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ON FREE EXTERIOR POWERS

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1. **Introduction.** We study modules over a commutative ring $\boldsymbol{R}$ with unity. We seek properties of a module $\boldsymbol{M}$ for which some exterior power $\bigwedge^p \boldsymbol{M}$ is free of finite rank one or finitely generated.

This work originated in discussions with Howard Osborn. Several of his conjectures are settled below.

2. **Preliminaries.** Let $\boldsymbol{R}$ be a commutative ring with unity and let $\boldsymbol{M}$ be an $\boldsymbol{R}$ module (always assumed unital: $1x = x$ for $x \in \boldsymbol{M}$).

LEMMA 1. *Suppose* $\bigwedge^n \boldsymbol{M} = \boldsymbol{0}$. *Then*

$$\bigwedge^{n+p} \boldsymbol{M} = \boldsymbol{0} \quad \textit{for } p = 1, 2, \ldots.$$

Proof. The map

$$(x_1 \wedge \cdots \wedge x_n, y_1 \wedge \cdots \wedge y_p) \to x_1 \wedge \cdots \wedge y_p$$

induces a surjection

$$(\bigwedge^n \boldsymbol{M}) \otimes (\bigwedge^p \boldsymbol{M}) \to \bigwedge^{n+p} \boldsymbol{M}.$$

The conclusion is immediate.

LEMMA 2. *Let* $\boldsymbol{S} = (s_1, \ldots, s_n)$ *and* $\boldsymbol{T}$ *be ideals of* $\boldsymbol{R}$ *such that* $\boldsymbol{S} + \boldsymbol{T} = \boldsymbol{R}$. *Then for each* p,

$$(s_1^p, \ldots, s_n^p) + \boldsymbol{T} = \boldsymbol{R}.$$

This is proved by induction starting with the standard fact in commutative ideal theory that $\boldsymbol{S}_1 + \boldsymbol{T} = \boldsymbol{S}_2 + \boldsymbol{T} = \boldsymbol{R}$ implies $\boldsymbol{S}_1\boldsymbol{S}_2 + \boldsymbol{T} = \boldsymbol{R}$.

3. **Vanishing of powers.** The following result was conjectured by H. Osborn in the free case.

THEOREM 1. *Suppose for some* p *that* $\bigwedge^p \boldsymbol{M}$ *is a cyclic* $\boldsymbol{R}$*-module. Then* $\bigwedge^{p+k} \boldsymbol{M} = \boldsymbol{0}$ *for* $k = 1, 2, \ldots$.

Proof. It suffices to prove that $\bigwedge^{p+1} \boldsymbol{M} = \boldsymbol{0}$. Let e be a generator of $\bigwedge^p \boldsymbol{M}$ and let $\boldsymbol{T}$ denote the ideal of annihilators of $\bigwedge^p \boldsymbol{M}$, i.e., annihilators of e. Then

$$e = \sum_1^n x_{i1} \wedge \cdots \wedge x_{ip}, \qquad x_{ij} \in \boldsymbol{M}.$$

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But $x_{i1}\wedge\cdots\wedge x_{ip}=s_ie$, $s_i\in \boldsymbol{R}$; hence $\sum s_ie=e$. It follows that $\boldsymbol{S}+\boldsymbol{T}=\boldsymbol{R}$ where $\boldsymbol{S}=(s_1,\ldots,s_n)$.

Now let $y_0\wedge\cdots\wedge y_p\in\bigwedge^{p+1}\boldsymbol{M}$, where $y_j\in\boldsymbol{M}$. If $t\in\boldsymbol{T}$, then

$$t(y_0\wedge\cdots\wedge y_p)=y_0\wedge[t(y_1\wedge\cdots\wedge y_p)]=0.$$

Hence, by Lemma 2, we shall have proved that $y_0\wedge\cdots\wedge y_p=0$ once we have shown that $s_i^2(y_0\wedge\cdots\wedge y_p)=0$. It suffices to consider $i=1$. Now

$$y_1\wedge\cdots\wedge y_p=re \quad\text{and}\quad y_0\wedge x_{11}\wedge\cdots\wedge x_{1,p-1}=r'e,$$

hence

$$\begin{aligned} s_1^2(y_0\wedge\cdots\wedge y_p) &= s_1^2r(y_0\wedge e)\\ &= s_1r(y_0\wedge x_{11}\wedge\cdots\wedge x_{1p})\\ &= s_1rr'(e\wedge x_{1p})\\ &= rr'(x_{11}\wedge\cdots\wedge x_{1p}\wedge x_{1p})\\ &= 0.\end{aligned}$$

The natural generalization of this result is the conjecture that if $\bigwedge^p\boldsymbol{M}$ can be generated by q elements, then $\bigwedge^{p+q}\boldsymbol{M}=\boldsymbol{0}$. It is doubtful that this is true; however, we do have the following.

