LIE ALGEBRAS OF CHARACTERISTIC 2(1)

BY
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1. **Introduction.** This paper presents a structure theorem for certain simple, restricted Lie algebras of characteristic 2. The work here extends Kaplansky's work on rank 2 simple, restricted Lie algebras of characteristic 2 to arbitrary ranks. See [6].

The work in this paper differs from that of Brown in that the roots and root spaces are given their usual definitions here. Their somewhat different than usual properties are then deduced by using the theory of quadratic forms and a different form of the root diagrams. Brown retains most of the common properties of root spaces and root diagrams by altering the definitions of root spaces and roots. See [1].

2. **Basic definitions and axioms.** A Lie algebra over a field is a vector space over the field possessing a bilinear product which is antisymmetric and satisfies the Jacobi identity. All Lie algebras considered here will be finite dimensional over algebraically closed fields of characteristic 2.

Subalgebras and ideals have their usual definitions. The normalizer of a subalgebra is the maximal subalgebra in which the given subalgebra is an ideal. A subalgebra A is nilpotent if the sequence $A, A^2, A^3, \ldots$ terminates in 0 where $A^{n+1}=A^nA$. A subalgebra is a Cartan subalgebra if it is nilpotent and equals its own normalizer. The condition that the field be algebraically closed implies the existence of a Cartan subalgebra. See [4]. The rank of a Lie algebra is the minimal dimension of a Cartan subalgebra.

Let H be a Cartan subalgebra of Lie algebra L, if $\alpha \in H^*$ (the dual space of H) we let $L_\alpha=\{x \in L : x(R_h-\alpha(h)I)^k=0$ for all $h \in H$ and some $k\}$, where $xR_h=xh$. If $L_\alpha \neq \{0\}$, then L_α is called a root space of L and α is called a root. The root spaces of L give a vector space decomposition of L of the following form

$$L = L_0 \oplus L_\alpha \oplus L_\beta \oplus \cdots \oplus L_\varepsilon.$$

This decomposition has the property that $L_\alpha L_\beta \subseteq L_{\alpha+\beta}$ where $L_{\alpha+\beta}=\{0\}$ if $\alpha+\beta$ is not a root. Because H is a Cartan subalgebra we have $L_0=H$. When the Lie algebra L is of characteristic $p \neq 0$, then R_x^p is a derivation and we say L is restricted if there exists $y \in L$ so that $R_x^p=R_y$, and we denote this y by $x^{[p]}$. If L is simple y is uniquely determined by x.

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(1) This work is essentially the author's doctoral thesis presented at the University of Chicago, September 1964.

A Cartan subalgebra H is diagonalizable if there is a basis $u_1, \ldots, u_n$ of H so that $u_i^{[p]}=u_i$ for $i=1, \ldots, n$. An invariant form f on L is a symmetric, bilinear form with the added property that $f(xy, z)=f(x, yz)$ for all x, y and z in L.

The Lie algebras, L, which will be classified in this paper must satisfy the following hypotheses:

(i) L is a simple, restricted Lie algebra over an algebraically closed field of characteristic 2.

(ii) L has an abelian diagonal Cartan subalgebra H.

(iii) L has a nonsingular invariant form f.

(iv) Let α be a root of H, and h_α the unique element in H such that $\alpha(h)=f(h, h_\alpha)$ for all $h \in H$. If L_α and L_β are 2 root spaces, $\alpha \neq \beta$, such that $L_\alpha L_\beta \neq \{0\}$ then $f(h_\alpha, h_\beta)=1$.

(v) If L_α, L_β, L_γ, and L_δ are distinct root spaces, no one containing the product of two others, such that $L_\alpha L_\beta \neq \{0\}$, $L_\alpha L_\gamma \neq \{0\}$, $L_\alpha L_\delta \neq \{0\}$, $L_\beta L_\gamma \neq \{0\}$ and $L_\beta L_\delta \neq \{0\}$ then $L_\gamma L_\delta \neq \{0\}$.

Unless specifically stated otherwise, when we use the term Lie algebra the hypotheses above are implied.

3. **Structure theory.** Our main result is the following:

STRUCTURE THEOREM. *Let L be a Lie algebra satisfying the general hypotheses of §2. The dimension r of the Cartan subalgebra is unique, r is even, and for $r=2$ exactly 3 nonisomorphic simple Lie algebras exist; for $r=2n$, $n>1$, 4 nonisomorphic simple Lie algebras exist.*

A matrix representation of these algebras will be displayed in §4.

The following general results will be used, but the proofs will not be given. See [4].

LEMMA 1. *Let f be a nonsingular, invariant form on L. Let H be a Cartan subalgebra of L and let L_α, $\alpha \neq 0$, be a root space relative to H. Then f is nonsingular on H and $L_{-\alpha}$ is the dual space of L_α relative to f.*

Notice that in characteristic 2, $L_\alpha = L_{-\alpha}$, which implies that f is nonsingular on L_α. Also the nonsingularity of f on H gives the existence of $h_\alpha \in H$ such that $f(h, h_\alpha)=\alpha(h)$ for all $h \in H$.

LEMMA 2. *If f is a nonsingular invariant form on L, $x \in L_\alpha$ and $y \in L_{-\alpha}$, then $xy=-f(x, y)h_\alpha$.*

In Lemmas 1 and 2 no additional hypothesis beyond that stated is required. The Lie algebras referred to in the following lemmas and theorems will satisfy the requirements of §2.

LEMMA 3. *If $L_\alpha L_\beta=0$, $\alpha \neq \beta$, then $f(h_\alpha, h_\beta)=0$.*

Proof. Take $w, x \in L_\alpha$ such that $wx = h_\alpha$, and $y, z \in L_\beta$ so that $yz = h_\beta$. Then

$$f(h_\alpha, h_\beta) = f(wx, yz) = f(w, x(yz)) = 0$$

since $x(yz) = (xy)z + y(xz) = 0$.

LEMMA 4. *If f is a nonsingular invariant form on a Lie algebra L of characteristic 2 such that $L^2 = L$, then f is alternate.*

Proof. Take $x \in L$ such that $x = yz$, for some $y, z \in L$. $f(x, x) = f(yz, yz) = f(y, yz^2) = f(yy, z^2) = 0$. The general case now follows from f's symmetry, since every element of L is of the form $\sum y_i z_i$.

LEMMA 5. *The rank of L is even.*

Proof. Since f is alternate and nonsingular on each root space, including H, then H and every other root space is even dimensional.

LEMMA 6. *If $f(h_\alpha, h_\beta) = 1$, then $f(h_\alpha, h_{\alpha+\beta}) = f(h_\beta, h_{\alpha+\beta}) = 1$.*

Proof. $f(h_\alpha, h_{\alpha+\beta}) = f(h_\alpha, h_\alpha + h_\beta) = 1$.

LEMMA 7. *If $L_\alpha L_\beta = 0$, then $L_\beta h_\alpha = 0$.*

Proof. Let x be an element of L_β. $xh_\alpha = \beta(h_\alpha)x = f(h_\alpha, h_\beta)x = 0$, since $f(h_\alpha, h_\beta) = 0$ from Lemma 3.

LEMMA 8. *For any root $\alpha \neq 0$ there is a root $\beta \neq 0$, and $\beta \neq -\alpha$, such that $L_\alpha L_\beta \neq 0$.*

Proof. Let $A = L_\alpha \oplus$ span of h_α. If every $L_\alpha L_\beta = 0$, then $AL_\beta = 0$, and also $AH \subseteq A$. Hence A is a nonzero ideal, so $A = L$, but then L is of rank 1, which is impossible by Lemma 5.

