THE AUTOMORPHISM GROUP OF A HOMOGENEOUS ALMOST COMPLEX MANIFOLD (*)

BY

JOSEPH A. WOLF

1. Introduction. Let $M$ be a compact simply connected manifold of nonzero Euler characteristic that carries a homogeneous almost complex structure. We determine the largest connected group $A_0(M)$ of almost analytic automorphisms of $M$.

Our hypotheses represent $M$ as a coset space $G/K$ where $G$ is a maximal compact subgroup of the Lie group $A_0(M)$ and $K$ is a closed connected subgroup of maximal rank in $G$. In §2 we collect some information, decomposing $M = M_1 \times \cdots \times M_t$ as a product of “irreducible” factors along the decomposition of $G$ as a product of simple groups; then every invariant almost complex structure or riemannian metric decomposes and every invariant riemannian metric is hermitian relative to any invariant almost complex structure; furthermore the decomposition is independent of $G$ in a certain sense. In §3 we choose an invariant riemannian metric and determine the largest connected groups $H_0(M_1)$ of almost hermitian isometries of the $M_i$. Then $A_0(M)$ is determined in §4. There it is shown that $A_0(M) = A_0(M_1) \times \cdots \times A_0(M_t)$, that $A_0(M_i) = H_0(M_i)$ if the almost complex structure on $M_i$ is not integrable, and that $A_0(M_i) = H_0(M_i)^c$ if the almost complex structure on $M_i$ is induced by a complex structure. $A_0(M)$ thus is a centerless semisimple Lie group whose simple normal analytic subgroups are just the $A_0(M_i)$.

2. Decomposition. Let $M$ be an effective coset space of a compact connected Lie group $G$ by a connected subgroup $K$ of maximal rank. In other words $M = G/K$ is compact, simply connected and of nonzero Euler characteristic; $G$ is a compact centerless semisimple Lie group, rank $K = \text{rank } G$, and $K$ contains no simple factor of $G$. Then

\begin{align}
(2.1a) \quad G &= G_1 \times \cdots \times G_t, \quad K = K_1 \times \cdots \times K_t \quad \text{and} \quad M = M_1 \times \cdots \times M_t
\end{align}

where

\begin{align}
(2.1b) \quad G_i \text{ is simple,} \quad K_i = K \cap G_i \quad \text{and} \quad M_i = G_i/K_i.
\end{align}

$G_i$ is a compact connected centerless simple Lie group, $K_i$ is a connected subgroup of maximal rank, and $M_i = G_i/K_i$ is a simply connected effective coset space of nonzero Euler characteristic. The decomposition of $M$ is unique up to order of the factors because it is determined by the decomposition of $G$.

Received by the editors November 5, 1966 and, in revised form, March 1, 1969.

(*) Research partially supported by N.S.F. Grants GP-5798 and GP-8008.
We call (2.1) the canonical decomposition of the coset space $M = G/K$. The factors $M_i = G_i/K_i$ are the irreducible factors of $M = G/K$. If there is just one irreducible factor, i.e. if $G$ is simple, then we say that $M = G/K$ is irreducible.

2.2. Proposition. Let $M$ be an effective coset space $G/K$ where $G$ is a compact connected Lie group and $K$ is a connected subgroup of maximal rank. Let $M = M_1 \times \cdots \times M_t$ be the canonical decomposition into irreducible factors $M_i = G_i/K_i$.

1. The $G$-invariant almost complex structures $J$ on $M$ are just the $J_1 \times \cdots \times J_t$ where $J_i$ is a $G_i$-invariant almost complex structure on $M_i$.

2. The $G$-invariant riemannian metrics $ds^2$ on $M$ are just the $ds^2_1 \times \cdots \times ds^2_t$ where $ds^2_i$ is a $G_i$-invariant riemannian metric on $M_i$; there each $(M_i, ds^2_i)$ is an irreducible riemannian manifold, so

$$(M, ds^2) = (M_1, ds^2_1) \times \cdots \times (M_t, ds^2_t)$$

is the de Rham decomposition.

3. Let $J$ be a $G$-invariant almost complex structure on $M$. If $ds^2$ is a $G$-invariant riemannian metric, then it is the real part of a $G_i$-invariant almost hermitian (for $J_i$) metric $h$ on $M_i$ and $h = h_1 \times \cdots \times h_t$ where $h_i$ is a $G_i$-invariant almost hermitian (for $J_i$) metric on $M_i$ and $ds^2_i$ is the real part of $h_i$.

