THE AUTOMORPHISM GROUP OF A HOMOGENEOUS ALMOST COMPLEX MANIFOLD (*)

BY

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1. Introduction. Let $M$ be a compact simply connected manifold of nonzero Euler characteristic that carries a homogeneous almost complex structure. We determine the largest connected group $A_0(M)$ of almost analytic automorphisms of $M$.

Our hypotheses represent $M$ as a coset space $G/K$ where $G$ is a maximal compact subgroup of the Lie group $A_0(M)$ and $K$ is a closed connected subgroup of maximal rank in $G$. In §2 we collect some information, decomposing $M = M_1 \times \cdots \times M_t$ as a product of "irreducible" factors along the decomposition of $G$ as a product of simple groups; then every invariant almost complex structure or riemannian metric decomposes and every invariant riemannian metric is hermitian relative to any invariant almost complex structure; furthermore the decomposition is independent of $G$ in a certain sense. In §3 we choose an invariant riemannian metric and determine the largest connected groups $H_0(M_i)$ of almost hermitian isometries of the $M_i$. Then $A_0(M)$ is determined in §4. There it is shown that $A_0(M) = A_0(M_1) \times \cdots \times A_0(M_t)$, that $A_0(M_i) = H_0(M_i)$ if the almost complex structure on $M_i$ is not integrable, and that $A_0(M_i) = H_0(M_i)^c$ if the almost complex structure on $M_i$ is induced by a complex structure. $A_0(M)$ thus is a centerless semisimple Lie group whose simple normal analytic subgroups are just the $A_0(M_i)$.

2. Decomposition. Let $M$ be an effective coset space of a compact connected Lie group $G$ by a connected subgroup $K$ of maximal rank. In other words $M = G/K$ is compact, simply connected and of nonzero Euler characteristic; $G$ is a compact centerless semisimple Lie group, rank $K = \text{rank } G$, and $K$ contains no simple factor of $G$. Then

(2.1a) \[ G = G_1 \times \cdots \times G_t, \quad K = K_1 \times \cdots \times K_t \quad \text{and} \quad M = M_1 \times \cdots \times M_t \]

where

(2.1b) \[ G_i \text{ is simple,} \quad K_i = K \cap G_i \quad \text{and} \quad M_i = G_i/K_i. \]

$G_i$ is a compact connected centerless simple Lie group, $K_i$ is a connected subgroup of maximal rank, and $M_i = G_i/K_i$ is a simply connected effective coset space of nonzero Euler characteristic. The decomposition of $M$ is unique up to order of the factors because it is determined by the decomposition of $G$.

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We call (2.1) the canonical decomposition of the coset space $M = G/K$. The factors $M_i = G_i/K_i$ are the irreducible factors of $M = G/K$. If there is just one irreducible factor, i.e. if $G$ is simple, then we say that $M = G/K$ is irreducible.

2.2. Proposition. Let $M$ be an effective coset space $G/K$ where $G$ is a compact connected Lie group and $K$ is a connected subgroup of maximal rank. Let $M = M_1 \times \cdots \times M_t$ be the canonical decomposition into irreducible factors $M_i = G_i/K_i$.

1. The $G$-invariant almost complex structures $J$ on $M$ are just the $J_1 \times \cdots \times J_t$ where $J_i$ is a $G_i$-invariant almost complex structure on $M_i$.

2. The $G$-invariant riemannian metrics $ds^2$ on $M$ are just the $ds_1^2 \times \cdots \times ds_t^2$ where $ds_i^2$ is a $G_i$-invariant riemannian metric on $M_i$; there each $(M_i, ds_i^2)$ is an irreducible riemannian manifold, so $$(M, ds^2) = (M_1, ds_1^2) \times \cdots \times (M_t, ds_t^2)$$ is the de Rham decomposition.

3. Let $J$ be a $G$-invariant almost complex structure on $M$. If $ds^2$ is a $G$-invariant riemannian metric, then it is the real part of a $G$-invariant almost hermitian (for $J$) metric $h$ on $M$, and $h = h_1 \times \cdots \times h_t$ where $h_i$ is a $G_i$-invariant almost hermitian (for $J_i$) metric on $M_i$ and $ds_i^2$ is the real part of $h_i$.

