigma([A]) = [AT] \), is an order preserving homomorphism of \( \mathcal{A}(R) \) onto \( \mathcal{A}(T) \).

**Proof.** Here, \( \mathcal{A}(T) \) denotes the semigroup of fractionary ideal classes of \( T \) formed with the family \( G \).

It is clear that \( \sigma \) is well defined. To see that \( \sigma \) is onto, let \( \mathcal{V} \) be any nonzero fractionary ideal of \( T \). Then \( \mathcal{V} = (1/d) \mathcal{B} \), where \( \mathcal{B} \) is an ideal of \( T \), \( d \in R \), \( d \neq 0 \). Put \( A = (1/d)B \), where \( B = \mathcal{V} \cap R \). It can be shown that \( v(B) = v(\mathcal{B}) \) for all \( v \in G \), and hence \( \sigma([A]) = [\mathcal{V}] \). It is straightforward to show that \( \sigma \) is a homomorphism which preserves order.

**Corollary 1.12.** If \( T \) is as in 1.11, then \( \mathcal{D}(T) \) is a homomorphic image of \( \mathcal{A}(R) \).
Let $T$ be as in 1.11, and consider the following diagram:

Diagram 1.13.

\[
\begin{array}{ccc}
\mathcal{A}(R) & \xrightarrow{\sigma} & \mathcal{A}(T) \\
\downarrow g_1 & & \downarrow g_2 \\
\mathcal{D}(R) & \xrightarrow{\rho} & \mathcal{D}(T)
\end{array}
\]

Here $\sigma$ is the homomorphism of 1.11, $g_1$ and $g_2$ are the canonical homomorphisms of 1.10. In general, this diagram may not be completed commutatively by a homomorphism $\rho$. For let $R$ be an AD-domain which is not Dedekind, and let $F$ denote the family of essential valuations of $R$. By a result in [6], $R$ contains at least one proper prime $P$ which is not divisoriel. Then $P < \bar{P}$, and hence $\bar{P} = R$ since $P$ is maximal. Since $R$ is AD, there is $v \in F$ such that $P = P(v)$, for some $v \in F$. Take $T = R_P = R_v$ and assume that $\rho$ completes the following diagram commutatively:

Diagram 1.14.

\[
\begin{array}{ccc}
\mathcal{A}(R) & \xrightarrow{\sigma} & \mathcal{A}(R_P) \\
\downarrow g_1 & & \downarrow g_2 \\
\mathcal{D}(R) & \xrightarrow{\rho} & \mathcal{D}(R_P)
\end{array}
\]

Then we must have $\rho(g_1([P])) = g_2(\sigma([P]))$. However, $g_1([P]) = \text{div}_R(P) = 0$ (since $\bar{P} = R$) so that $\rho(g_1([P])) = 0$; and on the other hand $\sigma([P]) = [PR_P]$. But since $R_P$ is a Dedekind domain with unique maximal ideal $PR_P$, we have that $\text{div}_{R_P}(PR_P) > 0$; i.e., $g_2(\sigma([P])) = \text{div}_{R_P}(PR_P) > 0$. Thus $g_2 \neq g_2 \sigma$, contradicting our assumption on $\rho$. This proves the assertion that, in general, Diagram 1.13 may not be completed commutatively.

Equivalent conditions for an AD-domain $R$ to be Dedekind are given in terms of $\mathcal{A}(R)$ in [6]. If we are to extend these results we need to know something about the inverses of elements of $\mathcal{A}(R)$ whenever they exist.

**Proposition 1.15.** If $[A] \in \mathcal{A}(R)$ has an inverse then $-[A] = [R:A]$.

**Proof.** Suppose $[A] \in \mathcal{A}(R)$ has an inverse $[B]$. Since the canonical map $g:\mathcal{A}(R) \to \mathcal{D}(R)$ is a homomorphism, we must have that $g(-[A]) = -g([A]) = -\text{div}_R(A) = \text{div}_R(R:A)$. Thus $g([B]) = g(-[A]) = \text{div}_R(R:A)$. But by definition of $g$, $g([B]) = \text{div}_R(B)$ so that $\text{div}_R(B) = \text{div}_R(R:A)$. Since $R:A$ is divisoriel we have $B = B = R:A = R:A$. Then $AB = A(R:A) \subseteq R$ so that $0 = [AB] \geq [A(R:A)] \geq 0$. Thus $0 = [A] + [B] = [A] + [R:A]$; i.e., $-[A] = [B] = [R:A]$.