Proof. The Lie algebras decompose uniquely as direct sums $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ and $\mathfrak{g}_i = \mathfrak{k}_i + \mathfrak{m}_i$, $\mathfrak{k} = \sum \mathfrak{k}_i$ and $\mathfrak{m} = \sum \mathfrak{m}_i$, with $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$ and $[\mathfrak{k}_i, \mathfrak{m}_i] \subset \mathfrak{m}_i$. Let $Z$ be the center of $K$, so $\mathfrak{k}$ is the centralizer of $Z$ in $\mathfrak{g}$. Then $Z = Z_1 \times \cdots \times Z_t$ where $Z_i$ is the center of $K_i$ and $\mathfrak{k}_i$ is the centralizer of $Z_i$ in $\mathfrak{g}_i$.

$\pi$ denotes the representation of $K$ on $\mathfrak{m}$ and $\pi_i$ is the representation of $K_i$ on $\mathfrak{m}_i$. Then $\pi = \pi_1 \oplus \cdots \oplus \pi_t$. Let $X = X_1 \cup \cdots \cup X_t$ be the set of nontrivial characters on $Z$ such that

$$(2.3a) \quad \mathfrak{m}^C = \sum_X \mathfrak{m}_x \quad \text{and} \quad \mathfrak{m}_i^C = \sum_{X_i} \mathfrak{m}_x$$

where $Z$ acts on $\mathfrak{m}_x$ by the character $\chi$. Each $\mathfrak{m}_x$ is ad $(K)$-stable, so $K$ acts on $\mathfrak{m}_x$ by a representation $\pi_x$, and

$$(2.3b) \quad \pi^C = \sum_X \pi_x \quad \text{and} \quad \pi_i^C = \sum_{X_i} \pi_x.$$ 

The point [7, Theorem 8.13.3] is that

$$(2.3c) \quad \text{the } \pi_x \text{ are irreducible and mutually inequivalent.}$$

We transform the complex decomposition (2.3) to a real decomposition. Let $X = S \cup T$, $S = S_1 \cup \cdots \cup S_t$ and $T = T_1 \cup \cdots \cup T_t$ where $S_i$ consists of the nonreal characters in $X_i$ and $T_i$ consists of the real ones. By real partition of $X$ we mean a disjoint $X = S' \cup S'' \cup T$ where $S'' = S'$. If $\chi \in S_i$ then $\bar{\chi} \in S_i$; thus the real partition
induces real partitions \( X_i = S'_i \cup S''_i \cup T_i \). If \(|S| = 2n\) then \( X \) has \( 2^n \) real partitions. Now choose a real partition \( X = S' \cup S'' \cup T \) and define

\[
\chi \in S': K \text{ acts on } \mathfrak{M}^S = \mathfrak{M} \cap (\mathfrak{M}_x + \mathfrak{M}_y) \quad \text{by } \pi^R_x \\
\chi \in T: K \text{ acts on } \mathfrak{M}^T = \mathfrak{M} \cap \mathfrak{M}_x \quad \text{by } \pi^R_x.
\]

Then (2.3abc) becomes

\[
(2.4a) \quad \mathfrak{M} = \sum_{S'} \mathfrak{M}^R + \sum_{T} \mathfrak{M}^R \quad \text{and} \quad \mathfrak{M}_x = \sum_{S_i} \mathfrak{M}^R + \sum_{T_i} \mathfrak{M}^R,
\]

\[
(2.4b) \quad \pi = \sum_{S'} \pi^R + \sum_{T} \pi^R \quad \text{and} \quad \pi_i = \sum_{S_i} \pi^R + \sum_{T_i} \pi^R,
\]

\[
(2.4c) \quad \text{the } \pi^R \text{ are real-irreducible and mutually inequivalent.}
\]

Let \( A \) be the commuting algebra of \( \pi \) on \( \mathfrak{M} \). By (2.4c), \( A = \sum_C + \sum R \), for \( \pi^R \) has commuting algebra \( C \) if \( \chi \in S' \), \( R \) if \( \chi \in T \). Invariant almost complex structures are in obvious correspondence with elements of square \(-I\) of the commuting algebra, which now are seen to exist if and only if \( T \) is empty, and (1) follows. Similarly, the decomposition of \( ds^2 \) in (2), and the existence and decomposition of \( h \) in (3), are immediate.