LEMMA 9. *Let $\mathscr{A}$ be a nonempty set whose elements are root spaces of L, with the property that if $L_\alpha \in \mathscr{A}$, and L_β is such that $L_\alpha L_\beta \neq 0$ then $L_{\alpha+\beta} \in \mathscr{A}$. Then $\mathscr{A}$ contains all the root spaces of L.*

Proof. Let $A =$ the direct sum of all the $L_\alpha \in \mathscr{A}$, let $B =$ the span of all h_α such that $L_\alpha \in \mathscr{A}$, and let $C = A \oplus B$. We will show that C is an ideal. Given any root space L_β, it must follow from the definition of $\mathscr{A}$, that $AL_\beta \subseteq C$. If $L_\beta \in \mathscr{A}$ then $BL_\beta = L_\beta \subseteq C$, and if $L_\beta \notin \mathscr{A}$, then $BL_\beta = 0$ by Lemma 7. $CH \subseteq C$ is clear, so that C is an ideal. Hence $C = L$ and $\mathscr{A}$ contains every root space of L.

DEFINITION. The bilinear form f^*, on the roots is defined by $f^*(\alpha, \beta) = f(h_\alpha, h_\beta)$. ($f^*$ is just the restriction to the roots of the dual of f on the dual space of H.)

At this point we can limit ourselves to working with the roots. The theory needed here uses only a vector space setting.

Let V be a vector space over the field of integers modulo 2, with a nonsingular, bilinear, alternate form f^*. Let Γ be a set of nonzero vectors in V with the following properties:

1. Γ spans V.
2. If $\alpha, \beta \in \Gamma$ and $f^*(\alpha, \beta) = 1$, then $\alpha + \beta \in \Gamma$.

3. If α, β, γ, and $\delta \in \Gamma$ are distinct, no 3 linearly dependent, with $f^*(\alpha, \beta)=f^*(\alpha, \gamma)=f^*(\alpha, \delta)=f^*(\beta, \gamma)=f^*(\beta, \delta)=1$ then $f^*(\gamma, \delta)=1$.

4. Γ is indecomposable under f^*, that is Γ cannot be partitioned into 2 nonempty, orthogonal subsets.

Notice that if Γ is taken as the set of nonzero roots of H and V is their span over the 2 element field, then all the above requirements are met by V, Γ, and f^* as defined on the roots. This follows because H^* has a basis of roots, and if β is any other root, β is a linear combination of this basis with coefficients from the 2 element field, since f^* is nonsingular and its values on roots are 0 or 1. Lemma 9 implies condition 4, since a partitioning of the roots into 2 nonempty, orthogonal subsets gives rise to 2 disjoint nonempty subsets $\mathscr{A}$ of root spaces.

DEFINITION. For $\alpha \in \Gamma$, let K_α be the number of elements $\beta \in \Gamma$ such that $f^*(\alpha, \beta)=1$. Lemma 6 shows for the Lie algebra example above, K_α is even, and this is easily seen to be generally true.

DEFINITION. A pair β, γ nonorthogonal to $\alpha \in \Gamma$ is a set of 2 elements β and $\gamma \in \Gamma$ such that $\alpha=\beta+\gamma$ and $f^*(\alpha, \beta)=f^*(\alpha, \gamma)=1$.

LEMMA 10. *If $\alpha, \beta \in \Gamma$ with $f^*(\alpha, \beta)=1$ and γ, δ is a pair nonorthogonal to α, then either γ or δ (but not both) is nonorthogonal to β.*

Proof. Because γ, δ is a pair nonorthogonal to α, we have $\gamma=\alpha+\delta$. If $f^*(\beta, \delta)=0$, then $f^*(\beta, \gamma)=1$ and if $f^*(\beta, \delta)=1$ then $f^*(\beta, \gamma)=0$.

THEOREM 1. *$K_\alpha=K_\delta$ for all α and $\delta \in \Gamma$.*

Proof. Assume first that $f^*(\alpha, \delta)=1$. Let S_α be the subset of Γ which is nonorthogonal to α. Then S_α is a disjoint union of nonorthogonal pairs (β_1, γ_1), $(\beta_2, \gamma_2), \ldots, (\beta_m, \gamma_m)$ where β_i is chosen so that $f^*(\delta, \beta_i)=1$. S_δ is similarly defined. The mapping $S_\alpha \to S_\delta$ given by $\beta_i \to \beta_i$ and $\gamma_i \to \beta_i+\delta$ is one to one. ($\beta_i=\beta_j+\delta$ would imply $1=f^*(\alpha, \beta_i)=f^*(\alpha, \beta_j)+f^*(\alpha, \delta)=1+1=0$.) So $K_\alpha \leqq K_\delta$, and symmetry then implies $K_\alpha=K_\delta$.

In the general case where $f^*(\alpha, \delta)=1$ is not assumed the 4th hypothesis for Γ says in effect that there are elements $\alpha_1, \alpha_2, \ldots, \alpha_n$ in Γ so that $f^*(\alpha_i, \alpha_{i+1})=1$, and $\alpha_1=\alpha$, $\alpha_n=\delta$. The first part of the proof implies $K_{\alpha_i}=K_{\alpha_{i+1}}$, hence $K_\alpha=K_\delta$.

From here on the common value of K_α will simply be referred to as K, or K_Γ if more than one subset such as Γ is under consideration.

DEFINITION. Pick an arbitrary element $\alpha \in \Gamma$. Define $A \subseteq \Gamma$ by

$$A = \{\beta \in \Gamma : f^*(\alpha, \beta) = 1\} \cup \{\alpha\}.$$

Define $B \subseteq \Gamma$ by $B=\Gamma-A$.

LEMMA 11. *For $K>4$, $B \neq \varnothing$.*

Proof. Let $\beta, \beta+\alpha$, and $\gamma, \gamma+\alpha$ be distinct nonorthogonal pairs for α, with $f^*(\beta, \gamma)=1$. Then $\beta+\gamma \in \Gamma$ and $f^*(\alpha, \beta+\gamma)=0$. If $\beta+\gamma \in A$ then $\alpha=\beta+\gamma$ which shows the nonorthogonal pairs are not distinct.

LEMMA 12. *If* $\gamma, \delta \in B$ *with* $f^*(\gamma, \delta)=1$ *then* $\gamma+\delta \in B$.

Proof. Because $\gamma, \delta \in B$ we have $f^*(\alpha, \gamma)=f^*(\alpha, \delta)=0$, which implies $f^*(\alpha, \gamma+\delta)=0$. If $\gamma+\delta \notin B$ then $\gamma+\delta=\alpha$, but then $1=f^*(\gamma, \delta)=f^*(\gamma, \gamma+\alpha)=0$, so we see $\gamma+\delta \in B$.

LEMMA 13. *Each element in B is nonorthogonal to some element in A.*

Proof. Since Γ is indecomposable at least one element of B, say γ, is nonorthogonal to some element of A, say β. Take $\delta \in B$ nonorthogonal to γ. Either $f^*(\beta, \delta)=1$ or $f^*(\beta, \gamma+\delta)=1$. In the latter case $\beta+\gamma+\delta \in \Gamma$, and since if $\beta+\gamma+\delta$ were in B, then β would be in B, hence $\beta+\gamma+\delta$ is in A and $f^*(\delta, \beta+\gamma+\delta)=1$.

The proof is now completed using the indecomposability of Γ once more. Let C be the subset of B consisting of all elements which are orthogonal to every element of A. The paragraph above shows that elements in C are also orthogonal to all elements in $B-C$. Hence $A \cup (B-C)$ and C decompose Γ, which implies that C is empty.