Proof. The Lie algebras decompose uniquely as direct sums $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ and $\mathfrak{g}_i = \mathfrak{k}_i + \mathfrak{m}_i$, $\mathfrak{k} = \sum \mathfrak{k}_i$ and $\mathfrak{m} = \sum \mathfrak{m}_i$ with $[\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}$ and $[\mathfrak{k}_i, \mathfrak{m}_i] \subseteq \mathfrak{m}_i$. Let $Z$ be the center of $K$, so $\mathfrak{k}$ is the centralizer of $Z$ in $\mathfrak{g}$. Then $Z = Z_1 \times \cdots \times Z_t$ where $Z_i$ is the center of $K_i$ and $\mathfrak{k}_i$ is the centralizer of $Z_i$ in $\mathfrak{g}_i$.

$\pi$ denotes the representation of $K$ on $\mathfrak{m}$ and $\pi_i$ is the representation of $K_i$ on $\mathfrak{m}_i$. Then $\pi = \pi_1 \oplus \cdots \oplus \pi_t$. Let $X = X_1 \cup \cdots \cup X_t$ be the set of nontrivial characters on $Z$ such that

\begin{equation}
(2.3a) \quad \mathfrak{m}_C = \sum_x \mathfrak{m}_x \quad \text{and} \quad \mathfrak{m}_i^C = \sum_{x_i} \mathfrak{m}_x
\end{equation}

where $Z$ acts on $\mathfrak{m}_x$ by the character $x$. Each $\mathfrak{m}_x$ is ad $(K)$-stable, so $K$ acts on $\mathfrak{m}_x$ by a representation $\pi_x$, and

\begin{equation}
(2.3b) \quad \pi^C = \sum_x \pi_x \quad \text{and} \quad \pi_i^C = \sum_{x_i} \pi_x.
\end{equation}

The point [7, Theorem 8.13.3] is that

\begin{equation}
(2.3c) \quad \text{the } \pi_x \text{ are irreducible and mutually inequivalent.}
\end{equation}

We transform the complex decomposition (2.3) to a real decomposition. Let $X = S \cup T$, $S = S_1 \cup \cdots \cup S_t$ and $T = T_1 \cup \cdots \cup T_t$ where $S_i$ consists of the nonreal characters in $X_i$ and $T_i$ consists of the real ones. By real partition of $X$ we mean a disjoint $X = S' \cup S'' \cup T$ where $S'' = S'$. If $x \in S_i$ then $\bar{x} \in S_i$; thus the real partition

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induces real partitions $X_i = S'_i \cup S''_i \cup T_i$. If $|S| = 2n$ then $X$ has 2$^n$ real partitions. Now choose a real partition $X = S' \cup S'' \cup T$ and define

\[ \chi \in S': \text{ } K \text{ acts on } \mathcal{M}_x^R = \mathcal{M} \cap (\mathcal{M}_x + \mathcal{M}_y) \text{ by } \pi_x^R \]

\[ \chi \in T: \text{ } K \text{ acts on } \mathcal{M}_x^R = \mathcal{M} \cap \mathcal{M}_y \text{ by } \pi_x^R. \]

Then (2.3abc) becomes

(2.4a) \[ \mathcal{M} = \sum_{S'} \mathcal{M}_x^R + \sum_{T} \mathcal{M}_x^R \text{ and } \mathcal{M}_i = \sum_{S_i} \mathcal{M}_x^R + \sum_{T_i} \mathcal{M}_y^R, \]

(2.4b) \[ \pi = \sum_{S'} \pi_x^R + \sum_{T} \pi_x^R \text{ and } \pi_i = \sum_{S_i} \pi_x^R + \sum_{T_i} \pi_y^R, \]

(2.4c) \[ \text{the } \pi_x^R \text{ are real-irreducible and mutually inequivalent.} \]

Let $A$ be the commuting algebra of $\pi$ on $\mathcal{M}$. By (2.4c), $A = \sum C + \sum R$, for $\pi_x^R$ has commuting algebra $C$ if $\chi \in S'$, $R$ if $\chi \in T$. Invariant almost complex structures are in obvious correspondence with elements of square $-I$ of the commuting algebra, which now are seen to exist if and only if $T$ is empty, and (1) follows. Similarly, the decomposition of $ds^2$ in (2), and the existence and decomposition of $h$ in (3), are immediate.