THEOREM 2. *Suppose for some value of p that $\bigwedge^p\boldsymbol{M}$ is a free module of finite rank q. Then $\bigwedge^{p+q}\boldsymbol{M}=\boldsymbol{0}$.*

Proof. Let $e_1,\ldots,e_q$ be a basis of $\bigwedge^p\boldsymbol{M}$. Then

$$e_i=\sum_j x_{ij},\qquad x_{ij}=x_{ij1}\wedge\cdots\wedge x_{ijp},$$

and

$$x_{ij}=\sum s_{ijk}e_k,\qquad s_{ijk}\in\boldsymbol{R}.$$

Hence

$$e_i=\sum_k\left(\sum_j s_{ijk}\right)e_k,\qquad \sum_j s_{ijk}=\delta_{ik}.$$

Here $1\leqq i,\ k\leqq q$. Now select any $(j_1,j_2,\ldots,j_q)=\boldsymbol{j}$. By Cramer's rule,

$$[\det(s_{ij_ik})_{i,k}]e_r\in\sum_{m=1}^{q}\boldsymbol{R}x_{mj_m}.$$

Call the determinant factor $\Delta_{\boldsymbol{j}}$. These determinants together generate the unit ideal $\boldsymbol{R}$ since

$$1=\det(\delta_{ij})=\det\left(\sum_j s_{ijk}\right)_{i,k}=\sum\Delta_{\boldsymbol{j}}.$$

Renumbering, we have obtained quantities $t_1, t_2, \ldots \in \boldsymbol{R}$ such that

(i) $(t_1, t_2, \ldots) = \boldsymbol{R}$,

(ii) for each i,

$$t_i \bigwedge^p \boldsymbol{M} \leqq \sum_{k=1}^{q} \boldsymbol{R} y_{ik},$$

where the y_{ik} are pure p-vectors. From (i) and Lemma 2 above, $(t_1^{q+1}, t_2^{q+1}, \ldots) = \boldsymbol{R}$. We shall prove that

$$t_i^{q+1}(\bigwedge^{p+q} \boldsymbol{M}) = \boldsymbol{0} \qquad (i = 1, 2, \ldots),$$

which implies $\bigwedge^{p+q} \boldsymbol{M} = \boldsymbol{0}$. It suffices to prove

$$t^{q+1}(\bigwedge^{p+q} \boldsymbol{M}) = \boldsymbol{0}$$

if $t(\bigwedge^p \boldsymbol{M})$ is contained in a submodule of $\bigwedge^p \boldsymbol{M}$ generated by q pure p-vectors $y_i = y_{i1} \wedge \cdots \wedge y_{ip}$ $(i = 1, \ldots, q)$. We resort to "abuse of language" to save notation in the calculation that follows:

$$\begin{aligned} t^{q+1}(\bigwedge^{p+q} \boldsymbol{M}) &= t^{q+1}(\bigwedge^p \boldsymbol{M}) \wedge (\bigwedge^q \boldsymbol{M}) \\ &\leqq t^q \sum \boldsymbol{R} y_i \wedge (\bigwedge^q \boldsymbol{M}) = t^q \sum \boldsymbol{R} y_{i1} \wedge (\bigwedge^p \boldsymbol{M}) \wedge (\bigwedge^{q-1} \boldsymbol{M}) \\ &\leqq t^{q-1} \sum \boldsymbol{R} y_{i1} \wedge y_j \wedge (\bigwedge^{q-1} \boldsymbol{M}) \\ &= t^{q-1} \sum \boldsymbol{R} y_{i1} \wedge y_{j1} \wedge (\bigwedge^p \boldsymbol{M}) \wedge (\bigwedge^{q-2} \boldsymbol{M}) \\ &\leqq \cdots = \cdots \\ &\leqq \sum \boldsymbol{R} y_{i1} \wedge y_{j1} \wedge \cdots \wedge y_{k1}. \end{aligned}$$

But each $(q+1)$-tuple of indices $(i, j, \ldots, k)$ contains a repetition, hence the last module written vanishes. This completes the proof.