LEMMA 14. *Each element in B is nonorthogonal to exactly* 4 *elements of A.*

Proof. Take $\gamma \in B$ and $\beta \in A$ so that $f^*(\beta, \gamma)=1$. $\alpha+\beta$ is in A and $f^*(\alpha+\beta, \gamma)=1$ also. Because $\gamma \notin A$, it follows that $\beta+\gamma \in A$ and $f^*(\gamma, \beta+\gamma)=1$ and $f^*(\gamma, \alpha+\beta+\gamma)=1$. Hence γ is nonorthogonal to 4 elements of A, which can easily be seen to be distinct.

Suppose another element $\delta \in A$ was nonorthogonal to γ. Then $\delta+\alpha$ would also be nonorthogonal to γ. Assume $f^*(\beta, \delta)=1$, (else $f^*(\beta, \delta+\alpha)=1$), then we have $f^*(\delta, \beta+\gamma)=0$. Now β, $\beta+\gamma$, and δ are nonorthogonal to α. Both $\beta+\gamma$ and δ are nonorthogonal to β. Also α, β, $\beta+\gamma$, and δ are distinct and no three are linearly dependent. So hypothesis 3 requires that $f^*(\delta, \gamma+\beta)=1$. This contradiction implies no δ can exist.

LEMMA 15. *Given* 2 *distinct elements in* Γ, *there is a third element in* Γ *nonorthogonal to both of them.*

Proof. Let $\beta, \gamma \in \Gamma$ be arbitrary, $\beta \neq \gamma$. If $f^*(\beta, \gamma)=1$ then we can use $\beta+\gamma$ as the element nonorthogonal to both β and γ. If $f^*(\beta, \gamma)=0$, use β as the arbitrary α to define A. Then $\gamma \in B$ and Lemma 13 shows the existence of $\delta \in A$ so that $f^*(\gamma, \delta)=1$ and $\delta \in A$ implies $f^*(\beta, \delta)=1$.

For the arbitrary, but fixed $\alpha \in \Gamma$ let $(\beta_1, \gamma_1), (\beta_2, \gamma_2), \ldots, (\beta_n, \gamma_n)$ be the $K/2$ pairs which are nonorthogonal to it, with the convention that $f^*(\beta_1, \beta_i)=1$, $i=2, \ldots, n$. Note that property 3 then shows $f^*(\beta_i, \beta_j)=1$ if $i \neq j$.

DEFINITION. $\Lambda=\{\alpha, \beta_1, \beta_2, \ldots, \beta_n\}$.

LEMMA 16. Λ *spans* V.

Proof. We need only show that Γ is a subset of the span of Λ. The set Λ spans all the elements in A since $\gamma_i=\alpha+\beta_i$. For $\delta \in B$, take a $\beta_i \in \Lambda$ so that $f^*(\beta_i, \delta)=1$,

(the proof of Lemma 14 shows this is always possible) then $\beta_i+\delta \in A$, hence δ is in the span of Λ.

LEMMA 17. *For* $1 \leqq i \leqq n$, $\beta_i+\alpha$ *is orthogonal to* $\beta_1, \beta_2, \ldots, \beta_{i-1}$ *and is nonorthogonal to* β_i.

Proof. $f^*(\alpha+\beta_i, \beta_j)=f^*(\alpha, \beta_j)+f^*(\beta_i, \beta_j)=0$ if $i \neq j$ and 1 if $i=j$.

LEMMA 18. $\beta_1, \beta_2, \ldots, \beta_n$ *is a linearly independent set.*

Proof. β_1 and β_2 are linearly independent since $f^*(\beta_1, \beta_2)=1$. Assume that $\beta_1, \beta_2, \ldots, \beta_{i-1}$ is an independent set, if $\beta_1, \beta_2, \ldots, \beta_i$ is a linearly dependent set then $\beta_i=\sum b_j\beta_j$, $1 \leqq j \leqq i-1$. But then $1=f^*(\alpha+\beta_i, \beta_i)=f^*(\alpha+\beta_i, \sum b_j\beta_j)=0$, a contradiction, hence $\beta_1, \ldots, \beta_i$ is a linearly independent set for $1 \leqq i \leqq n$.

THEOREM 2. *Assume a vector space* V *satisfying hypotheses* 1 *through* 4 *exists for a given* K. *Then if* K *is divisible by* 4 *the dimension of* V *is* $K/2$, *if* K *is not divisible by* 4 *the dimension of* V *is* $K/2+1$.

Proof. Lemma 18 shows that the dimension of V is at least $K/2$ and Lemma 16 shows that it is not greater than $K/2+1$. Now using the fact that f^* is alternate and nonsingular we see that the dimension of V is even.

LEMMA 19. *On an even dimensional space* V *with bases* $x_1, x_2, \ldots, x_n$ *over a field of characteristic* 2 *the bilinear form* f^* *defined as* $f^*(x_i, x_j)=0$ *if* $i=j$ *and* 1 *if* $i \neq j$ *is nonsingular.*

Proof. Suppose $\delta \in V$ with $f^*(\delta, x_i)=0$ for each i. Let $\delta=\sum d_i x_i$, then $f^*(\delta, x_1+x_j)=d_1+d_j$, so that $d_1=d_j$ for each j. Hence $\delta=d_1 \sum x_i$ and $d_1=f^*(\delta, x_1)=0$, showing that $\delta=0$.

The nonsingularity of f^* on V implies a Γ with $K=4$ is impossible. If a Γ with $K=4$ occurred then β_1, β_2 would be a basis of V. Also $f^*(\beta_2, \alpha)=f^*(\beta_1, \alpha)=1$ and $f^*(\beta_1+\beta_2, \beta_1)=f^*(\beta_1+\beta_2, \beta_2)=1$, showing that for $\delta=\alpha+\beta_1+\beta_2$, $f^*(\delta, \beta_i)=0$, so either f^* is singular, or $\delta=0$ which implies $\beta_2=\alpha+\beta_1$ showing that $K<4$.

LEMMA 20. *For* $K=6$, Γ *has exactly* 10 *elements. For* $K=8$, Γ *has exactly* 15 *elements.*

Proof. Assume $K=6$. Take $\alpha \in \Gamma$, let the 3 pairs nonorthogonal to α be β_1, γ_1; β_2, γ_2; and β_3, γ_3, with β_1 nonorthogonal β_2 and β_3. With this information Γ is already determined. The pairs nonorthogonal to elements of Λ will be displayed.

α	β_1	β_2	β_3
β_1, γ_1	α, γ_1	α, γ_2	α, γ_3
β_2, γ_2	β_2, δ_{12}	β_1, δ_{12}	β_1, δ_{13}
β_3, γ_3	β_3, δ_{13}	β_3, δ_{23}	β_2, δ_{23}

where $\delta_{ij}=\beta_i+\beta_j$, $i \neq j$.

Here $A=\{\alpha, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3\}$ and $B=\{\delta_{12}, \delta_{13}, \delta_{23}\}$. The checking that the conditions 1 through 4 are satisfied is straightforward and will be omitted.

For $K=8$ we see in a similar manner that the pairs nonorthogonal to the elements of Λ are

α	β_1	β_2	β_3	β_4
β_1, γ_1	α, γ_1	α, γ_2	α, γ_3	α, γ_4
β_2, γ_2	β_2, δ_{12}	β_1, δ_{13}	β_1, δ_{13}	β_1, δ_{14}
β_3, γ_3	β_3, δ_{13}	β_3, δ_{23}	β_2, δ_{23}	β_2, δ_{24}
β_4, γ_4	β_4, δ_{14}	β_4, δ_{24}	β_4, δ_{34}	β_3, δ_{34}

$A=\{\alpha_1, \beta_1, \beta_2, \beta_3, \beta_4, \gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ and $B=\{\delta_{12}, \delta_{13}, \delta_{14}, \delta_{23}, \delta_{24}, \delta_{34}\}$ so that Γ has 15 elements.