It remains only to show the $(M, ds^2)$ irreducible as riemannian manifolds in (2). That fact is known [3, §5.1], but in our present context we can give a short proof for the convenience of the reader. If $(M, ds^2)$ reduces, then it is a riemannian product $M' \times M''$ because it is complete and simply connected, so we have an ad $(K_i)$-stable decomposition $\mathcal{M}_i = \mathcal{M}' \oplus \mathcal{M}''$ with the properties

\[ [\mathcal{M}', \mathcal{M}''] \subset \mathfrak{g}_i, \quad \mathcal{M}'^C = \sum_{\mathcal{M}_x} \mathcal{M}_x, \quad \mathcal{M}''^C = \sum_{\mathcal{M}_y} \mathcal{M}_y, \quad X_i = X' \cup X''. \]

Here $X'$ and $X''$ are disjoint and self conjugate. If $\chi' \in X'$ and $\chi'' \in X''$ with $[\mathcal{M}_x', \mathcal{M}_x''] \neq 0$, then $\chi' \chi'' = 1$ so $\chi'' = \bar{\chi}' \in X''$ which is absurd. Thus $[\mathcal{M}', \mathcal{M}''] = 0$, and it follows that the simple Lie algebra $\mathfrak{g}_i$ is direct sum of ideals

\[ \mathfrak{g}' = \{ \mathfrak{g}_i \cap [\mathcal{M}', \mathcal{M}'] \} + \mathcal{M}' \text{ and } \mathfrak{g}'' = \{ \mathfrak{g}_i \cap [\mathcal{M}'', \mathcal{M}'] \} + \mathcal{M}''. \]

That being absurd, irreducibility is proved. Q.E.D.

2.5. Remark. In the notation of the proof of Proposition 2.2, $M$ has a $G$-invariant almost complex structure if and only if $X = S$, and then those structures $J$ correspond to the real partitions $X = S' \cup S''$ by: $\sum_{S'} \mathcal{M}_x$ and $\sum_{S''} \mathcal{M}_y$ are the $\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces of $J$ on $\mathcal{M}^C$.

3. Almost hermitian isometries. Let $M$ be a manifold with an almost hermitian metric $h$. Then $h = ds^2 + (-1)^{1/2} \omega$ where the riemannian metric $ds^2$ is the real part of $h$ and $\omega(u, v) = -ds^2(u, Jv)$ is the imaginary part; that determines the almost complex structure $J$. By almost hermitian isometry of $(M, h)$ we mean a diffeomorphism that preserves $h$, i.e. that is a riemannian isometry of $(M, ds^2)$ which preserves $J$. 

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Let $I(M)$ denote the (Lie) group of all isometries of $(M, ds^2)$, $H(M)$ the closed subgroup consisting of those isometries that preserve $J$. Then $H(M)$ is the (Lie) group of all almost hermitian isometries of $(M, h)$. In particular its identity component $H_0(M)$ is an analytic subgroup of the identity component $I_0(M)$ of $I(M)$. If $(M, h) = (M_1, h_1) \times \cdots \times (M_t, h_t)$ hermitian product, then the de Rham decomposition says that $I_0(M)$ preserves each noneuclidean factor, so those factors are stable under $H_0(M)$.

Let $M = G/K$ as in Proposition 2.2. Let $h$ be a $G$-invariant almost hermitian metric on $M$. The canonical decomposition induces $(M, h) = (M_1, h_1) \times \cdots \times (M_t, h_t)$ hermitian product where each $(M_i, ds_i^2)$, $ds_i^2 = \text{Re } h_i$, is an irreducible noneuclidean riemannian manifold. Thus $H_0(M) \times H_0(M_2) \times \cdots \times H_0(M_t)$, and $H(M)$ is generated by its subgroup $H(M_1) \times \cdots \times H(M_t)$ and permutations of mutually isometric $(M_i, h_i)$; so its determination is more or less reduced to the case where $M = G/K$ is irreducible. There the result is

3.1. PROPOSITION. Let $M$ be an effective coset space $G/K$ where $G$ is a compact connected simple Lie group and $K$ is a connected subgroup of maximal rank. Let $h$ be a $G$-invariant almost hermitian metric on $M$, so $M = H_0(M)/B$ where $G' = H_0(M)$ and $B \cap G = K$. If $G \neq H_0(M)$, then $(M, h)$ is an irreducible hermitian symmetric space of compact type listed below.

<table>
<thead>
<tr>
<th>Case</th>
<th>$G$</th>
<th>$K$</th>
<th>$H_0(M)$</th>
<th>$B$</th>
<th>$(M, h)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$G_2$</td>
<td>$U(2)$</td>
<td>$SO(7)$</td>
<td>$SO(5) \times SO(2)$</td>
<td>5-dimensional complex quadric</td>
</tr>
<tr>
<td>2</td>
<td>$Sp(r)/Z_2$</td>
<td>$Sp(r-1)\cdot U(1)$</td>
<td>$SU(2r)/Z_{2r}$</td>
<td>$U(2r-1)$</td>
<td>complex projective $(2r-1)$-space</td>
</tr>
<tr>
<td>3</td>
<td>$SO(2r+1)$</td>
<td>$U(r)$</td>
<td>$SO(2r+2)/Z_2$</td>
<td>$U(r+1)/Z_2$</td>
<td>unitary structures on $\mathbb{R}^{2r+2}$</td>
</tr>
<tr>
<td>3'</td>
<td>$Spin(7)/Z_2$</td>
<td>$U(3)$</td>
<td>$SO(8)/Z_2$</td>
<td>$SO(6)\cdot SO(2)$</td>
<td>6-dimensional complex quadric</td>
</tr>
</tbody>
</table>