Now consider the following diagram.

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Here \( g \) is the canonical homomorphism and \( f \) is the homomorphism of 1.4. \( g \) is surjective and \( f \) is injective.

**Proposition 1.17.** Diagram 1.16 may be completed commutatively by a homomorphism \( \lambda \) iff \( g \) is an isomorphism.

We can now prove the following theorem.

**Theorem 1.18.** Let \( R \) be an integral domain with quotient field \( K \), and let \( F \) be a family of valuations satisfying (i), (ii), (iii). The following statements are equivalent.

1. \( \mathcal{A}(R) \) is a group.
2. \( R:A = R:B \Rightarrow [A] = [B] \), for all \( A, B \in I(R) \).
3. \( v(A) = v(\tilde{A}) \) for all \( A \in I(R) \) and \( v \in F \).
4. The map \( g: \mathcal{A}(R) \rightarrow \mathcal{D}(R) \) is an isomorphism.

**Proof.** (1) \( \Rightarrow \) (2) Suppose \( \mathcal{A}(R) \) is a group. Let \( A, B \in I(R) \) be such that \( R:A = R:B \). By 1.15 we have \(-[A] = [R:A] = [R:B] = -[B]\) and hence \([A] = [B]\).

(2) \( \Rightarrow \) (3) We have \( R:A = R:\tilde{A} \) for all \( A \in I(R) \). If (2) holds then \([A] = [\tilde{A}]\) for all \( A \in I(R) \); i.e., \( v(A) = v(\tilde{A}) \) for all \( v \in F, A \in I(R) \).

(3) \( \Rightarrow \) (4) Consider Diagram 1.16. If \( v(\tilde{A}) = v(A) \) for all \( v \in F \) and \( A \in I(R) \), we can define \( \lambda: \mathcal{D}(R) \rightarrow X' \) by \( \lambda(\text{div}_R(A)) = (v_i(A))_{i \in I} \). It follows that \( \lambda \) is a homomorphism and that \( \lambda \circ g = f \). By 1.17, \( g \) is an isomorphism.

(4) \( \Rightarrow \) (1) obvious.

We observe that the converse of statement (2) in 1.18 is always true in \( R \). For if \([A] = [B]\), then \( A \in [B] \subseteq \text{div}_R(B) \), and \( B \in \text{div}_R(B) \) so that \( \text{div}_R(A) = \text{div}_R(B) \) and hence \( R:A = R:B \).

When the valuations in \( F \) are discrete we obtain a partial generalization of a result in [6] with the aid of the following lemma.

**Lemma 1.19.** Assume that each \( v \in F \) is discrete. Then for each \( v \in F \), if \( \text{div}_R(P(v)) \neq 0 \) then \( P(v) = (P(v))^\sim \).

**Proof.** We have \( P(v) \subseteq (P(v))^\sim \subseteq R \). If \( P(v) < (P(v))^\sim \), there is \( x \in (P(v))^\sim \), \( x \notin P(v) \). Then \( v(x) = 0 = v(P(v))^\sim \). Also, for \( w \in F, w \neq v \), we have \( 0 = w(P(v)) \geq w(P(v))^\sim \geq 0 \). Thus \([(P(v))^\sim] = 0 \). Since the canonical map \( g \) from \( \mathcal{A}(R) \) onto \( \mathcal{D}(R) \) is a homomorphism, we should have \( 0 = [(P(v))^\sim] = \text{div}_R(P(v))^\sim = \text{div}_R(P(v)) = 0 \). But \( \text{div}_R(P(v)) \neq 0 \) by assumption. Thus we must have \( (P(v))^\sim = P(v) \) if \( \text{div}_R(P(v)) \neq 0 \).

**Theorem 1.20.** Assume each \( v \in F \) is discrete. Then the canonical map \( g \) from \( \mathcal{A}(R) \) onto \( \mathcal{D}(R) \) is an isomorphism iff \( P(v) \) is divisoriel for each \( v \in F \).
Proof. (⇒) Suppose \( g \) is an isomorphism. If \( P = P(v) \) for some \( v \in F \), then \( [P] > 0 \) since \( v(P) = 1 \). If \( P \) is not divisoriel then \( g([P]) = \text{div}_R(P) = 0 \), by Lemma 1.19. But then \( g \) is not 1-1 and hence not an isomorphism. For \( [R] = 0 \neq [P] \).