LEMMA 21. *Let W be the subspace of V spanned by B. Then in W, B satisfies the same hypotheses as Γ did in V.*

Proof. The hypotheses will be taken in order.

1. B spans W is given.

2. If $\eta, \nu \in B$ with $f^*(\eta, \nu)=1$, then we have already seen by Lemma 12 that $\eta+\nu \in B$.

3. If γ, δ, η, and ν are distinct in B, no 3 linearly dependent, with $f^*(\gamma, \delta) =f^*(\gamma, \eta)=f^*(\gamma, \nu)=f^*(\delta, \eta)=f^*(\delta, \nu)=1$ then $f^*(\eta, \nu)=1$ since this held already in Γ.

4. B is indecomposable. To show this it is sufficient to show that for any 2 elements in B either they are nonorthogonal or there is a third element in B nonorthogonal to both of them. Let η, ν be 2 elements in B. Lemma 15 gives us a $\delta \in \Gamma$ nonorthogonal to η and ν. If $\delta \in B$ we are done, if $\delta \in A$ we will show $f^*(\eta, \nu)=1$. The elements α, δ, $\delta+\eta$, and $\delta+\nu$ satisfy hypothesis 3 in Γ. From this we conclude $f^*(\delta+\eta, \delta+\nu)=1$ which implies $f^*(\eta, \nu)=1$.

LEMMA 22. *If $K \geqq 10$, f^* is nonsingular on W.*

Proof. By Lemma 14, $K_B=K-4 \geqq 6$. Using the notation $\delta_{ij}=\beta_i+\beta_j$ for $i \neq j$, we see that the set $\{\delta_{12}, \delta_{13}, \ldots, \delta_{1n}\}$ forms a spanning set Λ_B in B. $f^*(\delta_{1i}, \delta_{1j}) = f^*(\beta_1+\beta_i, \beta_1+\beta_j)=1$, hence by Lemma 19 f^* is nonsingular on W.

THEOREM 3. *Γ has exactly $(K^2+6K+8)/8$ elements.*

Proof. We will use induction on K, where K is a positive even integer. The cases $K=6$ and $K=8$ are done in Lemma 20. Now assume $K \geqq 10$, and that the theorem is true for smaller values of K. By Lemmas 21 and 22, B satisfies the hypotheses 1 through 4 in W, hence B has

$$\frac{(K-4)^2+6(K-4)+8}{8}$$

elements. Since A has $K+1$ elements, then Γ has

$$K+1+\frac{(K-4)^2+6(K-4)+8}{8}=\frac{K^2+6K+8}{8}$$

elements.

THEOREM 4. *The structure of Γ relative to f^* is completely determined by K. I.e. let V_i, Γ_i and f_i^*, for $i=1, 2$ be respectively a vector space over the two element field, a subset of V_i, and a nonsingular, alternate, bilinear form. Let both of these systems satisfy the hypotheses* 1 *through* 4. *Let Γ_1 and Γ_2 have the same K value. Then there exists an isomorphism Φ of V_1 onto V_2 such that Γ_1 is mapped onto Γ_2 and if $\eta, \nu \in V_1$ then $f_1^*(\eta, \nu)=f_2^*(\Phi(\eta), \Phi(\nu))$.*

Proof. Define Λ_i as was done earlier, denoting Λ_i by $\{\alpha^i, \beta_1^i, \ldots, \beta_n^i\}$. Define $\Phi\colon \Lambda_1 \to \Lambda_2$ by $\Phi(\alpha^1)=\alpha^2$ and $\Phi(\beta_j^1)=\beta_j^2$. Extend Φ linearly to an isomorphism of V_1 onto V_2. (Note in the cases where Λ_1 and Λ_2 are not bases that $\alpha^i=\beta_1^i+\beta_2^i+\cdots+\beta_n^i$ so that it is always possible to extend Φ linearly.) For $\eta, \nu \in \Lambda_1$ we have $f_1^*(\eta, \nu)=f_2^*(\Phi(\eta), \Phi(\nu))$. The linearity of Φ and the bilinearity of f_1^* and f_2^* imply this same equality for $\eta, \nu \in V_1$.

The definition of Φ shows that $\Phi(\delta_{ij}^1)=\delta_{ij}^2$ where $\delta_{ij}=\beta_i+\beta_j$. Hence Φ carries $\Lambda_{B_1}(=\{\delta_{12}^1, \delta_{13}^1, \ldots, \delta_{1n}^1)$ onto Λ_{B_2}. Using induction on k, the extension $\Psi\colon W_1 \to W_2$ of Φ restricted to Λ_{B_1} has the property that B_1 is carried onto B_2. That A_1 is carried onto A_2 is seen by inspection, hence Φ carries Γ_1 onto Γ_2.

At this point we can summarize the relation between Γ, f^*, and K with the following notation which will be useful later.

$$A = \{\alpha, \beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_n\}, \qquad B = \{\delta_{12}, \delta_{13}, \ldots, \delta_{n-1,n}\}$$

where $n=K/2$, $\gamma_i=\alpha+\beta_i$ and $\delta_{ij}=\beta_i+\beta_j=\gamma_i+\gamma_j$.

$$f^*(\alpha, \beta_i) = f^*(\alpha, \gamma_i) = 1, \qquad f^*(\alpha, \delta_{ij}) = 0$$

$$f^*(\beta_i, \beta_j) = 1 \quad \text{if } i \neq j, \qquad f^*(\beta_i, \gamma_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$f^*(\beta_i, \delta_{jk}) = f^*(\gamma_i, \delta_{jk}) = \begin{cases} 0 & i, j, k \text{ all distinct} \\ 1 & i = \text{either } j \text{ or } k \end{cases}$$

$$f^*(\delta_{ij}, \delta_{kl}) = \begin{cases} 1 & i = \text{either } k \text{ or } l \text{ and } j = \text{neither } k \text{ nor } l; \text{ or} \\ & j = \text{either } k \text{ or } l \text{ and } i = \text{neither } k \text{ nor } l; \\ 0 & \text{all other conditions.} \end{cases}$$

LEMMA 23. *If $\eta \in \Gamma$ then the annihilator of η under f^* has a basis of vectors from Γ.*

Proof. Let α be this given η. W is contained in the annihilator of α and has spanning set $\delta_{12}, \delta_{13}, \ldots, \delta_{1n}$; α is also contained in the annihilator of α. Hence the subspace spanned by $\alpha, \delta_{12}, \ldots, \delta_{1n}$ is contained in the annihilator of α, however this subspace has dimension 1 less than the dimension of V, hence it is the annihilator of α.

At this point we return to the Lie algebra context. Γ is the set of roots of the Cartan subalgebra H. If $K/2$ is even, H has dimension $K/2$; otherwise H has dimension $K/2+1$.

THEOREM 5. *If $a \in L_\alpha$, then a^2 is a scalar multiple of h_α.*

Proof. Take h_β such that $f^*(\alpha, \beta)=0$, $h_\beta = xy$; $xy \in L_\beta$. Then

$$f(a^2, h_\beta) = f(a^2, xy) = f(ax, ay) = 0$$

because $ax=0$ since $f(h_\alpha, h_\beta)=0$ implies $L_\alpha L_\beta = 0$. We also see that $a^2 \in H$ since a^2 is in the normalizer of H, because $ha^2 = \alpha(h)aa = 0$. a^2 is orthogonal to every h_β which is orthogonal to h_α. Lemma 23 then implies that a^2 is a scalar multiple of h_α.