It remains only to show the \((M_i, ds^2)\) irreducible as riemannian manifolds in (2). That fact is known [3, §5.1], but in our present context we can give a short proof for the convenience of the reader. If \((M_i, ds^2)\) reduces, then it is a riemannian product \( M' \times M'' \) because it is complete and simply connected, so we have an \( \text{ad} (K_i) \)-stable decomposition \( \mathfrak{M}_x = \mathfrak{M}' \oplus \mathfrak{M}'' \) with the properties

\[
[\mathfrak{M}', \mathfrak{M}''] \subset \mathfrak{g}_i, \quad \mathfrak{M}'^C = \sum_x \mathfrak{M}'_x, \quad \mathfrak{M}''^C = \sum_x \mathfrak{M}_x, \quad X_i = X' \cup X''.
\]

Here \( X' \) and \( X'' \) are disjoint and self conjugate. If \( \chi' \in X' \) and \( \chi'' \in X'' \) with \( [\mathfrak{M}'_x, \mathfrak{M}_x] \neq 0 \), then \( \chi'\chi'' = 1 \) so \( \chi' = \chi'' \in X'' \) which is absurd. Thus \([\mathfrak{M}', \mathfrak{M}''] = 0\), and it follows that the simple Lie algebra \( \mathfrak{g}_i \) is direct sum of ideals

\[
\mathfrak{g}' = \{ \mathfrak{g}_i \cap [\mathfrak{M}', \mathfrak{M}'' ] \} + \mathfrak{M}' \quad \text{and} \quad \mathfrak{g}'' = \{ \mathfrak{g}_i \cap [\mathfrak{M}'', \mathfrak{M}'' ] \} + \mathfrak{M}''.
\]

That being absurd, irreducibility is proved. Q.E.D.

2.5. Remark. In the notation of the proof of Proposition 2.2, \( M \) has a \( G \)-invariant almost complex structure if and only if \( X = S \), and then those structures \( J \) correspond to the real partitions \( X = S' \cup S'' \) by: \( \sum_{S'} \mathfrak{M}_x \) and \( \sum_{S''} \mathfrak{M}_x \) are the \( \sqrt{-1} \) and \( -\sqrt{-1} \) eigenspaces of \( J \) on \( \mathfrak{M}^C \).

3. Almost hermitian isometries. Let \( M \) be a manifold with an almost hermitian metric \( h \). Then \( h = ds^2 + (-1)^{1/2} \omega \) where the riemannian metric \( ds^2 \) is the real part of \( h \) and \( \omega (u, v) = ds^2(u, Jv) \) is the imaginary part; that determines the almost complex structure \( J \). By almost hermitian isometry of \((M, h)\) we mean a diffeomorphism that preserves \( h \), i.e. that is a riemannian isometry of \((M, ds^2)\) which preserves \( J \).
Let $I(M)$ denote the (Lie) group of all isometries of $(M, ds^2)$, $H(M)$ the closed subgroup consisting of those isometries that preserve $J$. Then $H(M)$ is the (Lie) group of all almost hermitian isometries of $(M, h)$. In particular its identity component $H_0(M)$ is an analytic subgroup of the identity component $I_0(M)$ of $I(M)$. If $(M, h) = (M_1, h_1) \times \cdots \times (M_t, h_t)$ hermitian product, then the de Rham decomposition says that $I_0(M)$ preserves each noneuclidean factor, so those factors are stable under $H_0(M)$.

Let $M = G/K$ as in Proposition 2.2. Let $h$ be a $G$-invariant almost hermitian metric on $M$. The canonical decomposition induces $(M, h) = (M_1, h_1) \times \cdots \times (M_t, h_t)$ hermitian product where each $(M_i, ds_i^2), ds_i^2 = \Re h_i$ is an irreducible noneuclidean riemannian manifold. Thus $H_0(M) = H_0(M_1) \times \cdots \times H_0(M_t)$, and $H(M)$ is generated by its subgroup $H(M_1) \times \cdots \times H(M_t)$ and permutations of mutually isometric $(M_i, h_i)$; so its determination is more or less reduced to the case where $M = G/K$ is irreducible. There the result is

3.1. PROPOSITION. Let $M$ be an effective coset space $G/K$ where $G$ is a compact connected simple Lie group and $K$ is a connected subgroup of maximal rank. Let $h$ be a $G$-invariant almost hermitian metric on $M$, so $M = H_0(M)/B$ where $G = H_0(M)$ and $B \cap G = K$. If $G \neq H_0(M)$, then $(M, h)$ is an irreducible hermitian symmetric space of compact type listed below.