(⇐) Suppose \( P(v) \) is divisoriel for each \( v \in F \). Then if \( A \in I(R) \) is such that \( \text{div}_R(A) = 0 \) we must have \( A \subseteq R \) (for this result see [1, bottom of p. 4]). Moreover, \( A \notin P(v) \) for any \( v \in F \). If \( A \subseteq P(v) \) for some \( v \in F \), then \( \text{div}_R(A) \geq \text{div}_R(P) > 0 \), a contradiction. Thus \( g([A]) = 0 \) iff \( [A] = 0 \). Now suppose \([A], [B] \in \mathcal{A}(R)\) are such that \( g([A]) = g([B]) \). Then \( \text{div}_R(A) = \text{div}_R(B) \) so that \( \text{div}_R(A) - \text{div}_R(B) = 0 = \text{div}_R(B) - \text{div}_R(A) \); i.e., \( \text{div}_R(A:B) = 0 = \text{div}_R(B:A) \). Since \( g([A:B]) = g([B:A]) = 0 \), we must have \([A:B] = 0 = [B:A]\). Since each \( v \in F \) is discrete, for each \( v \in F \) there is \( x \in A : B \) such that \( v(x) = v(A:B) = 0 \). Now \( xB \subseteq A \) (by definition of \( A : B \)) so that \( v(x) + v(B) = v(xB) \geq v(A) \); i.e., \( v(B) \geq v(A) \). Thus \( v(B) \geq v(A) \) for all \( v \in F \).

Similarly \( v(A) \geq v(B) \) for all \( v \in F \), and \([A] = [B]\). This shows that \( g \) is 1-1 and hence an isomorphism.

When \( R \) is AD, the author has shown in [6] that \( P(v) \) is divisoriel for each \( v \in F \) iff \( R \) is Dedekind. To date, however, the author has been unable to prove the following conjecture: If \( R \) is AK and \( P(v) \) is divisoriel for each \( v \in F \), then \( R \) is a Krull domain.

When \( R \) is AK, we do have the following theorem.

Theorem 1.21. Let \( R \) be an AK-domain with family \( F \) of essential valuations and let \( \Delta \) denote the collection of maximal ideals of \( R \). Every minimal prime of \( R \) is divisoriel iff \( \Delta = \bigcap_{M \in \mathcal{A}(AR_M)} \) for every ideal \( A \) of \( R \).

Proof. Here \( \Delta = \bigcap_{A \in R \times R} Rx \) and \( (AR_M)^\sim = \bigcap_{s \in R_M} R_M \). For any maximal ideal \( M \) of \( R \), \( F_M \) denotes the family of essential valuations of the Krull domain \( R_M \). Recall that \( F_M \in F \).

(⇒) Let \( A \) be an ideal of \( R \). If \( M \) is any maximal ideal of \( R \) then \( v(A) = v(AR_M) \) for all \( v \in F_M \). Since \( R_M \) is a Krull domain, \( v(AR_M) = v(AR_M)^\sim \) for all \( v \in F_M \) so that \( v(A) = V(AR_M)^\sim \) for all \( v \in F_M \).

Case 1. \( v(A) = 0 \) for all \( v \in F \).

Then \( F < A \) for every minimal prime \( P \) of \( R \). In this case \( \Delta = R = \bigcap_{M \in \mathcal{A}} R_M = \bigcap_{M \in \mathcal{A}} (AR_M)^\sim \).

Case 2. \( v(A) > 0 \) for some \( v \in F \).

For each maximal ideal \( M \) of \( R \), if there is \( v \in F_M \) such that \( 0 < v(A) = v(AR_M)^\sim \), then we can write

\[
(AR_M)^\sim = \bigcap_{v_i \in F_M \mid v_i(A) > 0} Q_i(\mathbf{n}_i),
\]