The next series of lemmas will lead to the theorem showing that each root space has dimension 2, 4, or 8. For notational convenience let A, B, and C be root spaces of L so that $0 \neq AB \subseteq C$.

LEMMA 24. *A, B, and C all have the same dimension and $AB=C$.*

Proof. Take $c \in C$ so that $c^2 = h_\gamma$, where $C = L_\gamma$. Note that such a c can be found, for if $c^2=0$ for each $c \in C$ then for $c, d \in C$ we have $x(c+d)^2 = (x(c+d))(c+d) = xc^2 + (xc)d + (xd)c + xd^2$ so that $(xc)d + (xd)c = 0$. Then by the Jacobi identity we have $x(cd)=0$, hence $CC=0$ contradicting the nonsingularity of f on C. h_γ acts as the identity on A and B so we have

$$A \xrightarrow{R_c} Ac \xrightarrow{R_c} (Ac)c = Ac^2 = A.$$

Since $Ac \subseteq B$ we see that $\dim B \geqq \dim A$, and reversing the roles of A and yields equality. Permuting A, B, and C shows that all dimensions are equal and that $AB=C$.

LEMMA 25. *All root spaces have the same dimension.*

Proof. Apply Lemma 9.

The following four lemmas follow Kaplansky's work in [6] very closely. We will continue to use the notation of Lemma 24 for root spaces A, B, and C. As shown before, if X is a root space then XX is the span of h_η where $X = L_\eta$. We will now index that h by the lower case letter corresponding to X, i.e. XX=span of h_x. For each root space X, define the map $g: X \to F$ (the field) by $y^2 = g(y)h_x$ for $y \in X$.

LEMMA 26. *g is a quadratic form on each root space.*

Proof. $g(\lambda y)h_x = (\lambda y)^2 = \lambda^2 y^2 = \lambda^2 g(y)h_x$ hence $g(\lambda y) = \lambda^2 g(y)$. For $y, z \in X$ we have

$$f(y, z)h_x = yz = (y+z)^2 + y^2 + z^2 = (g(y+z) + g(y) + g(z))h_x.$$

Hence $g(y+z)+g(y)+g(z)$ is a bilinear form, in fact it is the bilinear f that we started with.

LEMMA 27. *If x is in one of the root spaces A, B, or C; and y is in a different one; then $g(xy)=g(x)g(y)$.*

Proof. Assume $x \in A$ and $y \in B$.

$$\begin{aligned} g(xy)x &= g(xy)h_c x = x(xy)^2 = (x(xy))(xy) = (g(x)h_a y)(xy) \\ &= g(x)xy^2 = g(x)g(y)h_b x = g(x)g(y)x. \end{aligned}$$

A similar proof for B, C or A, C can be constructed.

Since f is nonsingular on each root space we can take $b \in B$ and $c \in C$ so that $g(b)=1=g(c)$. In A we define a new product $*$ by

$$a_1 * a_2 = (a_1 b)(a_2 c).$$

LEMMA 28. *A has a 2-sided identity relative to $*$-multiplication.*

Proof. Let $a=bc$, and $x \in A$.

$$\begin{aligned} x * a &= (xb)(ac) = (xb)(bc^2) = (xb)b = xb^2 = x. \\ a * x &= (ab)(xc) = (cb^2)(xc) = c(xc) = xc^2 = x. \end{aligned}$$

THEOREM 6. *In L all root spaces have the same dimension, which is* 2, 4, *or* 8.

Proof. On A with $*$-multiplication, the quadratic form, g, admits composition.

$$\begin{aligned} g(a_1 * a_2) &= g((a_1 b)(a_2 c)) = g(a_1 b)g(a_2 c) \\ &= g(a_1)g(b)g(a_2)g(c) \\ &= g(a_1)g(a_2). \end{aligned}$$

Hence the results of [5] apply. Therefore A has dimension 1, 2, 4, or 8; but 1 is impossible.

THEOREM 7. *Let L and $\bar{L}$ be 2 Lie algebras of characteristic 2 meeting our general hypotheses. Let each of them have the same K value and the same root space dimension. Then L is isomorphic to $\bar{L}$.*

Proof. We will construct an isomorphism $\theta: L \to \bar{L}$. Decompose L and $\bar{L}$ into Cartan subalgebras and root spaces. Since K determines the dimension of the Cartan subalgebras and the number of root spaces, it follows that L and $\bar{L}$ have the same dimension. The roots of L, Γ are denoted as before

$$\{\alpha, \beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_n, \delta_{12}, \ldots, \delta_{n-1,n}\}$$

and the corresponding root spaces of L, A, $B_1, \ldots, B_n$, $C_1, \ldots, C_n$, $D_{12}, \ldots, D_{n-1,n}$. Use the same notation with a bar to denote the roots and root spaces of $\bar{L}$.

From the theory of quadratic forms admitting composition there is a map $\Phi: A \to \bar{A}$ which preserves $*$-multiplication and the quadratic form. Let $a \in A$ be such that $g(a)=1$ and let $\bar{a}=\Phi(a)$, so that $\bar{g}(\bar{a})=1$. Choose $b_i \in B_i$ so that $g(b_i)=1$, define $c_i=ab_i$, and $d_{ij}=b_i b_j$, for $i \neq j$. Because of Lemma 27, $g(c_i)=g(d_{ij})=1$.

These elements are the $*$-multiplication identities in the various root spaces in which they lie. We also have the relations $ac_i=b_i$; $ad_{ij}=0$; $b_ic_j=0$, $i\neq j$; $b_id_{ij}=b_j$, $i\neq j$; $c_ic_j=d_{ij}$, $i\neq j$; $c_id_{ij}=c_j$, $i\neq j$; $d_{ij}d_{ik}=d_{jk}$, $i\neq k$; and $d_{ij}d_{kl}=0$ if all the indices are distinct. In a similar manner pick the $*$-multiplication identities in the root spaces of $\bar{L}$.

The question, to which $*$-multiplication do we refer to, can be raised here. The fact is that it makes no difference once the b_i, c_i, and d_{ij}'s have been chosen. For example, if we use b_1 and c_1 to define $*_1$-multiplication in A, or b_2 and c_2 to define $*_2$-multiplication on A we will have $x *_1 y=x *_2 y$ for all $x, y\in A$. However, in this proof we need only assume that $*$-multiplication on A is defined by b_1 and c_1 and similarly for $\bar{A}$.

Construct θ as follows

$$\begin{aligned}
\theta(x) &= \Phi(x), && x\in A\\
\theta(x) &= \bar{c}_i\theta(c_ix), && x\in B_i\\
\theta(x) &= \bar{b}_i\theta(b_ix), && x\in C_i\\
\theta(x) &= \bar{b}_j\theta(b_jx), && x\in D_{ij},\ i<j.
\end{aligned}$$

Let $h_0=a^2$ and $h_i=b_i^2$, then $h_0, h_1, \ldots, h_n$ span H; and repeat this with bars in $\bar{H}$. Define

$$\theta(h_i) = \bar{h}_i.$$

$\theta: L\to\bar{L}$ is the linear extension of θ as defined above. We wish to show that θ is an isomorphism. θ preserves addition by definition. The verification that θ preserves multiplication will be broken into several cases.