<table>
<thead>
<tr>
<th>Case</th>
<th>$G$</th>
<th>$K$</th>
<th>$H_0(M)$</th>
<th>$B$</th>
<th>$(M, h)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$G_2$</td>
<td>$U(2)$</td>
<td>$SO(7)$</td>
<td>$SO(5) \times SO(2)$</td>
<td>5-dimensional complex quadric</td>
</tr>
<tr>
<td>2</td>
<td>$Sp(r)/Z_2$</td>
<td>$Sp(r-1)/U(1)$</td>
<td>$SU(2r)/Z_{2r}$</td>
<td>$U(2r-1)$</td>
<td>complex projective $(2r-1)$-space</td>
</tr>
<tr>
<td>3</td>
<td>$SO(2r+1)$</td>
<td>$U(r)$</td>
<td>$SO(2r+2)/Z_2$</td>
<td>$U(r+1)/Z_2$</td>
<td>unitary structures on $R^{2r+2}$</td>
</tr>
<tr>
<td>3'</td>
<td>$Spin(7)/Z_2$</td>
<td>$U(3)$</td>
<td>$SO(8)/Z_2$</td>
<td>$SO(6) \cdot SO(2)$</td>
<td>6-dimensional complex quadric</td>
</tr>
</tbody>
</table>

Remark 1. In the exceptional cases above, $K$ is not $R$-irreducible on the tangent space, so $M$ has another $G$-invariant almost hermitian metric for which $G = H_0(M)$.

Remark 2. The proof is easily reduced to the case where $B$ is the centralizer of a toral subgroup of $H_0(M)$, and then the result can be extracted from [2, Table 5] and the Bott-Borel-Weil Theorem. But here it is convenient to reduce the proof to some classifications of Oniščik [4].

Proof. As $M$ has nonzero Euler characteristic, $B$ has maximal rank in $H_0(M)$, so $H_0(M)/B = G/K$ is one of the following entries in Oniščik’s list [4, Table 7].

(i) $A_{2n-1}/A_{2n-2} \cdot T = C_n/C_{n-1} \cdot T$ (our Case 2),
(ii) $B_3/B_2 \cdot T = G_2/A_1 \cdot T$ (our Case 1),
(iii) $B_3/D_3 = G_2/A_2$ ($B_3$ does not preserve $J$ here),

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
The assertions follow with the observation that \( H_0(M)/B \) is an irreducible hermitian symmetric coset space of compact type in each of the admissible cases. Q.E.D.

4. Almost analytic automorphisms. Let \( M \) be a manifold with almost complex structure \( J \). By almost analytic automorphism of \( M \), we mean a diffeomorphism of \( M \) which preserves \( J \). The set of all such diffeomorphisms forms a group \( A(M) \). If \( M \) is compact, then \([1]\) in the compact-open topology, \( A(M) \) is a Lie transformation group of \( M \). We denote its identity component by \( A_0(M) \). If, further, we have an almost hermitian metric on \( M \), then \( H(M) \) is a compact subgroup of \( A(M) \). That will be our main tool in studying \( A(M) \).

4.1. Theorem. Let \( M=G/K \) be a simply connected effective coset space of nonzero Euler characteristic where \( G \) is a compact connected Lie group. Let \( J \) be a \( G \)-invariant almost complex structure on \( M \). Let \( M=M_1 \times \cdots \times M_t \) be the canonical decomposition into irreducible coset spaces, and decompose \( J=J_1 \times \cdots \times J_t \) where \( J_i \) is a \( G_i \)-invariant almost complex structure on \( M_i \). Then

1. \( A_0(M)=A_0(M_1) \times \cdots \times A_0(M_t) \).

2. \( M \) has a \( G \)-invariant riemannian metric \( ds^2=ds_1^2 \times \cdots \times ds_t^2 \) for which \( H_0(M) \) is a maximal compact subgroup of \( A_0(M) \).

3. If \( J_i \) is integrable then \( A_0(M_i)=H_0(M_i) \). If \( J_i \) is not integrable then \( A_0(M_i)=H_0(M_i) \).

Proof. For the second statement, enlarge \( G \) to a maximal compact subgroup \( H \) of \( A_0(M) \) and choose an \( H \)-invariant riemannian metric \( ds^2 \) on \( M \). Then \( ds^2=ds_1^2 \times \cdots \times ds_t^2 \) as required, by Proposition 2.2, and \( H=H_0(M) \) by construction.

We simplify notation for the proofs of the first and third statement by enlarging \( G \) to \( H_0(M) \) and writing \( A \) for \( A_0(M) \). That does not change the canonical decomposition of \( M \), for the latter is the de Rham decomposition for \( ds^2 \) according to Proposition 2.2. Now \( G/K=M=A/B \) where \( G \subset A \) is a maximal compact subgroup and \( K=G \cap B \).

