CORRECTION AND ADDENDUM TO "ON ALGEBRAS OF FINITE REPRESENTATION TYPE"

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In this note we retain the notation and terminology of our original paper [2]. Our purpose is to repair the statement and proof of Theorem 2.3 of [2], and using its generalization (Theorem A below) as a prototype, we can by use of an extension of Corollary 3.3 of [2] (Theorem B below) remove the weighty hypothesis of "large kernels" from Theorems 3.4, 3.5, 3.6 and Corollary 3.7 of [2]. This latter task derives some urgency from the conjecture of J. P. Jans [3] that an algebra with large kernels has finite module type. This conjecture, if true, would of course annihilate any importance of the above-mentioned results of [2] in its original form as far as the Brauer-Thrall conjecture is concerned(1).

CORRECTION 1. The statement of Theorem 2.3 of [2] should read as follows:

THEOREM 2.3. *Let A be a QF ring with large kernels having the property that if M is an indecomposable finitely generated left A-module, then the homogeneous components of the socle of M are cyclic (hence simple) as right modules over the endomorphism ring $R=\mathrm{End}_A(M)$. Then A has ω-finite module type.*

The argument presented in the paper for the proof of this result is insufficient for the reduction to the stable case, and the following remarks will correct this oversight.

Recall that a submodule A of a module B is called small (or superfluous) if for any submodule H of B, $A+H=B$ implies $H=B$.

LEMMA. *Let A be a ring with unit and P a projective left A-module. If L is a small submodule of P then an automorphism of P/L is induced by one of P.*

Proof. By smallness of L, an automorphism φ of P/L lifts to an epimorphism $\Phi: P \to P$. But Φ splits and since $\mathrm{Ker}\,\Phi \subseteq L$ is small with $P=\mathrm{Ker}\,\Phi \oplus N$, say, $\mathrm{Ker}\,\Phi=0$.

COROLLARY A. *If A is a QF ring and M is a module of finite length, an automorphism of $E(M)/M$ lifts to one of $E(M)$.*

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(1) After submitting these additions and corrections, the author learned that the Brauer-Thrall conjecture has been proved by A. V. Roiter, *Unboundedness of the degrees of indecomposable representations of algebras having infinitely many indecomposable representations*, Izv. Akad. Nauk SSR Ser. Mat. **32** (1968), 1275–1282. (Russian)

Proof. If M is indecomposable M is small in $E(M)$ by Proposition 2.1 of [2], and if M is indecomposable the same is true since smallness commutes with finite direct sums.

COROLLARY B. *Let M be a module of finite length and let φ be an automorphism of $E(M)/M$ taking $S=E/M$ onto $S'=E'/M$ where $S\approx S'$. Then φ lifts to an automorphism Φ of $E(M)$ such that $\Phi(E)=E'$.*

Proof. That $\Phi(E)\subset E'$ is clear. Equality holds since Φ preserves lengths.

Proof of Theorem 2.3. We show by induction that for any module M of length n there are only finitely many nonisomorphic submodules E of $E(M)$ which contain M and have a simple quotient by M. If M has length zero this is clear. If M is indecomposable and $M\subset E$, $E'\subset E(M)$ with $E/M\approx E'/M$ then by hypothesis there is an endomorphism (by large kernels necessarily an automorphism) of the indecomposable $E(M)/M$ mapping E/M onto E'/M so by Corollary B we have $E\approx E'$. If M is decomposable, write $M=M_1\oplus\cdots\oplus M_n$ where M_i are indecomposable. Then $E(M)/M\approx E(M_1)M_1\oplus\cdots\oplus E(M_n)/M_n$. In this case choose generators x, x', respectively for E/M, E'/M, and write $x=x_1+\cdots+x_n$, $x'=x_i'+\cdots+x_n'$, where without loss of generality all the x_i and x_i' are nonzero. Then there are automorphisms φ_i $(1\leqq i\leqq n)$ of the $E(M_i)/M_i$ such that $x_i\varphi_i=x_i'$ and according to Corollary B the φ_i lift to automorphisms Φ_i of $E(M_i)$ such that the automorphism $\Phi=\Phi_1\oplus\cdots\oplus\Phi_n$ maps E onto E' as desired.

CORRECTION 2. The last line of the proof of Theorem 2.5 of [2] should read "conclude that either $E(M)/M$ has square-free socle or A has non-ω-finite module type."