Besides the conjecture mentioned before the statement of the theorem, one might also ask whether

$$\bigwedge^{q+1}(\bigwedge^p \boldsymbol{M}) = \boldsymbol{0} \qquad (p \geqq 1, q \geqq 0)$$

implies

$$\bigwedge^{p+q} \boldsymbol{M} = \boldsymbol{0}.$$

4. **Duality.** Let $\boldsymbol{M}$ be an $\boldsymbol{R}$-module, and denote its conjugate space by $\boldsymbol{M}^* = \mathrm{Hom}\,(\boldsymbol{M}, \boldsymbol{R})$. Recall the natural homomorphisms:

$$\bigwedge^p \boldsymbol{M}^* \to (\bigwedge^p \boldsymbol{M})^*, \qquad \boldsymbol{M} \to (\boldsymbol{M}^*)^*.$$

The first is based on the pairing

$$\langle f_1 \wedge \cdots \wedge f_p, z_1 \wedge \cdots \wedge z_p \rangle = \det\,(\langle f_i, z_j \rangle)$$

of

$$(\bigwedge^p \boldsymbol{M}^*) \times (\bigwedge^p \boldsymbol{M}) \to \boldsymbol{R}.$$

If $\bigwedge^p M$ is free of rank one with basis e, then there are further natural (relative to e) homomorphisms:

$$\bigwedge^r M \to (\bigwedge^{p-r} M)^* \qquad (r = 0, \ldots, p),$$

based on

$$(z_1 \wedge \cdots \wedge z_r) \wedge (z_{r+1} \wedge \cdots \wedge z_p) = f(z_{r+1} \wedge \cdots \wedge z_p)e.$$

The main result of this section, the theorem below, settles several conjectures of H. Osborn.

THEOREM 3. *Let M be an R-module such that $\bigwedge^p M$ is free of rank one for some value of p. Then the following hold:*

(i) *M and M^* are finitely generated.*

(ii) *$\bigwedge^p M^*$ is free of rank one.*

(iii) *The natural map $\bigwedge^p M^* \to (\bigwedge^p M)^*$ is an isomorphism.*

(iv) *M is reflexive: the natural map $M \to M^{**}$ is an isomorphism.*

(v) *For each $r=0, \ldots, p$, the natural map $\bigwedge^r M^* \to (\bigwedge^r M)^*$ is an isomorphism. In addition, each module $\bigwedge^r M$, $\bigwedge^r M^*$ is reflexive:*

$$\bigwedge^r M \approx (\bigwedge^r M)^{**}, \qquad \bigwedge^r M^* \approx (\bigwedge^r M^*)^{**}.$$

(vi) *For each $r=0, \ldots, p$, the natural map (relative to a basis of $\bigwedge^p M$)*

$$\bigwedge^r M \to (\bigwedge^{p-r} M)^*$$

is an isomorphism.

(vii) *The modules M and M^* are projective.*

The proof will involve several steps.

LEMMA 3. *If $F \in (\bigwedge^p M)^*$ and $z_0, \ldots, z_p \in M$, then*

$$\sum (-1)^j F(z_0 \wedge \cdots \wedge z_{j-1} \wedge z_{j+1} \wedge \cdots \wedge z_p) z_j = 0.$$

Proof. The mapping

$$(z_0, \ldots, z_p) \to \sum (-1)^j F(z_0 \wedge \cdots \wedge z_{j-1} \wedge z_{j+1} \wedge \cdots \wedge z_p) z_j$$

is alternating multilinear on $\times^{p+1} M \to M$, hence vanishes identically by Theorem 1.

For the next steps it is convenient to fix certain elements of R, M, M^*, and their exterior powers.