We check that \( A \) is a centerless semisimple Lie group. If \( L \) is a closed normal analytic subgroup of \( A \) with \( G \cap L \) discrete, then \( G \cdot L \subset A \) is effective on

\[(G \cdot L)/(K \cdot L) = M, \text{ so } L = \{1\}.
\]

Let \( L \) be the radical of \( A \): now \( A \) is semisimple. Let \( \mathfrak{g} \) be the orthocomplement of \( \mathfrak{g} \) in a maximal compactly embedded subalgebra of \( \mathfrak{g} \): now \( A \) has finite center, so the centerless group \( G \) contains the center of \( A \), so \( A \) is centerless.

Let \( A^\alpha \), \( 1 \leq \alpha \leq r \), be the simple normal analytic subgroups of \( A \). So \( A=A^1 \times \cdots \times A^r \) with \( A^\alpha \) centerless simple. Now \( G=G^1 \times \cdots \times G^r \), \( K=K^1 \times \cdots \times K^r \) and \( M=M^1 \times \cdots \times M^r \) where

\[G^\alpha = G \cap A^\alpha, \quad K^\alpha = K \cap G^\alpha, \quad M^\alpha = G^\alpha/K^\alpha.\]
If \( \alpha \neq \beta \) then \( A^\alpha \) acts trivially on \( M^\beta \). For every \( a \in A^\alpha \) centralizes the transitive transformation group \( G^\beta \) of \( M^\beta \), hence induces some transformation \( \tilde{a} \) of \( M^\beta \) that is trivial or fixed point free. As \( A^\alpha \) is connected, \( \tilde{a} \) is homotopic to 1 so its Lefschetz number is the (nonzero) Euler characteristic of \( M^\beta \); that shows \( a = 1 \). Now \( M^a = A^a/B^a \), \( B^a = B \cap A^a \), with \( B = B^1 \times \cdots \times B^r \).

According to Oniščik [5, Table 1] the only possibilities for \( G^a/K^a = M^a = A^a/B^a \), \( A^\alpha \) noncompact, are given in the following table.

<table>
<thead>
<tr>
<th>( A^\alpha )</th>
<th>( M^a = G^a/K^a )</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( SL(2n, R)/Z_2 )</td>
<td>( SO(2n)/SO(2n_1) \times \cdots \times SO(2n_\alpha) )</td>
<td>( n = \sum n_i &gt; 1 )</td>
</tr>
<tr>
<td>( SL(2n+1, R) )</td>
<td>( SO(2n+1)/SO(2n_1) \times \cdots \times SO(2n_\alpha-1) \times SO(2n_\alpha+1) )</td>
<td>( n = \sum n_i )</td>
</tr>
<tr>
<td>( GL(n, Q)/Z_2 )</td>
<td>( Sp(n)/Sp(n_1) \times \cdots \times Sp(n_\alpha) \times U(1)^a )</td>
<td>( n = q + \sum n_i )</td>
</tr>
<tr>
<td>( SO(1, 2n-1)/Z_2 )</td>
<td>( SO(2n-1)/SO(2n_1) \times \cdots \times SO(2n_\alpha) \times U(m_1) \times \cdots \times U(m_\beta) )</td>
<td>( n - 1 = \sum n_i + \sum m_i )</td>
</tr>
<tr>
<td>( E_6, E_7, E_8 )</td>
<td>( Sp(4)/Sp(2) \times Sp(2) ) and ( Sp(4)/[Sp(1)] )</td>
<td>none</td>
</tr>
<tr>
<td>( (G^\alpha)^C )</td>
<td>( G^a/K^a ) where ( K^a ) is the centralizer of a nontrivial toral subgroup of ( G^a )</td>
<td>( G^\alpha ) compact centerless simple</td>
</tr>
</tbody>
</table>

Note that \( G^a \) is simple except in Case 1 with \( n = 2 \). There \( M^a \) is the product of two Riemann spheres, so \( A^\alpha \) is the product of two copies of \( SL(2, C)/Z_2 \), contradicting the table entry for \( A^\alpha \). Thus we always have \( G^a \) simple, so each \( M^a \) is an \( M_a \), and the first statement of our theorem is proved with \( A^a = A_0(M^a) \).

Now we may, and do, assume \( M \) irreducible. Thus \( A \) and \( G \) are simple.