**Remark 1.** In the exceptional cases above, $K$ is not $R$-irreducible on the tangent space, so $M$ has another $G$-invariant almost hermitian metric for which $G = H_0(M)$.

**Remark 2.** The proof is easily reduced to the case where $B$ is the centralizer of a toral subgroup of $H_0(M)$, and then the result can be extracted from [2, Table 5] and the Bott-Borel-Weil Theorem. But here it is convenient to reduce the proof to some classifications of Oniščik [4].

**Proof.** As $M$ has nonzero Euler characteristic, $B$ has maximal rank in $H_0(M)$, so $H_0(M)/B = G/K$ is one of the following entries in Oniščik’s list [4, Table 7].

(i) $A_{2n-1}/A_{2n-2} \cdot T = C_n/C_{n-1} \cdot T$ (our Case 2),
(ii) $B_3/A_2 \cdot T = G_2/A_1 \cdot T$ (our Case 1),
(iii) $B_3/D_3 = G_2/A_2$ ($B_3$ does not preserve $J$ here),
(iv) $D_{n+1}/A_n \cdot T = B_n/A_{n-1} \cdot T$ (our Case 3),
(v) $D_4/D_3 \cdot T = B_3/A_2 \cdot T$ (our Case 3').

The assertions follow with the observation that $H_0(M)/B$ is an irreducible hermitian symmetric coset space of compact type in each of the admissible cases. Q.E.D.

4. Almost analytic automorphisms. Let $M$ be a manifold with almost complex structure $J$. By almost analytic automorphism of $M$, we mean a diffeomorphism of $M$ which preserves $J$. The set of all such diffeomorphisms forms a group $A(M)$. If $M$ is compact, then $[1]$ in the compact-open topology, $A(M)$ is a Lie transformation group of $M$. We denote its identity component by $A_0(M)$. If, further, we have an almost hermitian metric on $M$, then $H(M)$ is a compact subgroup of $A(M)$. That will be our main tool in studying $A(M)$.

4.1. Theorem. Let $M = G/K$ be a simply connected effective coset space of nonzero Euler characteristic where $G$ is a compact connected Lie group. Let $J$ be a $G$-invariant almost complex structure on $M$. Let $M = M_1 \times \cdots \times M_t$ be the canonical decomposition into irreducible coset spaces, and decompose $J = J_1 \times \cdots \times J_t$ where $J_i$ is a $G_i$-invariant almost complex structure on $M_i$. Then

1. $A_0(M) = A_0(M_1) \times \cdots \times A_0(M_t)$.
2. $M$ has a $G$-invariant riemannian metric $ds^2 = ds_1^2 \times \cdots \times ds_t^2$ for which $H_0(M)$ is a maximal compact subgroup of $A_0(M)$.
3. If $J_i$ is integrable then $A_0(M_i) = H_0(M_i)^c$. If $J_i$ is not integrable then $A_0(M) = H_0(M)$.

Proof. For the second statement, enlarge $G$ to a maximal compact subgroup $H$ of $A_0(M)$ and choose an $H$-invariant riemannian metric $ds^2$ on $M$. Then $ds^2 = ds_1^2 \times \cdots \times ds_t^2$ as required, by Proposition 2.2, and $H = H_0(M)$ by construction.

We simplify notation for the proofs of the first and third statement by enlarging $G$ to $H_0(M)$ and writing $A$ for $A_0(M)$. That does not change the canonical decomposition of $M$, for the latter is the de Rham decomposition for $ds^2$ according to Proposition 2.2. Now $G/K = M = A/B$ where $G \subset A$ is a maximal compact subgroup and $K = G \cap B$.

We check that $A$ is a centerless semisimple Lie group. If $L$ is a closed normal analytic subgroup of $A$ with $G \cap L$ discrete, then $G \cdot L \subset A$ is effective on

$$(G \cdot L)/(K \cdot L) = M,$$

so $L = \{1\}$.