Let e be a basis of $\bigwedge^p M$. Then $e=\sum_1^n b_i x_i$, where $b_i \in R$, $x_i = x_{i1} \wedge \cdots \wedge x_{ip}$. Since e is a basis, $x_i = s_i e$, $\sum b_i s_i = 1$. Set

$$y_{ij} = (-1)^{j-1} x_{i1} \wedge \cdots \wedge x_{i,j-1} \wedge x_{i,j+1} \wedge \cdots \wedge x_{ip} \in \bigwedge^{p-1} M,$$

so $x_i = x_{ij} \wedge y_{ij}$. Define $f_{ij} \in M^*$ by

$$\langle f_{ij}, z \rangle e = z \wedge y_{ij} \qquad (1 \leqq i \leqq n,\ 1 \leqq j \leqq p).$$

Then $\langle f_{ij}, x_{ij}\rangle = s_i$, $\langle f_{ij}, x_{ik}\rangle = 0$ $(k \neq j)$. Set $f_i = f_{i1}\wedge\cdots\wedge f_{ip} \in \bigwedge^p \boldsymbol{M}^*$. Then

$$\langle f_i, x_i\rangle = \det\,(\langle f_{ij}, x_{ik}\rangle)_{j,k} = s_i^p.$$

Also by a simple determinant computation, $\langle f_i, z\wedge y_{ij}\rangle = s_i^{p-1}\langle f_{ij}, z\rangle$.

The key to the whole proof is the following

LEMMA 4. *For $z \in \boldsymbol{M}$ and for $g \in \boldsymbol{M}^*$ we have*

$$\left.\begin{aligned} s_i z &= \sum_{j=1}^{p} \langle f_{ij}, z\rangle x_{ij}, \\ s_i g &= \sum_{j=1}^{p} \langle g, x_{ij}\rangle f_{ij} \end{aligned}\right. \qquad (1 \leqq i \leqq n).$$

Proof. Since $\bigwedge^p \boldsymbol{M}$ is free with basis e, there is an $F \in (\bigwedge^p \boldsymbol{M})^*$ such that $F(e)=1$. Apply Lemma 3 to F; $z, x_{i1}, \ldots, x_{ip}$:

$$F(x_i)z = \sum_j F(z\wedge y_{ij})x_{ij}.$$

But $F(x_i) = F(s_i e) = s_i$ and $F(z\wedge y_{ij}) = F(\langle f_{ij}, z\rangle e) = \langle f_{ij}, z\rangle$, so the first relation follows.

Now apply g to this relation:

$$s_i\langle g, z\rangle = \sum_j \langle f_{ij}, z\rangle\langle g, x_{ij}\rangle.$$

Since this is true for all $z \in \boldsymbol{M}$, the second relation follows.

LEMMA 5. *If $z \in \boldsymbol{M}$ and if $g \in \boldsymbol{M}^*$, then*

$$z = \sum_{i,j} b_i\langle f_{ij}, z\rangle x_{ij}, \qquad g = \sum_{i,j} b_i\langle g, x_{ij}\rangle f_{ij}.$$

Proof. Multiply each relation in Lemma 4 by b_i and sum.

Proof of (i), **Theorem 3.** By Lemma 5, the x_{ij} $(1\leqq i\leqq n,\ 1\leqq j\leqq p)$ span $\boldsymbol{M}$ and the f_{ij} span $\boldsymbol{M}^*$.

By Lemma 2, $(s_1^p, \ldots, s_n^p) = \boldsymbol{R}$. Hence there are $a_i \in \boldsymbol{R}$ satisfying

$$\sum_1^n a_i s_i^p = 1.$$

Set

$$f = \sum_1^n a_i s_i f_i \in \textstyle\bigwedge^p \boldsymbol{M}^*.$$

LEMMA 6. $s_i\langle f_i, e\rangle = s_i^p$, $\langle f, e\rangle = 1$.

Proof.

$$s_i\langle f_i, e\rangle = \langle f_i, s_i e\rangle = \langle f_i, x_i\rangle = s_i^p,$$
$$\langle f, e\rangle = \sum a_i s_i\langle f_i, e\rangle = \sum a_i s_i^p = 1.$$

LEMMA 7. *For each* $g \in \bigwedge^p M^*$, $g = \langle g, e\rangle f$.

Proof. It suffices to prove this for g pure, $g = g_1 \wedge \cdots \wedge g_p$. By Lemma 4, $s_i g_k = \sum_j \langle g_k, x_{ij}\rangle f_{ij}$. Multiply these together:

$$s_i^p g = \langle g, x_i\rangle f_i = \langle g, e\rangle s_i f_i.$$

Multiply by a_i and sum to complete the proof.