4.2. Lemma. The invariant almost complex structure \( J \) is integrable if and only if \( A = G^C \). In that case \( B \) is a complex parabolic subgroup of \( A \) and \( J \) is induced either from the natural complex structure on \( A/B \) or from the conjugate structure.

Proof of lemma. Let \( J \) be integrable; we check \( \Theta \subseteq \mathfrak{g} \). For if \( \xi \in \Theta \) and \( \xi^* \) denotes the holomorphic vector field induced on \( M \), then \( J(\xi^*) = \mathfrak{g} \) is holomorphic. Thus \( \Theta \subseteq \mathfrak{g} \) acts on \( M \) by \( \xi + i\eta \rightarrow \xi^* + J(\eta^*) \), and this action integrates to \( G^C \) because \( M \) is compact; that shows \( G^C \subseteq A \) so \( \Theta \subseteq \mathfrak{g} \).

Let \( \mathfrak{g} = \Theta \). As \( \Theta \) is its own normalizer in \( \mathfrak{g} \) because it has maximal rank, \( \Theta \) is its own normalizer in \( \mathfrak{g} \), and \( B \) is a \( \mathfrak{R} \)-algebraic subgroup of \( A \). Thus \( A \) has an Iwasawa decomposition \( GSN \) with \( G = KSN \). As \( \mathfrak{g} = \Theta \), the group \( S^C \) is a complex Cartan subgroup of \( A \), so \( N \) is a complex unipotent subgroup. Now \( K^C S^C N \) is the complex group generated by \( B \) and it has intersection \( K \) with \( G \); thus \( M = A/B \rightarrow A/K^C S^C N = G/K \) is trivial so \( B \) is a complex subgroup of \( A \). As \( A/B \) is compact now \( B \) is a complex parabolic subgroup.
Decompose $B = B' \cdot B^u$ into reductive and unipotent parts. Let $Z$ be the identity component of the center of $B'$, complex subtorus of $S^c$. Let $D$ be the set of characters $\chi \neq 1$ on $Z$ that are restrictions of positive roots, so $\mathfrak{B}^u = \sum D \mathfrak{A}_\chi$. Define $\mathfrak{B}^{-u} = \sum D \mathfrak{A}_{-\chi}$ so that $\mathfrak{A}$ is the direct sum of its subspaces $\mathfrak{B}'$, $\mathfrak{B}^u$ and $\mathfrak{B}^{-u}$. $\mathfrak{B} \cap (\mathfrak{B}^u + \mathfrak{B}^{-u})$ represents the real tangent space of $M$, and $\mathfrak{B}^u + \mathfrak{B}^{-u}$ represents the complexified tangent space. If $\pm \chi \in D$, then $\mathfrak{A}_\chi$ is an irreducible representation space of $B'$, so $J$ acts on $\mathfrak{A}_\chi$ either as $\sqrt{-1}$ or as $-\sqrt{-1}$. Let $\mathfrak{U}^+$ (resp. $\mathfrak{U}^-$) denote the image in $\mathfrak{B}/\mathfrak{B}$ of the $\mathfrak{U}_\chi$, $-\chi \in D$, on which $J$ acts as $\sqrt{-1}$ (resp. $-\sqrt{-1}$). Then $\text{ad}(\mathfrak{B}) \cdot \mathfrak{U}^+ \subseteq \mathfrak{U}^+$ by invariance of $J$ under $B$. If $v$ is the restriction to $Z$ of the highest root, then $\mathfrak{A}/\mathfrak{B} = \sum n \geq 0 \text{ad}(\mathfrak{B})^n \cdot (\mathfrak{A}_-, \text{mod } \mathfrak{B})$, because $\mathfrak{A}$ is simple, so $\mathfrak{A}/\mathfrak{B}$ is the one of $\mathfrak{U}^+$ or $\mathfrak{U}^-$ into which $\mathfrak{A}_-$ maps. Thus either $J$ acts on $\mathfrak{B}^{-u}$ as $\sqrt{-1}$ and the natural complex structure of $A/B$ induces $J$, or $J$ acts on $\mathfrak{B}^{-u}$ as $-\sqrt{-1}$ and the natural structure induces $-J$. In either case $J$ is integrable.

In general suppose $\mathfrak{B}^C \subseteq \mathfrak{A}$. Then $M = G^C/B \cap G^C$ is a complex flag manifold on which $A$ is the largest connected group of analytic automorphisms. Thus $A$ is a centerless complex semisimple group, hence the complexification of its maximal compact subgroup $G$.