Let $L$ be the radical of $A$: now $A$ is semisimple. Let $\mathfrak{g}$ be the orthocomplement of $\mathfrak{g}$ in a maximal compactly embedded subalgebra of $\mathfrak{g}$: now $A$ has finite center, so the centerless group $G$ contains the center of $A$, so $A$ is centerless.

Let $A^\alpha$, $1 \leq \alpha \leq r$, be the simple normal analytic subgroups of $A$. So $A = A^1 \times \cdots \times A^r$ with $A^\alpha$ centerless simple. Now $G = G^1 \times \cdots \times G^r$, $K = K^1 \times \cdots \times K^r$ and $M = M^1 \times \cdots \times M^r$ where

$$G^\alpha = G \cap A^\alpha, \quad K^\alpha = K \cap G^\alpha, \quad M^\alpha = G^\alpha/K^\alpha.$$
If \( \alpha \neq \beta \) then \( A^\alpha \) acts trivially on \( M^\beta \). For every \( a \in A^\alpha \) centralizes the transitive transformation group \( G^\beta \) of \( M^\beta \), hence induces some transformation \( \tilde{a} \) of \( M^\beta \) that is trivial or fixed point free. As \( A^\alpha \) is connected, \( \tilde{a} \) is homotopic to 1 so its Lefschetz number is the (nonzero) Euler characteristic of \( M^\beta \); that shows \( \tilde{a} = 1 \). Now \( M^\alpha = A^\alpha / B^\alpha \), \( B^\alpha = B \cap A^\alpha \), with \( B = B_1 \times \cdots \times B^n \).

According to Oniščik [5, Table 1] the only possibilities for \( G^\alpha / K^\alpha = M^\alpha = A^\alpha / B^\alpha \), \( A^\alpha \) noncompact, are given in the following table.

<table>
<thead>
<tr>
<th>( A^\alpha )</th>
<th>( M^\alpha = G^\alpha / K^\alpha )</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( SL(2n, \mathbb{R}) / \mathbb{Z}_2 )</td>
<td>( SO(2n_1) \times \cdots \times SO(2n_\alpha) ) ( n = \sum n_i &gt; 1 )</td>
<td></td>
</tr>
<tr>
<td>( SL(2n+1, \mathbb{R}) )</td>
<td>( SO(2n_1) \times \cdots \times SO(2n_{n-1}) \times SO(2n_\alpha+1) ) ( n = \sum n_i )</td>
<td></td>
</tr>
<tr>
<td>( GL(n, \mathbb{Q}) / \mathbb{Z}_2 )</td>
<td>( Sp(n_1) \times \cdots \times Sp(n_\alpha) \times U(1)^n ) ( n = q + \sum n_i )</td>
<td></td>
</tr>
<tr>
<td>( SO(1, 2n-1) / \mathbb{Z}_2 )</td>
<td>( SO(2n) \times \cdots \times SO(2n_\alpha) \times U (m_1) \times \cdots \times U (m_j) ) ( n-1 = \sum m_1 + \sum m_j )</td>
<td></td>
</tr>
<tr>
<td>( E_6, E_7 )</td>
<td>( Sp(4) \times Sp(2) \times Sp(2) ) and ( Sp(4) / [Sp(1)]^4 )</td>
<td>none</td>
</tr>
<tr>
<td>( E_6, F_4 )</td>
<td>( F_4 / Sp(4), F_4 / Spin(8), F_4 / U(4) ) and ( F_4 / [SU(2)]^4 )</td>
<td>none</td>
</tr>
<tr>
<td>( (G^\alpha)^C )</td>
<td>( G^\alpha / K^\alpha ) where ( K^\alpha ) is the centralizer of a nontrivial toral subgroup of ( G^\alpha )</td>
<td></td>
</tr>
</tbody>
</table>

Note that \( G^\alpha \) is simple except in Case 1 with \( n = 2 \). There \( M^\alpha \) is the product of two Riemann spheres, so \( A^\alpha \) is the product of two copies of \( SL(2, \mathbb{C}) / \mathbb{Z}_2 \), contradicting the table entry for \( A^\alpha \). Thus we always have \( G^\alpha \) simple, so each \( M^\alpha \) is an \( M_\alpha \), and the first statement of our theorem is proved with \( A^\alpha = A_0(\alpha) \).

Now we may, and do, assume \( M \) irreducible. Thus \( A \) and \( G \) are simple.

4.2. **Lemma.** The invariant almost complex structure \( J \) is integrable if and only if \( A = G^C \). In that case \( B \) is a complex parabolic subgroup of \( A \) and \( J \) is induced either from the natural complex structure on \( A / B \) or from the conjugate structure.