(i) $x, y\in A$.

$$\theta(xy) = \theta(f(x, y)h_0) = \bar{f}(\Phi(x), \Phi(y))\bar{h}_0 = \Phi(x)\Phi(y) = \theta(x)\theta(y).$$

(ii) x in any root space, L_η.

$$\theta(xh_i) = \theta(f(h_i, h_\eta)x) = \bar{f}(\bar{h}_i, \bar{h}_\eta)\theta(x) = \bar{h}_i\theta(x) = \theta(x)\theta(h_i).$$

(iii) $x, y\in B_i$.

$$\begin{aligned}
\theta(xy) &= \theta(f(x, y)h_i) = f(x, y)\theta(h_i) = f(xc_i^2, y)\theta(h_i)\\
&= f(xc_i, yc_i)\bar{h}_i = \bar{f}(\Phi(xc_i), \Phi(yc_i))\bar{h}_i\\
&= \bar{f}(\Phi(xc_i)\bar{c}_i^2, \Phi(yc_i))\bar{h}_i = \bar{f}(\Phi(xc_i)c_i, \Phi(yc_i)c_i)\bar{h}_i\\
&= \bar{f}(\theta(x), \theta(y))\bar{h}_i = \theta(x)\theta(y).
\end{aligned}$$

A parallel argument shows $\theta(xy)=\theta(x)\theta(y)$ for $x, y\in C_i$.

(iv) $x\in B_i$, $y\in C_i$.

$$\begin{aligned}
\theta(xy) = \Phi(xy) &= \Phi(yx) = \Phi((yb_i^2)(xc_i^2)) = \Phi((yb_i) * (xc_i))\\
&= \Phi(yb_i) \bar{*} \Phi(xc_i) = (\Phi(yb_i)\bar{b}_i)(\Phi(xc_i)\bar{c}_i)\\
&= \theta(y)\theta(x) = \theta(x)\theta(y).
\end{aligned}$$

(v) $x \in A$, $y \in B_i$.

$$\begin{aligned}\theta(xy) &= \Phi((xy)b_i)\bar{b}_i = [\Phi((xb_i)y)+\Phi(x(yb_i))]\bar{b}_i\\ &= [\theta((xb_i)y)+\theta(x(yb_i))]\bar{b}_i\\ &= [\theta(xb_i)\theta(y)+\theta(x)\theta(yb_i)]\bar{b}_i\\ &= [(\theta(x)\bar{b}_i)\theta(y)+\theta(x)(\theta(y)\bar{b}_i)]\bar{b}_i\\ &= (\theta(x)\bar{b}_i^2)\theta(y)+(\theta(x)\bar{b}_i)(\theta(y)\bar{b}_i)+(\theta(x)\bar{b}_i)(\theta(y)\bar{b}_i)+\theta(x)(\theta(y)\bar{b}_i^2)\\ &= \theta(x)\theta(y).\end{aligned}$$

Note $\theta(b_i)=\theta(b_ic_i)\bar{c}_i=\theta(a)\bar{c}_i=\bar{a}\bar{c}_i=\bar{b}_i$, and $\theta(xb_i)=\theta(xb_i^2)\bar{b}_i=\theta(x)\bar{b}_i$ for $x \in A$.

(vi) $x, y \in B_i$.

$$\begin{aligned}f(x, y) &= f(xc_i^2, y) = f(xc_i, yc_i) = \bar{f}(\Phi(xc_i), \Phi(yc_i))\\ &= \bar{f}(\Phi(xc_i)\bar{c}_i^2, \Phi(yc_i)) = f(\theta(x), \theta(y)).\end{aligned}$$

(vii) $x, y \in D_{ij}$.

$$\begin{aligned}\theta(xy) &= \theta(f(x, y)(h_i+h_j)) = f(x, y)\theta(h_i+h_j) = f(xb_j^2, y)(\bar{h}_i+\bar{h}_j)\\ &= f(xb_j, yb_j)(\bar{h}_i+\bar{h}_j) = \bar{f}(\theta(xb_j), \theta(yb_j))(\bar{h}_i+\bar{h}_j)\\ &= \theta(x)\theta(y).\end{aligned}$$

(viii) $x \in B_i$, $y \in B_j$.

$$\begin{aligned}\theta(xy) &= \theta((xy)b_j)\bar{b}_j = (\Phi(((xy)b_j)c_i)\bar{c}_i)\bar{b}_j = (\Phi(((xy)b_j)c_i)\bar{c}_i)\bar{b}_j\\ &= (\Phi(((xc_i)y)b_j)\bar{c}_i)\bar{b}_j = ((\theta((xc_i)y)\theta(b_j))\bar{c}_i)\bar{b}_j\\ &= (((\theta(xc_i)\theta(y))\bar{b}_j)\bar{c}_i)\bar{b}_j = (((\theta(x)\bar{c}_i)\theta(y))\bar{b}^2)\bar{c}_i\\ &= ((\theta(x)\bar{c}_i)\theta(y))\bar{c}_i = (\theta(x)\bar{c}_i^2)\theta(y) = \theta(x)\theta(y).\end{aligned}$$

$x \in C_i$, $y \in C_j$ is similar.

(ix) $x \in B_i$, $y \in C_j$, $i \neq j$.

$$\theta(xy) = 0 = \theta(x)\theta(y).$$

$x \in D_{ij}$, $y \in D_{kl}$ all indices distinct, $x \in B_i$, $y \in D_{jk}$, or $x \in C_i$, $y \in D_{jk}$ with all indices distinct also have 0 as their product.

(x) $x \in D_{ij}$, $x = zb_i$, $z \in B_j$.

$$\begin{aligned}\theta(xb_i) &= \theta((zb_i)b_i) = \theta(zb_i^2) = \theta(z)\bar{h}_i = \theta(z)\bar{b}_i^2 = (\theta(z)\theta(b_i))\bar{b}_i\\ &= \theta(zb_i)\bar{b}_i = \theta(x)\bar{b}_i.\end{aligned}$$

$\theta(xb_j)=\theta(x)\bar{b}_j$, $\theta(xc_i)=\theta(x)\bar{c}_i$, and $\theta(xc_j)=\theta(x)\bar{c}_j$ are all similar.

(xi) $x \in D_{ij}$, $y \in D_{ik}$, $j \neq k$.

$$\begin{aligned}\theta(xy) &= \theta((xy)b_k)\bar{b}_k = (\theta(((xy)b_k)c_j)\bar{c}_j)\bar{b}_k = (\theta((xc_j)(yb_k))\bar{c}_j)\bar{b}_k\\ &= (((\theta(x)\bar{c}_j)(\theta(y)\bar{b}_k))\bar{c}_j)\bar{b}_k = (\theta(x)\bar{c}_j^2)(\theta(y)\bar{b}_k^2) = \theta(x)\theta(y).\end{aligned}$$

(xii) $x \in B_i$, $y \in D_{ij}$.

$$\begin{aligned}\theta(xy) &= \theta((xy)c_j)\bar{c}_j = \theta(x(yc_j))\bar{c}_j = (\theta(x)(\theta(y)\bar{c}_j))\bar{c}_j\\ &= \theta(x)(\theta(y)\bar{c}_j^2) = \theta(x)\theta(y).\end{aligned}$$

$x \in C_i$, $y \in D_{ij}$ is similar.

Hence θ preserves products. That θ is 1-1 and onto follows directly from the fact that Φ is. It must also be verified that θ preserves the bilinear form. However, since distinct root spaces are orthogonal under f in L, and under $\bar{f}$ in $\bar{L}$, this verification is trivial. This concludes the proof of the theorem.