where \( n_i = v_i(AR_M)^\sim \) and \( Q_i \) is the center of \( v_i \) on \( R_M \). Then for each \( i \) such that \( v_i(A) > 0 \) we have \( Q_i = P_i R_i \) where \( P_i = P(v_i) \) in \( R \). Thus \( (AR_M)^\sim = \bigcap_i (P_i R_M)^{n_i} = \bigcap_i ((P_i R_M)^{n_i} R_M) \cap R_M = \bigcap_i (P_i^n R_i \cap R_M) \) where \( i \) runs over all indices such that \( v_i \in F_M \) and \( v_i(A) > 0 \), and \( n_i = v_i(A) \) for each such \( i \). It can then be shown that, \( C = \bigcap_{M \in \mathcal{A}} (AR_M)^\sim = \bigcap \{ P_i^{n_i} \mid v_i \in F, v_i(A) > 0 \} \). Then \([C] = [A] \). Since \([A] = \text{div}_R(A) \), it follows that \( \Delta = C \).
Suppose $P$ is a minimal prime of $R$. If $M \subseteq \Delta$, then either $P \subseteq M$ or $P \not\subseteq M$. If $P \not\subseteq M$ then $PR_M = M$. If $P \subseteq M$, then $PR_M$ is a minimal prime of the Krull domain $R_M$ and thus $(PR_M)^- = PR_M$. Since $P$ is contained in some maximal ideal $M$ we have $\overline{P} = \bigcap_{M \in \Delta} (PR_M)^- = \bigcap_{M \in \Delta} PR_M = P$.

We now drop the assumption that $R$ is an AK-domain and assume only that $F$ satisfies axioms (i), (ii), (iii) at the beginning of this section. The next lemma tells us more about the elements of $\mathcal{A}(R)$ which have inverses and enables us to partially describe $\mathcal{D}(R)$ in certain cases where $\mathcal{A}(R)$ may not be a group.

**Lemma 1.22.** If $[A] \in \mathcal{A}(R)$ is such that $[A]$ has an inverse then $[A] = [\overline{A}]$.

**Proof.** If $[A] \in \mathcal{A}(R)$ has an inverse then $[-A] = [R:A]$ by Proposition 1.15. Now $A \subseteq \overline{A}$ and $R: A = R: \overline{A}$. Thus $A(R: A) \subseteq \overline{A}(R: A) = \overline{A}(R: \overline{A}) \subseteq R$. These containment relations yield the following: $0 = [A(R: A)] \geq [\overline{A}(R: \overline{A})] \geq 0$. Thus $0 = [A] + [R:A] = [\overline{A}] + [R:A]$ and $[A] = [\overline{A}]$.

**Corollary 1.23.** If $[A], [B]$ have inverses in $\mathcal{A}(R)$, then $[A] + [B] = [AB]^{-1}$; i.e., $v(\overline{AB}) = v(AB)^{-1}$ for all $v \in F$.

**Proof.** By 1.22 above, if $[A], [B]$ have inverses then $[A] = [\overline{A}]$ and $[B] = [\overline{B}]$. Moreover, $[A] + [B]$ has an inverse. Thus $[A] + [B] = [AB]^{-1}$ by 1.22; i.e., $[\overline{A}] + [\overline{B}] = [AB]^{-1}$.

Now consider the map $\rho: \mathcal{D}(R) \to X^I$ defined by $\rho(div_R(A)) = (v_\mathcal{I}(A))_{\mathcal{I}}$. $\rho$ is well defined, for if $div_R(A) = div_R(B)$ then $\overline{A} = \overline{B}$. Conversely, if $(v_\mathcal{I}(A))_{\mathcal{I}} = (v_\mathcal{I}(B))_{\mathcal{I}}$ then $[\overline{B}] = [\overline{A}]$ and so $div_R(A) = div_R(\overline{A}) = div_R(\overline{B}) = div_R(B)$ by the remark following the proof of 1.18. Thus $\rho$ is 1-1. We can now give a description of $\mathcal{D}(R)$ when $R$ is fairly well behaved.

**Theorem 1.24.** The map $\rho: \mathcal{D}(R) \to X^I$ defined by $\rho(div_R(A)) = (v_\mathcal{I}(A))_{\mathcal{I}}$ is 1-1. Furthermore $\rho$ is a homomorphism iff $[\overline{A}] \in \mathcal{A}(R)$ has an inverse for all $A \in I(R)$.

**Proof.** The first assertion is proved in the immediately preceding remarks. We now prove the second assertion.

$(\Rightarrow)$ Suppose $\rho$ is a homomorphism. Then since $\mathcal{D}(R)$ is a group, for $div_R(A) \in \mathcal{D}(R)$, $-div_R(A) = div_R(R:A)$. Thus $\rho(div_R(A) + div_R(R:A)) = 0 = \rho(div_R(A)) + \rho(div_R(R:A)) = (v_\mathcal{I}(A))_{\mathcal{I}} + (v_\mathcal{I}(R:A))_{\mathcal{I}}$. It follows that $[\overline{A}]$ has an inverse in $\mathcal{A}(R)$.