**Proof of lemma.** Let \( J \) be integrable; we check \( \Theta^C \subset \mathcal{U} \). For if \( \xi \in \Theta \) and \( \xi^* \) denotes the holomorphic vector field induced on \( M \), then \( J(\xi^*) \) is holomorphic. Thus \( \Theta^C \) acts on \( M \) by \( \xi + i\eta \to \xi^* + J(\eta^*) \), and this action integrates to \( G^C \) because \( M \) is compact; that shows \( G^C \subset A \) so \( \Theta^C \subset \mathcal{U} \).

Let \( \mathcal{U} = \Theta^C \). As \( \Theta \) is its own normalizer in \( \Theta \) because it has maximal rank, \( \Theta \) is its own normalizer in \( \mathcal{U} \), so \( B \) is an \( R \)-algebraic subgroup of \( A \). Thus \( A \) has an Iwasawa decomposition \( GSN \) with \( B = KSN \). As \( \mathcal{U} = \Theta^C \), the group \( SC \) is a complex Cartan subgroup of \( A \), so \( N \) is a complex unipotent subgroup. Now \( K^C SC \) is the complex group generated by \( B \) and it has intersection \( K \) with \( G \); thus \( M = A / B \to A / K^C SC N = G / K \) is trivial so \( B \) is a complex subgroup of \( A \). As \( A / B \) is compact now \( B \) is a complex parabolic subgroup.
Decompose $B = B'^{\cdot}B'^{\cdot}$ into reductive and unipotent parts. Let $Z$ be the identity component of the center of $B'$, complex subtorus of $S'$. Let $D$ be the set of characters $\chi \neq 1$ on $Z$ that are restrictions of positive roots, so $\mathbb{B}^u = \sum D \mathfrak{H}_\chi$. Define $\mathbb{B}^{-u} = \sum D \mathfrak{H}_{-\chi}$ so that $\mathfrak{H}$ is the direct sum of its subspaces $\mathbb{B}^r$, $\mathbb{B}^u$ and $\mathbb{B}^{-u}$. 

$(\mathfrak{g} \cap (\mathbb{B}^u + \mathbb{B}^{-u}))$ represents the real tangent space of $M$, and $\mathfrak{g}^u + \mathfrak{g}^{-u}$ represents the complexified tangent space. If $\pm \chi \in D$, then $\mathfrak{H}_\chi$ is an irreducible representation space of $B'$, so $J$ acts on $\mathfrak{H}_\chi$ either as $\sqrt{-1}$ or as $-\sqrt{-1}$. Let $\mathfrak{h}^+$ (resp. $\mathfrak{h}^-$) denote the image in $\mathfrak{g}^{/\mathbb{B}}$ of the $\mathfrak{g}^{/\mathbb{B}}_{-\chi}$, on which $J$ acts as $\sqrt{-1}$ (resp. $-\sqrt{-1}$). Then $\text{ad} (\mathfrak{g}) \cdot \mathfrak{h}^+ \subseteq \mathfrak{h}^+$ by invariance of $J$ under $B$. If $v$ is the restriction to $Z$ of the highest root, then $\mathfrak{g}^{/\mathbb{B}} = \sum_{n \geq 0} \text{ad} (\mathfrak{g})^n (\mathfrak{g}^{/\mathbb{B}}_{-v})$, because $\mathfrak{g}$ is simple, so $\mathfrak{H}/\mathbb{B}$ is the one of $\mathfrak{h}^+$ or $\mathfrak{h}^-$ into which $\mathfrak{H}_{-v}$ maps. Thus either $J$ acts on $\mathfrak{g}^{-u}$ as $\sqrt{-1}$ and the natural complex structure of $A/B$ induces $J$, or $J$ acts on $\mathfrak{g}^{-u}$ as $-\sqrt{-1}$ and the natural structure induces $-J$. In either case $J$ is integrable.

In general suppose $\mathfrak{g}^C \subset \mathfrak{g}$. Then $M = G^C/B \cap G^C$ is a complex flag manifold on which $A$ is the largest connected group of analytic automorphisms. Thus $A$ is a centerless complex semisimple group, hence the complexification of its maximal compact subgroup $G$.

Lemma 4.2 is proved.