We can now see that for each K there are at most 3 nonisomorphic Lie algebras, one with 2-dimensional root spaces, one with 4-dimensional root spaces, and one with 8-dimensional root spaces. The next line of attack will be to show that 8-dimensional root spaces occur only for $K=2$. For $K=4$ we have already seen that there is no Lie algebra.

DEFINITION. For L with $K>4$ given, we define an algebra L^* as follows.

1. Take root spaces A, B_1, B_2, C_1, C_2, and D_{12} using the notation of Theorem 7.
2. Take the subspace $\bar{H}$ of H spanned by h_0, h_1, and h_2.

Let $\bar{L}=\bar{H}\oplus A\oplus B_1\oplus C_1\oplus B_2\oplus C_2\oplus D_{12}$. The center C of $\bar{L}$ is spanned by $h_0+h_1+h_2$. $L^*=\bar{L}/C$, and $H^*=\bar{H}/C$ is the Cartan subalgebra for L^*.

We will show that L^* is simple and restricted, and hence, using [5] again, will arrive at the conclusion that L^* has root spaces of dimension 2, 4, or 8.

LEMMA 29. *L^* is restricted.*

Proof. We establish this result in 2 steps. First $\bar{L}$ is restricted, which is seen by noting that for each $x\in A$, $x^2=$a scalar multiple of h_0, and $h_0^2=h_0$, and similarly for the other root spaces. Then since $\bar{L}$ is restricted we will have L^* restricted because we have factored out only the center.

We wish to find the root spaces of L^*. Because h_0, h_1, and h_2 do not distinguish the pairs of root spaces A, D_{12}; B_1, C_2; and B_2, C_1 the root spaces in L^* will be

$$A^* = (A\oplus D_{12})\pi, \qquad B^* = (B_1\oplus C_2)\pi, \quad \text{and} \quad C^* = (B_2\oplus C_1)\pi$$

where π denotes the coset map from $\bar{L}$ onto L^*.

LEMMA 30. *L^* is simple.*

Proof. $L^*L^*\neq 0$. Let I^* be a proper ideal in L^*, and let $I=(I^*)\pi^{-1}$. I is a proper ideal in $\bar{L}$, we will show that $I\subseteq$the center of $\bar{L}$. Let $t\in I$, $t=a+b+c+h$ with $a\in(A^*)\pi^{-1}$, $b\in(B^*)\pi^{-1}$, $c\in(C^*)\pi^{-1}$, and $h\in\bar{H}$. Assume $a\neq 0$, then $(th_1)h_2=a$. By picking $a_1\in(A^*)\pi^{-1}$ such that $aa_1=h_0$ or $aa_1=h_1+h_2$ we see that $(B^*)\pi^{-1}$ or $(C^*)\pi^{-1}$ is in I, hence h_1 or h_2 is also in I, and then $I=\bar{L}$ will follow. So instead assume $I\subseteq\bar{H}$, but then if there is an $h\in I$, $h\notin C$, one of $(A^*)\pi^{-1}$, $(B^*)\pi^{-1}$, or $(C^*)\pi^{-1}$ will again be in I and $I=\bar{L}$ will follow. Therefore $I\subseteq C$, and $I^*=0$.

THEOREM 8. *For $K>4$ no Lie algebra exists with 8-dimensional root spaces.*

Proof. Assume L is a Lie algebra with $K>4$ and root spaces of dimension 8. Then L^* formed from this L has root spaces with dimension 16, which contradicts Theorem 6. Hence no such L exists.

At this point we have the following information. For $K=2$ three nonisomorphic Lie algebras can exist, for $K=4$ there are none, and for $K>4$, and even, two Lie algebras are possible. § 4 will exhibit an algebra of each of these types.

4. **Representations of the algebras.** In this section it will be shown that the characteristic 2 analogues of A_m and C_m are the desired algebras.

$$A_m \text{ for characteristic } 2, \qquad m \geqq 3.$$

Let $\mathfrak{L}$ be the Lie algebra of all $(m+1)\times(m+1)$ matrices, $\mathfrak{L}'$ is all matrices of $\mathfrak{L}$ which have trace 0, and C is all scalar matrices contained in $\mathfrak{L}'$. Then $A_m=\mathfrak{L}'/C$. It is known that A_m is simple, a Cartan subalgebra is the diagonal matrices, and e_{ij}, e_{ji} for $i\neq j$ span a root space when $m>3$ (see [2], [3]). For $m>3$, the K for this algebra is $2m-2$.

For $m=3$ we get an L^* type algebra with root spaces spanned by

$e_{12}, e_{21}, e_{34}, e_{43}$;

$e_{13}, e_{31}, e_{24}, e_{42}$; and

$e_{14}, e_{41}, e_{23}, e_{32}$; where each e_{ij} is taken as the representative of the coset which contains it.

It is well known that A_m is simple. See [3].

For $a, b \in A_m$ define $f(a, b)=\text{trace}\,(\bar{a}\bar{b})$ where $a=\bar{a}+C$ and $b=\bar{b}+C$. Since f is a trace form it is invariant.

LEMMA 31. *f is nonsingular on A_m.*

Proof. Since A_m is simple f is either nonsingular or identically 0; hence it is nonsingular.

Let L_{ij} denote the root space containing e_{ij} and e_{ji}.

LEMMA 32. *If distinct root spaces L_{ij} and L_{kl} multiply nontrivially, then $f(h_{ij}, h_{kl})=1$, where $h_{ij}=e_{ii}+e_{jj}$ and $\bar{h}_{kl}=e_{kk}+e_{ll}$.*

Proof. In order for $L_{ij}L_{kl}\neq 0$ there must be a common index, so assume $k=j$.

$$f(h_{ij}, h_{jl}) = f([e_{ij}, e_{ji}], h_{jl}) = f(e_{ij}, [e_{ji}, h_{jl}]) = f(e_{ij}, e_{ji}) = 1.$$

LEMMA 33. *Let root space A multiply nontrivially root spaces B, C, and D; and B multiply nontrivially C and D; and no one equal the product of 2 others. Then $CD\neq 0$.*

Proof. Let $A=L_{ij}$ and $B=L_{ik}$. Then C and D must be of the form L_{il} and L_{im}; hence $CD\neq 0$. Note neither C nor D can be of the form L_{jk} since $L_{jk}=L_{ij}L_{ik}$.

LEMMA 34. *Let $\bar{H}$ be another abelian, diagonal Cartan subalgebra of A_m. Then dimension $\bar{H}$ = dimension H; moreover H and $\bar{H}$ are conjugate under an automorphism of A_m.*

Proof. By a change of basis the matrices mapping into $\overline{H}$ can be simultaneously put in diagonal form. This change of basis sends A_m onto A_m, since trace 0 is preserved and the identity goes into the identity. Hence $\overline{H}$ is contained in the image of all diagonal matrices of A_m after the automorphism. Therefore $\overline{H}$ is the image of all diagonal matrices of A_m.

This lemma will be used to show later that the algebras of the A_m series are nonisomorphic, with the exception of rank 2, to the algebras in the C_m series. The following theorem summarizes our results for A_m.

THEOREM 9. *For rank $r, r \geqq 4$ and even, A_{r+1} and A_r are nonisomorphic Lie algebras satisfying our general hypotheses and having root spaces of dimension* 2. *For $r=2$, A_2 and A_3 also satisfy our general hypotheses and have root spaces of dimension* 2 *and* 4 *respectively.*

$$C_m \textit{ for characteristic } 2, \qquad m \geqq 6.$$

This Lie algebra can be realized in the following way. Let M be an m-dimensional vector space over an algebraically closed field F of characteristic 2. Let g be a nondegenerate alternate bilinear form on M. This g is easily seen to be symmetric $(0=g(x+y, x+y)=g(x, x)+g(x, y)+g(y, x)+g(y, y)=g(x, y)+g(y, x))$ and M must have even dimension.