Lemma 4.2 is proved.

4.3. Lemma. If $B^C$ is parabolic in $A^C$, then $J$ is integrable so $A = G^C$.

Proof of lemma. $J$ is an element of square $-I$ in the commuting algebra of $\text{ad}(\mathfrak{B})$ on $\mathfrak{A}/\mathfrak{B}$. Thus it induces an element $J^C$ of square $-I$ in the commuting algebra of $\text{ad}(\mathfrak{B}^C)$ on $\mathfrak{A}_C/\mathfrak{B}_C$. Now suppose $B^C$ parabolic in $A^C$, so $M^C = A^C/B^C$ is compact and of positive Euler characteristic with invariant almost complex structure $J^C$.

If $A$ is complex then $A = G^C$ and Lemma 4.2 says that $J$ is integrable. Thus we may assume $A$ not complex so that $A^C$ is simple. Then Lemma 4.2 says that $J^C$ is integrable, and in fact that either $J^C$ or $-J^C$ is induced by the natural complex structure on $A^C/B^C$. Replace $J$ by $-J$ if necessary; that does not alter integrability of $J$, but it replaces $J^C$ by $-J^C$, allowing us to assume $J^C$ induced by the natural complex structure of $A^C/B^C$.

Decompose $B = B' \cdot B^u$ into reductive and unipotent parts, so $\mathfrak{B} = \mathfrak{B}' + \mathfrak{B}^u$ and $\mathfrak{A} = \mathfrak{B} + \mathfrak{B}^{-u}$ where $\mathfrak{B}^{\pm u}$ are subalgebras normalized by $\mathfrak{B}'$. Let $\mathfrak{B}^{-u}$ represent the real tangent space to $M$. Note that $J^C$ acts on $(\mathfrak{B}^{-u})^C$ as $\sqrt{-1}$. That contradicts our arrangement that the action of $J^C$ on $(\mathfrak{B}^{-u})^C$ is induced by the action of $J$ on $\mathfrak{B}^{-u}$. Thus $A$ cannot be noncomplex. Lemma 4.3 is proved.

We complete the proof of Theorem 4.1. As in the second paragraph of the proof of Lemma 4.2, $B$ is a real algebraic subgroup of $A$, so there is a semidirect product decomposition $B = B' \cdot B^u$ into reductive and unipotent parts. If rank $B' < \text{rank } A$, then any Cartan subalgebra of $\mathfrak{A}$ has an element $\xi$ not contained in any isotropy subalgebra of $\mathfrak{A}$ on $M$ so it induces a nonvanishing vector field $\xi^*$ on $M$. The
existence of a nonvanishing vector field $\xi^*$ says that $M$ has Euler characteristic zero. That contradiction proves rank $B'=\text{rank } A$.

Let $\sigma$ be the Cartan involution of $\mathfrak{g}$ with fixed point set $\mathfrak{g}$ and let $\mathfrak{g} = \mathfrak{g} + \mathfrak{h}$ be the Cartan decomposition. We may assume $\sigma(\mathfrak{h}') = \mathfrak{h}'$, so $\mathfrak{h}' = \mathfrak{k} + (\mathfrak{h} \cap \mathfrak{h}')$. That gives compact real forms

$$\mathfrak{K} = \mathfrak{g} + \sqrt{-1} \mathfrak{h} \quad \text{and} \quad \mathfrak{K}' = \mathfrak{h} + \sqrt{-1} (\mathfrak{h} \cap \mathfrak{h}').$$

Let $A_c$ denote the centerless group with Lie algebra $\mathfrak{g}_c$ and let $B'_c$ be the analytic subgroup for $\mathfrak{K}'$. Then rank $B'_c = \text{rank } B' = \text{rank } A = \text{rank } A_c$ tells us that $X = A_c/B'_c$ is a compact simply connected manifold of positive Euler characteristic. If $A = G$ then $B = B' = K$, so $A_c = G$ and $B'_c = K$, whence $X = M$.