PROPOSITION. *Let M be an indecomposable left A-module of finite length such that each homogeneous component of the socle of M is uniserial as a right module over $R=\mathrm{End}_A(M)$. Then there are a finite number of simple submodules $S_0,\ldots,S_n$ of M such that any simple submodule of M maps onto one of the S_i $(0\leqq i\leqq n)$ by an automorphism of M.*

Proof. Let $\mathcal{N}=\mathrm{Rad}\,R$. Let $0\neq x_i\in H\mathcal{N}^i$, $x_i\notin H\mathcal{N}^{i+1}$, $(0\leqq i\leqq n)$ where H is the S-homogeneous component of the socle of M and $H\mathcal{N}^{n+1}=0$. Let $x\neq 0$ generate a simple left A-submodule of H, and let i be the first integer $\geqq 0$ such that $x\in H\mathcal{N}^i$, $x\notin H\mathcal{N}^{i+1}$. Then $xR=H\mathcal{N}^i$ which follows since $H\mathcal{N}^{i+1}$ is small in $H\mathcal{N}^i$. Since $H\mathcal{N}^i/H\mathcal{N}^{i+1}$ is R-simple, $xR=x_iR$ and in particular there is an endomorphism φ of M such that $x\varphi=x_i$. If φ were nilpotent, it would follow that $x_i=x\varphi\in H\mathcal{N}^{i+1}$, contradictory to the choice of x_i. Thus $(Ax)\varphi=Ax_i$ and in particular it follows that Ax_i is simple $(0\leqq i\leqq n)$.

Using the Proposition together with a trivial modification of the proof of Theorem 2.3 above we have its generalization

THEOREM A. *Let A be a QF ring having the property that for any indecomposable left A-module M of finite length, the homogeneous components of the socle of M are uniserial as right modules over the endomorphism ring $R=\mathrm{End}_A(M)$. Then A has ω-finite module type.*

The ring A having minimum condition on left ideals is said to be basic, if $A/\mathrm{Rad}\, A$ is a direct sum of division rings [1]. Then it follows that a simple module is one-dimensional over its endomorphism division ring, and if A is a finite-dimensional algebra over an algebraically closed field K Schur's lemma implies that simple A-modules are one-dimensional over K. Also, if A is a basic ring, each generator of a simple module has the same left annihilator ideal so it follows that every maximal left ideal of A is two-sided and moreover, that any cyclic submodule of a homogeneous completely reducible left A-module is simple.

Since the category of finitely generated left modules over a ring with minimum condition is isomorphic to the category of finitely generated left modules over a basic ring with the isomorphism preserving the properties of finite length and indecomposability, we may restrict ourselves to basic rings. Also, note that if A is a QF ring then the basic ring will be QF [4].

We utilize the basic algebra in the next theorem since it greatly facilitates the checking of the interlacing equations in the proof.

THEOREM B. *Let A be a finite-dimensional basic algebra over an algebraically closed field K having bounded module type. Let M be an indecomposable left A-module of finite length and S a simple submodule of M such that* $\mathrm{Ext}_R^1(M, S)=0$. *Then the S-homogeneous component H of M is uniserial as a right R-module, $R=\mathrm{End}_A(M)$.*

Proof. It suffices to show that H_i/H_{i-1} is a simple R-module, where $H_i=\{x\in H \mid x\mathcal{N}^i=0\}$ $(0\leqq i\leqq n+1)$, $H_0=0$, $H\mathcal{N}^{n+1}=0$. For $x\in H_1$ we have $Ax+xR=Ax+xK=Ax=Kx$, the latter equality holding since A is basic. Theorem 3.1 of [2] shows that H_1 is one-dimensional over K. Assume for induction that for $1\leqq m<k$ and $x, y\in H_m$, $x, y\notin H_{m-1}$, there is an automorphism of M taking x onto y. Now let $x, y\in H_k$ but $x, y\notin H_{k-1}$. We read the interlacing equations on page 138 of [2] as congruences modulo H_{k-1} and conclude $Kx\equiv Ky(\mathrm{mod}\ H_{k-1})$. Hence $y=\xi x+h$, $0\neq\xi\in K$, $h\in H_{k-1}$. Suppose that $h\in H_p$, $h\notin H_{p-1}$, $0\leqq p\leqq k-1$. Since $x\mathcal{N}^{k-1}\neq 0$ there is a nilpotent endomorphism η such that $x\eta\in H_p$, $x\eta\notin H_{p-1}$. Thus by induction hypothesis there is an automorphism ψ with $h=x\eta\psi$. Then $\xi 1_M+\eta\psi$ is scalar plus nilpotent (hence an automorphism) which takes x into y. It follows that H_k/H_{k-1} is one-dimensional over K for $1\leqq k\leqq n+1$ and H is uniserial.

REMARK. We shall not repeat the statements of Theorems 3.4, 3.5, 3.6 and Corollary 3.7 of [2], but simply make the blanket assertion that with the results now in hand the assumption of "large kernels" may be eliminated from their proofs, using the extension of Theorem 2.3 to Theorem A as a model.

References

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