C_m is defined as follows. Let $\mathfrak{L}$ be all linear transformations T on M such that $g(xT, y)=g(x, yT)$. Let $Z=$center of $[[\mathfrak{L}\mathfrak{L}]\mathfrak{L}]$, then $C_m=[[\mathfrak{L}\mathfrak{L}]\mathfrak{L}]/Z$. Let $l=m/2$. Pick a basis $u_1, \ldots, u_l, v_1, \ldots, v_l$ for M such that $g(u_i, u_j)=0=g(v_i, v_j)$ and $g(u_i, v_j)=0$ if $i \neq j$ and $=1$ if $i=j$. Relative to this basis in the order shown $\mathfrak{L}$ is represented by matrices

$$\begin{pmatrix} A & B \\ C & A' \end{pmatrix}$$

where each of A, B, and C is $l \times l$ and B and C are symmetric. $[\mathfrak{L}\mathfrak{L}]$ is represented by these matrices with the additional property that all the diagonal entries of B and C are zero. Abstractly $[\mathfrak{L}\mathfrak{L}]$ is all linear transformations T such that $g(xT, x)=0$. $[[\mathfrak{L}\mathfrak{L}]\mathfrak{L}]$ is represented by the matrices in $[\mathfrak{L}\mathfrak{L}]$ which have the additional property that trace $A=0$. Finally the center of $[[\mathfrak{L}\mathfrak{L}]\mathfrak{L}]$ consists only of the scalar multiples of the identity which are in $[[\mathfrak{L}\mathfrak{L}]\mathfrak{L}]$.

LEMMA 35. *C_m is restricted.*

Proof. The restricted product in C_m is to be the one it inherits from $[[\mathfrak{L}\mathfrak{L}]\mathfrak{L}]$. Notice $A^2=(A+I)^2$ modulo I, in the field. So the question to answer is whether $[[\mathfrak{L}\mathfrak{L}]\mathfrak{L}]$ is restricted. If $T \in \mathfrak{L}$, $g(xT^2, x)=g(xT, xT)=0$ for all x, so $T^2 \in [\mathfrak{L}\mathfrak{L}]$. If $U \in [\mathfrak{L}\mathfrak{L}]$ then, for

$$U = \begin{pmatrix} A & B \\ C & A' \end{pmatrix},$$

we get

$$\operatorname{tr}(BC) = \sum_{i,j} b_{ij}c_{ji} = \sum_{i<j} b_{ij}c_{ji} + \sum_{i>j} b_{ij}c_{ji}$$
$$= \sum_{i<j} b_{ij}c_{ji} + \sum_{j<i} b_{ji}c_{ij} = 0$$

using the symmetry of B and C and the fact that the diagonals are 0. If $U \in [[\mathfrak{L}\mathfrak{L}]\mathfrak{L}]$ then trace $A = 0 = (\text{trace } A)^2 = \text{trace } A^2$. Hence trace $(A^2 + BC) = 0$, therefore $U^2 \in [[\mathfrak{L}\mathfrak{L}]\mathfrak{L}]$, and C_m is restricted.

As in A_m the images of diagonal matrices in C_m form an abelian, diagonal Cartan subalgebra. The verification of this is straightforward.

LEMMA 36. *For $m \neq 8$ the root spaces of C_m are 4-dimensional.*

$$e_{ij} + e_{l+j,l+i}, \qquad e_{ji} + e_{l+i,l+j}, \qquad e_{i,l+j} + e_{j,l+i}, \quad \textit{and} \quad e_{l+i,j} + e_{l+j,i}$$

modulo the center span the root spaces for $i, j = 1, 2, \ldots, l$, which we will denote L_{ij}.

Proof. Direct computation verifies that the image of each diagonal matrix in C_m acts as the same scalar on the 4 spanning elements of L_{ij}. $(e_{ij} + e_{l+j,l+i}) + Z$ and $(e_{ik} + e_{k+l,i+l}) + Z$, i, j, k all different can be distinguished by

$$(e_{ii} + e_{kk} + e_{l+i,l+i} + e_{l+k,l+k}) + Z.$$

$(e_{ij} + e_{l+j,l+i}) + Z$ and $(e_{st} + e_{l+t,l+s}) + Z$, i, j, s, t all distinct, can be distinguished by $(e_{ii} + e_{uu} + e_{l+i,l+i} + e_{l+u,l+u}) + Z$, u different from i, j, s, and t. The other cases are similar to this.

It is critical that $m \neq 8$, otherwise when 5 distinct indices are needed they do not exist. For C_8 we get an L^* algebra with 8-dimensional root spaces. The lemmas preceding Theorem 8 show how the usual 4-dimensional root spaces are combined to get 8-dimensional ones. For the simplicity of C_m, see [3], [7].

The form f on C_m will be defined in the following way. It will be nonsingular on the Cartan subalgebra and on each root space. When one variable is in one root space and the other variable is in a different root space or in the Cartan subalgebra, the value of the form will be zero; hence the form will be nonsingular on C_m. For $m \neq 8$ on L_{ij} define f as follows:

$$f(e_{ij} + e_{l+j,l+i}, e_{ji} + e_{l+i,l+j}) = 1$$
$$f(e_{i,l+j} + e_{j,l+i}, e_{l+i,j} + e_{l+j,i}) = 1.$$

The form is symmetric, its value on other pairs of basis elements is zero. Let r be the dimension of H. Define

$$h_i = (e_{ii} + e_{ll} + e_{l+i,l+i} + e_{2l,2l}) + Z, \qquad i = 1, 2, \ldots, r.$$

On this basis of H we define our form as

$$f(h_i, h_j) = 0 \quad \text{if } i = j$$
$$= 1 \quad \text{if } i \neq j.$$

The invariance of this form can be shown by direct and tedious computation. For root spaces L_α and L_β which multiply nontrivially the proof that $f(h_\alpha, h_\beta)=1$ is the same as Lemma 32.

The final hypothesis that C_m must satisfy to be included in the algebras the structure theorem permits, is that given root spaces A, B, C, and D, with AB, AC, AD, BC, and BD all nonzero and no one root space the product of 2 others, then $CD \neq 0$. This is done in the same way as done for A_m in Lemma 33.

For rank 2 we have C_6 and C_8, with C_6 isomorphic to A_3. The algebras A_2, A_3, and C_8 are nonisomorphic since they have different dimensions. For rank $r>2$ we have C_{2r+2} and C_{2r+4} of dimensions $2r^2+3r$, and $2r^2+7r+4$ respectively. Also for rank $r>2$ we have A_r and A_{r+1} of dimensions r^2+2r and r^2+4r+2. The only possible equality between these dimensions is $r^2+4r+2=2r^2+3r$ which has the solution $r=2$. Hence for $r>2$ no isomorphisms occur between C_m's and A_m's of different rank, so as stated for each rank >2, there are 4 nonisomorphic Lie algebras.

THEOREM 10. *For* rank 2, C_6 *and* C_8 *are nonisomorphic Lie algebras with root space dimensions* 4 *and* 8 *respectively. For* rank $r>2$ *but even,* C_{2r+2} *and* C_{2r+4} *are nonisomorphic Lie algebras with root space dimension* 4. *With the exception of* $A_3 \approx C_6$, *there are no isomorphisms between any of these algebras in the* A_m *series and the* C_m *series.*

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