As in the second paragraph of the proof of Lemma 4.2 we have Iwasawa decompositions $A = GSN$ and $B = KSN$. Choose a torus subgroup $T \subset K$ such that $H = T \times S \subset B'$ is a Cartan subgroup of $A$. Let $\Delta$ be the root system. Now $\Delta = D \cup E \cup -E$ disjoint, and $\mathfrak{g} = \mathfrak{h}' \oplus \mathfrak{b}^u + \mathfrak{b}^{-u}$ direct, where

$$\mathfrak{b}' = \mathfrak{h} + \mathfrak{h}' \quad \text{and} \quad \mathfrak{b}^u = \mathfrak{h} \cap \{ \sum_{\phi \in \Delta^+} \mathfrak{g}_\phi \}, \quad \mathfrak{b}^{-u} = \mathfrak{h} \cap \{ \sum_{\phi \in \Delta^-} \mathfrak{g}_\phi \}.$$

Observe that $\sigma$ interchanges $\mathfrak{b}^u$ and $\mathfrak{b}^{-u}$. For $\mathfrak{b}^u \cap N$ because $N = N' \cdot B^u$ where $B' = KSN'$, and the dual space of $\mathfrak{n}$ has an ordering such that

$$\mathfrak{g}^C = \sum_{\phi|_S > 0} \mathfrak{g}_\phi, \quad \text{and} \quad \phi|_S > 0 \iff \sigma \phi|_S < 0.$$

View the invariant almost complex structure $J$ of $M$ as an element of square $-I$ in the commuting algebra of $\text{ad}(\mathfrak{h})$ on $\mathfrak{h}/\mathfrak{g}$, hence in the commuting algebra of $\text{ad}(\mathfrak{b}')$ on $\mathfrak{b}' \subset \mathfrak{b}/\mathfrak{h}$; then extend $J$ to an element $J'$ of square $-I$ in the commuting algebra of $\text{ad}(\mathfrak{b}')$ on $\mathfrak{b}^u + \mathfrak{b}^{-u}$ by the formula

$$J'(\xi + \eta) = \sigma J(\sigma \xi) + J(\eta) \quad \text{where } \xi \in \mathfrak{b}^u, \eta \in \mathfrak{b}^{-u}.$$ 

Now $J'$ is an $A$-invariant $\sigma$-invariant almost complex structure on $A/B'$, so [6, Proposition 7.7] it defines an $A_c$-invariant $\sigma$-invariant almost complex structure on $A_c/B'_c$. We have proved that $X = A_c/B'_c$ has an invariant almost complex structure.

Suppose $A \neq G$. Note that [6, Theorem 4.10] eliminates lines 5 and 6 of the Oniščik table above, so either $A = G^C$ or $A$ is absolutely simple and of classical type. Suppose $A \neq G^C$ so $A_c$ is simple and of classical type. Then [6, Theorem 4.10] shows that $B'_c$ is the centralizer of a torus in $A_c$. Let $\mathfrak{g}_c$ denote the center of $\mathfrak{K}'$. Then $\sigma(\mathfrak{K}') = \mathfrak{K}'$ implies $\sigma(\mathfrak{g}_c) = \mathfrak{g}_c$, so $\mathfrak{g}_c = U + (-1)^{1/2} \mathfrak{b}$ with $U \subset \mathfrak{h}$ and $\mathfrak{b} \subset \mathfrak{b} \cap \mathfrak{b}'$. Now $\mathfrak{b}'$ has center $\mathfrak{g} = \mathfrak{h} + \mathfrak{b} \subset \mathfrak{b} \cap \mathfrak{g}$, and $\mathfrak{b}'$ is the centralizer of $\mathfrak{g}$ in $\mathfrak{g}$. We order the root system $\Delta$ so that a root $\phi > 0$ whenever $\phi|_S > 0$ and $\phi|_S > 0$. Then $\mathfrak{g}^C$ contains the Borel subalgebra $\mathfrak{g}_c + \sum_{\phi > 0} \mathfrak{g}_\phi$ of $\mathfrak{g}^C$ for that ordering, so $\mathfrak{g}_c$ is a parabolic subalgebra of $\mathfrak{g}^C$. Then Lemma 4.3 says $A = G^C$. We have proved that $A \neq G$ implies $A = G^C$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
If $J$ is integrable then Lemma 4.2 says $A = GC$. If $J$ is not integrable then Lemma 4.2 says $A \neq GC$, so we cannot have $A \neq G$, and that forces $A = G$. Theorem 4.1 is proved. Q.E.D.

4.3. Remark. Theorem 4.1 extends the scope of [8, Theorem 17.4(3)], but that result remains incomplete because, as remarked at the end of [8, §17], it is not known whether

$$A_0(E_6/\text{ad } SU(3))$$

is $E_6$ rather than $E_6^C$ or whether

$$A_0(SO(n^2-1)/\text{ad } SU(n))$$

is $SO(n^2-1)$ rather than $SO(n^2-1, C)$, $SL(n^2-1, R)$, or $SO(1, n^2-1)$.

REFERENCES