4.3. Lemma. If $B^C$ is parabolic in $A^C$, then $J$ is integrable so $A = G^C$.

Proof of lemma. $J$ is an element of square $-I$ in the commuting algebra of \text{ad} ($\mathfrak{g}$) on $\mathfrak{g}/\mathbb{B}$. Thus it induces an element $J^C$ of square $-I$ in the commuting algebra of \text{ad} ($\mathfrak{g}^C$) on $\mathfrak{g}^{C/\mathbb{B}^C}$. Now suppose $B^C$ parabolic in $A^C$, so $M^C = A^C/B^C$ is compact and of positive Euler characteristic with invariant almost complex structure $J^C$.

If $A$ is complex then $A = G^C$ and Lemma 4.2 says that $J$ is integrable. Thus we may assume $A$ not complex so that $A^C$ is simple. Then Lemma 4.2 says that $J^C$ is integrable, and in fact that either $J^C$ or $-J^C$ is induced by the natural complex structure on $A^C/B^C$. Replace $J$ by $-J$ if necessary; that does not alter integrability of $J$, but it replaces $J^C$ by $-J^C$, allowing us to assume $J^C$ induced by the natural complex structure of $A^C/B^C$.

Decompose $B = B'^{\cdot}B'^{\cdot}$ into reductive and unipotent parts, so $\mathbb{B} = \mathbb{B}^r + \mathbb{B}^u$ and $\mathfrak{H} = \mathfrak{H} + \mathfrak{H}^{-u}$ where $\mathbb{B}^{\pm u}$ are subalgebras normalized by $\mathbb{B}$. Let $\mathbb{B}^{-u}$ represent the real tangent space to $M$. Note that $J^C$ acts on $(\mathbb{B}^{-u})^C$ as $\sqrt{-1}$. That contradicts our arrangement that the action of $J^C$ on $(\mathbb{B}^{-u})^C$ is induced by the action of $J$ on $\mathbb{B}^{-u}$. Thus $A$ cannot be noncomplex. Lemma 4.3 is proved.

We complete the proof of Theorem 4.1. As in the second paragraph of the proof of Lemma 4.2, $B$ is a real algebraic subgroup of $A$, so there is a semidirect product decomposition $B = B'^{\cdot}B'^{\cdot}$ into reductive and unipotent parts. If rank $B'^{\cdot}$ $<$ rank $A$, then any Cartan subalgebra of $\mathfrak{H}$ has an element $\xi$ not contained in any isotropy subalgebra of $\mathfrak{H}$ on $M$ so it induces a nonvanishing vector field $\xi^*$ on $M$. The
existence of a nonvanishing vector field \( \xi^* \) says that \( M \) has Euler characteristic zero. That contradiction proves rank \( B^r = \text{rank } A \).

Let \( \sigma \) be the Cartan involution of \( \mathfrak{g} \) with fixed point set \( \mathfrak{s} \) and let \( \mathfrak{a} = \mathfrak{g} + \mathfrak{b} \) be the Cartan decomposition. We may assume \( \sigma(\mathfrak{b}') = \mathfrak{b}' \), so \( \mathfrak{b}' = \mathfrak{g} + (\mathfrak{b} \cap \mathfrak{b}') \). That gives compact real forms

\[
\mathfrak{a}_c = \mathfrak{g} + \sqrt{-1} \mathfrak{b} \quad \text{and} \quad \mathfrak{b}_c = \mathfrak{g} + \sqrt{-1} (\mathfrak{b} \cap \mathfrak{b}').
\]

Let \( A_c \) denote the centerless group with Lie algebra \( \mathfrak{a}_c \) and let \( B'_c \) be the analytic subgroup for \( \mathfrak{b}_c \). Then rank \( B'_c = \text{rank } B^r = \text{rank } A = \text{rank } A_c \) tells us that \( X = A_c / B'_c \) is a compact simply connected manifold of positive Euler characteristic. If \( A = G \) then \( B = B^r = K \), so \( A_c = G \) and \( B'_c = K \), whence \( X = M \).

As in the second paragraph of the proof of Lemma 4.2 we have Iwasawa decompositions \( A = GSN \) and \( B = KSN \). Choose a torus subgroup \( T \subseteq K \) such that \( H = T \times S \subseteq B^r \) is a Cartan subgroup of \( A \). Let \( \Delta \) be the root system. Now \( \Delta = D \cup E \cup -E \) disjoint, and \( A = \mathfrak{b}' + \mathfrak{b}^u + \mathfrak{b}^{-u} \) direct, where

\[
\mathfrak{b}' = \mathfrak{g} + \mathfrak{s} \quad \text{and} \quad \mathfrak{b}^u = \mathfrak{a} \cap \left\{ \sum \mathfrak{a}_\phi \right\}, \quad \mathfrak{b}^{-u} = \mathfrak{a} \cap \left\{ \sum \mathfrak{a}_{-\phi} \right\}.
\]