$(\Leftarrow)$ Suppose that $[\overline{A}]$ has an inverse for all $A \in I(R)$. By Corollary 1.23, we have that $[\overline{AB}] = [AB]^{-1}$ for all $A, B \in I(R)$. Thus for $A, B \in I(R)$ we have $\rho(div_R(AB)) = (v_\mathcal{I}(AB))_{\mathcal{I}} = (v_\mathcal{I}(A))_{\mathcal{I}} + (v_\mathcal{I}(B))_{\mathcal{I}} = \rho(div_R(A)) + \rho(div_R(B))$ and $\rho$ is a homomorphism.

Now, let $R$ be an AK-domain. Then $R_P$ is a Krull domain for any prime ideal $P$ of $R$. However, these are not the only Krull domains $T$ such that $R \subseteq T \subseteq K$. For if $\Delta = \{P_1, \ldots, P_n\}$ is any finite collection of prime ideals of $R$ then $T = \bigcap_{P \in \Delta} R_P$ is also a Krull domain. Thus there is a large class of Krull domains $T$ such that

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FAMILIES OF VALUATIONS

When $R$ is an AK-domain in which every minimal prime is divisoriel we always have that $\mathcal{D}(T)$ is a homomorphic image of $\mathcal{D}(R)$, where $T$ is an AK-domain such that $R \subseteq T \subseteq K$. For, $\mathcal{A}(T)$ is a homomorphic image of the group $\mathcal{A}(R)$ and so is a group. Then $\mathcal{A}(R) \subseteq \mathcal{D}(R)$ and $\mathcal{A}(T) \subseteq \mathcal{D}(T)$. When $T$ is a Krull domain and $R$ is an AK-domain for which the map $\rho$ of Theorem 1.24 is a homomorphism we also get that $\mathcal{D}(T)$ is a homomorphic image of $\mathcal{D}(R)$ as follows.

**Proposition 1.25.** Let $R$ be an AK-domain and let $T$ be a Krull domain such that $R \subseteq T \subseteq K$. If $[A]$ has an inverse for every $[A] \in \mathcal{A}(R)$ then the map $\tau: \mathcal{D}(R) \rightarrow \mathcal{D}(T)$, defined by $\tau(\text{div}_R(A)) = \text{div}_T(AT)$, is a homomorphism of $\mathcal{D}(R)$ onto $\mathcal{D}(T)$.

**Proof.** $\tau$ is well defined, for if $\text{div}_R(A) = \text{div}_R(B)$ then $A = B$ so that $AT = BT$. Now consider the following diagram.

![Diagram 1.26](image)

Here, $I$ is the index set for the family of essential valuations of $R$; $J$ is the index set for the family of essential valuations of $T$; $\pi$ is the projection of $Z^I$ onto $Z^J$; $\rho$ is the (injective) homomorphism of 1.24; $\gamma$ is the injection of 1.4. It is well known that $\gamma$ is also surjective; i.e., $\gamma$ is an isomorphism. Consider the map $\gamma^{-1} \circ \pi \circ \rho: \mathcal{D}(R) \rightarrow \mathcal{D}(T)$. We have, for $\text{div}_R(A) \in \mathcal{D}(R)$, $(\gamma^{-1} \circ \pi \circ \rho)(\text{div}_R(A)) = (\gamma^{-1} \circ \pi)(\text{div}_R(A)) \times (v_j(A))_{j \in J} = \gamma^{-1}((v_j(A))_{j \in J})$. Since $T$ is a Krull domain we have that $v(B) = v(B)$ for all fractionary ideals $B$ and all essential valuations $v$. Thus $(v_j(AT))_{j \in J} = (v_j(AT)^{-1})_{j \in J}$ so that $\gamma^{-1}((v_j(A))_{j \in J}) = \text{div}_R(AT)$. Then $\gamma^{-1} \circ \pi \circ \rho = \tau$ and $\tau$ is a homomorphism since $\tau$ is a composition of homomorphisms. To see that $\tau$ is surjective it is sufficient to show that for every divisoriel fractionary ideal $\mathcal{U}$ of $T$ there is a divisoriel fractionary ideal $A$ of $R$ such that $\tau(\text{div}_R(A)) = \text{div}_T(\mathcal{U})$. So let $\mathcal{U}$ be a fractionary ideal of $T$. There are elements $x, y \in K$ such that $\mathcal{U} = Tx \cap Ty$ [1, p. 13]. Let $A = Rx \cap Ry$. Then $A$ is divisoriel and $v_i(A) = v_i(\mathcal{U})$ for all $j \in J$. Following Gilmer in [3], we make the following definition.