Observe that \( \sigma \) interchanges \( \mathfrak{b}^u \) and \( \mathfrak{b}^{-u} \). For \( \mathfrak{b}^u \subset N \) because \( N = N' \cdot B^u \) where \( B^r = KSN' \), and the dual space of \( \mathfrak{s} \) has an ordering such that

\[
\mathfrak{a}_C = \sum_{\phi|\mathfrak{s} > 0} \mathfrak{a}_\phi, \quad \text{and} \quad \phi|\mathfrak{s} > 0 \iff \sigma \phi|\mathfrak{s} < 0.
\]

View the invariant almost complex structure \( J \) of \( M \) as an element of square \(-I\) in the commuting algebra of \( \text{ad}(\mathfrak{b}) \) on \( \mathfrak{a}/\mathfrak{b} \), hence in the commuting algebra of \( \text{ad}(\mathfrak{b}') \) on \( \mathfrak{b}^{-u} \subset \mathfrak{a}/\mathfrak{b} \); then extend \( J \) to an element \( J' \) of square \(-I\) in the commuting algebra of \( \text{ad}(\mathfrak{b}') \) on \( \mathfrak{b}^u + \mathfrak{b}^{-u} \) by the formula

\[
J'(\xi + \eta) = \sigma J(\sigma \xi) + J(\eta) \quad \text{where} \quad \xi \in \mathfrak{b}^u, \ \eta \in \mathfrak{b}^{-u}.
\]

Now \( J' \) is an \( A \)-invariant \( \sigma \)-invariant almost complex structure on \( A/B^r \), so [6, Proposition 7.7] it defines an \( A_c \)-invariant \( \sigma \)-invariant almost complex structure on \( A_c/B'_c \). We have proved that \( X = A_c/B'_c \) has an invariant almost complex structure.

Suppose \( A \neq G \). Note that [6, Theorem 4.10] eliminates lines 5 and 6 of the Oniščik table above, so either \( A = G^C \) or \( A \) is absolutely simple and of classical type. Suppose \( A \neq G^C \) so \( A_c \) is simple and of classical type. Then [6, Theorem 4.10] shows that \( B'_c \) is the centralizer of a torus in \( A_c \). Let \( \mathfrak{z}_c \) denote the center of \( \mathfrak{b}_c \). Then \( \sigma(\mathfrak{b}_c) = \mathfrak{b}_c \) implies \( \sigma(\mathfrak{z}_c) = \mathfrak{z}_c \), so \( \mathfrak{z}_c = \mathfrak{u} + \mathfrak{z}^{1/2} \mathfrak{b} \) with \( \mathfrak{u} \subseteq \mathfrak{b} \) and \( \mathfrak{z} \subseteq \mathfrak{b} \cap \mathfrak{b}' \). Now \( \mathfrak{b}' \) has center \( \mathfrak{z} = \mathfrak{u} + \mathfrak{b} \subseteq \mathfrak{z} + \mathfrak{b} = \mathfrak{z} \), and \( \mathfrak{b}' \) is the centralizer of \( \mathfrak{b} \) in \( \mathfrak{b}' \). We order the root system \( \Delta \) so that a root \( \phi > 0 \) whenever \( \phi|\mathfrak{z} > 0 \) and \( \phi|\mathfrak{z} > 0 \).

Then \( \mathfrak{b}^C \) contains the Borel subalgebra \( \mathfrak{b}_C + \sum_{\phi > 0} \mathfrak{a}_\phi \) of \( \mathfrak{a}^C \) for that ordering, so \( \mathfrak{b}^C \) is a parabolic subalgebra of \( \mathfrak{a}^C \). Then Lemma 4.3 says \( A = G^C \). We have proved that \( A \neq G \) implies \( A = G^C \).
If $J$ is integrable then Lemma 4.2 says $A = G^C$. If $J$ is not integrable then Lemma 4.2 says $A \neq G^C$, so we cannot have $A = G$, and that forces $A = G$. Theorem 4.1 is proved. Q.E.D.

4.3. Remark. Theorem 4.1 extends the scope of [8, Theorem 17.4(3)], but that result remains incomplete because, as remarked at the end of [8, §17], it is not known whether

$$A_0(E_6/\text{ad } SU(3))$$

is $E_6$ rather than $E_6^C$ or whether

$$A_0(SO(n^2-1)/\text{ad } SU(n))$$

is $SO(n^2-1)$ rather than $SO(n^2-1, C)$, $SL(n^2-1, R)$, or $SO(1, n^2-1)$.

REFERENCES

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