**Definition 2.1.** We say that $R$ satisfies property $(\star)$ with respect to $F$ iff for distinct subsets $F_1, F_2$ of $F$ we have that $\bigcap_{u \in F_1} R_u \neq \bigcap_{u \in F_2} R_u$.

When $R$ is a Prüfer domain and $F$ is the family of valuations induced by the collection of maximal ideals, then property $(\star)$ is the same as property $(\star)$ in [2].
For \( v \in F \), we let \( F_v = F - \{v\} \).

**Proposition 2.2.** \( R \) has property (\( * \)) with respect to \( F \) iff for each \( v \in F \), 
\[
\bigcap_{u \in F_v} R_u \not\subseteq R_v.
\]

**Proof.** The proof is substantially the same as that of Lemma 1 in [2] and is omitted.

**Corollary 2.3.** If \( R \) satisfies (\( * \)) with respect to \( F \) and if \( G \) is any nonempty subset of \( F \), then \( T = \bigcap_{u \in G} R_u \) satisfies (\( * \)) with respect to \( G \).

We note that if \( R \) satisfies (\( * \)) with respect to \( F \) then \( P(v) \not\subseteq P(w) \) for \( v \neq w \). For if \( P(v) \subseteq P(w) \) for some \( w \neq v \), then \( R_{P(w)} \subseteq R_{P(v)} \); i.e., \( R_v \subseteq R_w \). Then we have the following: 
\[
\bigcap_{u \in F - \{v, w\}} R_u \cap (R_v \cap R_w) = (\bigcap_{u \in F} R_u) \cap R_w = \bigcap_{u \in F_v} R_u \text{ and } F \neq F_v, \text{ a contradiction.}
\]

**Proposition 2.4.** If \( F \) is of finite character and is such that \( P(u) \not\subseteq P(v) \) if \( u \neq v \), then \( R \) satisfies (\( * \)) with respect to \( F \).

**Proof.** Let \( v \in F \) and let \( x \in R, x \neq 0 \), be such that \( v(x) > 0 \). Let \( v_1, \ldots, v_n \) be the distinct (from \( v \) and each other) valuations such that \( v_i(x) \neq 0, i = 1, \ldots, n \). There exists \( y \in \bigcap_{i=1}^n P(v_i) - P(v) \). For if \( P(v) \subseteq P(v_i) \), then \( \prod_{i=1}^n P(v_i) \subseteq P(v) \) and so \( P(v) \subseteq P(v) \) for some \( j, 1 \leq j \leq n \), contradicting our hypothesis. Choose \( n \) large enough so that \( w(y^m/x) \geq 0 \) for \( w \in F_v \). This is possible since \( F \) is of finite character and \( w(y^m/x) \geq 0 \) for all \( w \in F_v \). Then \( w(y^m/x) \geq 0 \) for all \( w \in F_v \) and \( v(y^m/x) = -v(x) < 0 \). Thus \( y^m/x \in \bigcap_{u \in F_v} R_u - R_v \); i.e., \( \bigcap_{u \in F_v} R_u \not\subseteq R_v \). So \( R \) satisfies (\( * \)) with respect to \( F \) by 2.2.

Let \( R \) be an integral domain with family \( F \) of valuations satisfying (1) and (2) listed at the beginning of this section. \( R \) is called a generalized Krull domain if \( F \) satisfies the following two additional properties (see [5]).

(3) Each \( v \in F \) has rank one.

(4) \( F \) is of finite character.

**Corollary 2.5.** If \( R \) is a Krull domain, or a generalized Krull domain with family \( F \) of valuations, then \( R \) satisfies (\( * \)) with respect to \( F \).

**Proof.** In this case \( F \) is a family of rank one valuations of finite character, so that if \( u, v \in F, u \neq v \), then \( P(v) \not\subseteq P(u) \).

**Proposition 2.6.** Let \( R \) be an AD-domain. The following conditions on \( R \) are equivalent.

(1) \( R \) satisfies (\( * \)) with respect to \( F \), the family of essential valuations of \( R \).

(2) \( R \) is Dedekind.

(3) Every minimal prime of \( R \) is divisorial.

**Proof.** (1) \( \Rightarrow \) (2) is Theorem 3 of [3].

(2) \( \Rightarrow \) (3) is found in [6].
Thus we see that in the case of almost-Dedekind domains, the divisoriel property of the minimal prime ideals completely determines whether or not $R$ satisfies property $(\star)$. We shall see that the divisoriel property of the minimal primes is always sufficient for $R$ to satisfy $(\star)$.

**Proposition 2.7.** Let $R$ be an integral domain with family $F$ of valuations such that

(i) Each $v \in F$ has rank one.

(ii) $R = \bigcap_{v \in F} R_v$.

(iii) $R_v = R_{P(v)}$ for each $v \in F$.

If $P(v)$ is divisoriel for each $v \in F$, then $R$ satisfies $(\star)$ with respect to $F$.

**Proof.** We note that since $R$ is the intersection of rank one valuation rings, $R$ is completely integrally closed and hence $\mathcal{D}(R)$ is a group. If each $P(v)$ is divisoriel, then each $v \in F$ is discrete. For if $P = P(v)$ is divisoriel, we must have $P^2 < P$. For if $P^2 = P$, then $\text{div}_R(P^2) = \text{div}_R(P)$; i.e., $2 \text{div}(P) = \text{div}(P)$. Thus $\text{div}(P) = 0$ and $\bar{P} = R \neq P$, contradicting $\bar{P} = P$.

Since $P^2 < P$, we have $P^2 R_P < PR_P$ and so $R_P$ is a discrete valuation ring. We now show that $\{P(v) \mid v \in F\}$ is the set of all minimal divisoriel primes of $R$. Clearly, $\{P(v) \mid v \in F\}$ is contained in the set of all divisoriel minimal primes. Now let $P$ be a minimal, divisoriel prime of $R$. If $P \neq P(v)$ for any $v \in F$, then $P \neq P(v)$ for any $v \in F$ and so $v(P) = 0$ for all $v \in F$; i.e., $[P] = 0$. But then we would have $g([P]) = 0$; i.e., $\text{div}(P) = 0$; i.e., $\bar{P} = R$, contradicting $\bar{P} = P < R$. So we must have that $\{P(v) \mid v \in F\}$ is the set of all divisoriel minimal primes of $R$. Now let $G$ be any subset of $F$ such that $R = \bigcap_{u \in G} R_u$. $P(u)$ is divisoriel for each $u \in G$ since $G \subseteq F$. By what we have just shown, $\{P(u) \mid u \in G\}$ is the collection of all minimal divisoriel primes of $R$; i.e., $G = F$. Thus for any $v \in F$, $\bigcap_{u \in G} R_u \neq R_v$ and so $R$ satisfies $(\star)$ with respect to $F$.

The first part of the proof of Proposition 2.7 shows that if $P$ is the center of a rank one valuation $v$, then $v$ is discrete if $P$ is divisoriel. This enables us to characterize Krull domains in the class of generalized Krull domains as follows.

**Corollary 2.8.** Let $R$ be a generalized Krull domain with family $F$ of valuations. $R$ is a Krull domain iff $P(v)$ is divisoriel for each $v \in F$.

Let $R$ be an integral domain with quotient field $K$ and let $F$ be a family of valuations on $K$ satisfying conditions (1) and (2) stated at the beginning of this section. Let $x$ be an indeterminate and let $F'$ denote the family of valuations on $K(x)$ which are canonical extensions of elements of $F$. Let $G$ denote the family of $p(x)$-adic valuations on $K(x)$, where $p(x)$ is a nonconstant irreducible polynomial in $K[x]$. Then $F' \cup G$ is a family of valuations on $K(x)$ satisfying (1) and (2) with $R[x]$ in place of $R$. 

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Proposition 2.9. If $R$ satisfies ($\star$) with respect to $F$, then $R[x]$ satisfies ($\star$) with respect to $F' \cup G$.

Proof. Let $w \in F' \cup G$. If $w \in G$, then $w$ is a $p(x)$-adic valuation for some nonconstant irreducible polynomial $p(x) \in K[x]$. Without loss of generality we may assume that $p(x) \in R[x]$. Suppose $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, $a_i \in R$. Let $b = \prod a_k \neq 0$. Then $b \neq 0$ since $a_n \neq 0$, and $v(b) = \min_{0 \leq i \leq n} v(a_i)$ for all $v \in F$ since $b \in R$ and $a_k \in R$ for all $k = 0, 1, \ldots, n$, and every $v \in F$ is non-negative on $R$. Then for $v' \in F'$, $v'(b/p(x)) = v(b) - v'(p(x)) = v(b) - v(a_0) = v(b) \geq 0$.

If $w \in G$ and $u \neq w$, then $u$ is a $q(x)$-adic valuation for some nonconstant irreducible polynomial $q(x)$ such that $q(x) \nmid p(x)$. Then $u(b/p(x)) = 0$. Thus $b/p(x) \in \bigcap_{w \in (F' \cup G)_w} (R[x])_w$. Since $w(b/p(x)) = -1 < 0$. Thus if $w \in G \cap \bigcap_{w \in (F' \cup G)_w} (R[x])_w \neq (R[x])_w$. On the other hand, if $w \in F'$, then $w = v'$ for some $v \in F$. Since $\bigcap_{w \in (F' \cup G)_w} (R[x])_w = (R[x])_w$, and $a \neq 0$. Then $a \notin R[x]_v$, for $v'(a) = v(a) < 0$. Thus for every $w \in F' \cup G$ we have $\bigcap_{w \in (F' \cup G)_w} (R[x])_w \neq (R[x])_w$ and thus $R[x]$ satisfies ($\star$) with respect to $F' \cup G$ by 2.2.

3. In [6] it was shown that if $R$ is an almost-Dedekind domain with family $F$ of essential valuations, then $R$ is Dedekind if every minimal prime of $R$ is divisoriel. Thus in an AD-domain $R$, every minimal prime is divisoriel iff $F$ is of finite character. In §1 it was conjectured that if $R$ is an AK-domain with family $F$ of essential valuations, then $R$ is Krull if $P(v)$ is divisoriel for each $v \in F$; i.e., $F$ is of finite character if $P(v)$ is divisoriel for each $v \in F$. In this section we give an example to show that this conjecture is false if the AK requirement is dropped. We also give an example of an AK-domain which is neither a Krull domain nor an AD-domain.

Let $R$ denote the set of entire functions, $C$ denote the set of complex numbers, $Z$ denote the additive group of integers. It is well known that $R$ is an integral domain under the usual pointwise definitions of addition and multiplication. For $a \in C$ we define $v_a: R \to Z$ by $v_a(f(z)) = n$ if $a$ is a zero of $f(z)$ of order $n$. If $a$ is not a zero of $f(z)$ then $v_a(f(z)) = 0$. If $f(z) \equiv 0$ we put $v_a(f(z)) = +\infty$ for each $z \in C$. It is easy to show that each $v_a$ can be extended to a valuation on the quotient field of $R$. We let $F$ denote this family of valuations. $F$ has the following properties: (i) Each $v \in F$ has rank one and is discrete; (ii) $R = \bigcap \{R_v \mid v \in F\}$; (iii) $R_v = R_{P(v)}$ for each $v \in F$; (iv) For $a \in C$, $P(v_a) = (z-a)R$, and hence is divisoriel; (v) $F$ is not of finite character. Furthermore, $P(v)$ is maximal for each $v \in F$. However, these are not all the maximal ideals of $R$. For let $\{z_n\}_{n=1}^\infty$ be a sequence of complex numbers such that $\lim z_n = \infty$. For each positive integer $m$, let $f_m(z)$ be an entire function whose zeros are exactly $\{z_m, z_{m+1}, \ldots\}$. The ideal generated by $\{f_1(z), f_2(z), \ldots\}$ is proper and is contained in a maximal ideal $M$. However, $R_M$ is not a Krull domain. It follows that $R$ is not AK.

It was shown in [7] that if $R$ is AK and $X_1, \ldots, X_n$ are indeterminates, then $R[X_1, \ldots, X_n]$ is AK. Let $R$ be an AD-domain which is not a Dedekind domain.
Such a domain is given in example 2 of [2]. Then \( R[X_1, \ldots, X_n] \) is an AK-domain which is neither a Krull domain nor an AD-domain. We observe that example 1 of [2] is a generalized Krull domain which is not a Krull domain.

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**References**


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