LEMMA 8. *For each* i, $s_i f_i = s_i^p f$.

Proof. $s_i f_i = \langle s_i f_i, e\rangle f = \langle f_i, s_i e\rangle f = \langle f_i, x_i\rangle f = s_i^p f$.

Proof of (ii), **Theorem 3.** By Lemma 7, f spans $\bigwedge^p M^*$. By Lemma 6, if $cf = 0$, then $c = c\langle f, e\rangle = 0$, hence f is free, $\bigwedge^p M^*$ is free of rank one with basis f.

Proof of (iii), **Theorem 3.** By Lemma 6, $f \to$ generator of $(\bigwedge^p M)^*$, so the map is an isomorphism.

LEMMA 9. *The natural map* $M \to M^{**}$ *is injective.*

Proof. Suppose $z \in M$ and $g(z) = 0$ for all $g \in M^*$. By the first formula in Lemma 5, $z = 0$.

To complete the proof of (iv) we must study the homomorphisms of M^* induced by the x_{ij}. Let

$$x_{ij} \to \phi_{ij} \in M^{**} \qquad \phi_{ij}(g) = \langle g, x_{ij}\rangle.$$

LEMMA 10. *For each* $\phi \in M^{**}$,

$$s_i\phi = \sum_{j=1}^{p} \phi(f_{ij})\phi_{ij} \qquad (1 \leqq i \leqq n),$$

$$\phi = \sum_{i,j} b_i \phi(f_{ij})\phi_{ij}.$$

Proof. Let $g \in M^*$. Apply ϕ to the second formula of Lemma 4:

$$s_i\phi(g) = \sum_j \langle g, x_{ij}\rangle \phi(f_{ij}) = \sum_j \phi(f_{ij})\phi_{ij}(g).$$

Hence the first formula of Lemma 10; the second formula easily follows.

Proof of (iv), **Theorem 3.** The map $M \to M^{**}$ is injective by Lemma 9 and surjective by Lemma 10 (since $x_{ij} \to \phi_{ij}$), hence is an isomorphism.

Now we prepare for the proofs of (v) and (vi). The cases $r = 0$, $r = p$ are trivial. Fix r, $1 \leqq r \leqq p-1$. As before, let H run over r-element subsets of $\{1, \ldots, p\}$ and let H' denote the complement of H. Set

$$x_{iH} = x_{ih_1} \wedge \cdots \wedge x_{ih_r} \in \textstyle\bigwedge^r M, \quad \text{etc.}$$

LEMMA 11. *If* $z \in \bigwedge^r M$, *then*

$$s_i^r z = \sum_H \langle f_{iH}, z\rangle x_{iH}.$$

If $g \in \bigwedge^r \boldsymbol{M}^*$, *then*

$$s_i^r g = \sum_H \langle g, x_{iH}\rangle f_{iH},$$

both relations for $1 \leqq i \leqq n$.

These formulas follow easily from Lemma 4 applied to the special cases of z, g pure.

LEMMA 12. *The* x_{iH} *span* $\bigwedge^r \boldsymbol{M}$ *and the* f_{iH} *span* $\bigwedge^r \boldsymbol{M}^*$.

This follows from Lemma 11 and the consequences of Lemma 2: $(s_1^r, \ldots, s_n^r) = \boldsymbol{R}$.

Proof of (v), **Theorem 3.** Each $g \in \bigwedge^r \boldsymbol{M}^*$ induces an element of $(\bigwedge^r \boldsymbol{M})^*$ via $z \to \langle g, z\rangle$. The resulting map $\bigwedge^r \boldsymbol{M}^* \to (\bigwedge^r \boldsymbol{M})^*$ is injective. For suppose $\langle g, z\rangle = 0$ for all z. By Lemma 11, $s_i^r g = 0$ for $1 \leqq i \leqq n$. But $\boldsymbol{R} = (s_1^r, \ldots, s_n^r)$, hence $g = 0$.

The map is also surjective. For let $F \in (\bigwedge^r \boldsymbol{M})^*$. Apply F to the first formula of Lemma 11:

$$s_i^r F(z) = \sum_H \langle f_{iH}, z\rangle F(x_{iH}).$$

Select c_i so $\sum c_i s_i^r = 1$. Then

$$F(z) = \sum_{i,H} c_i F(x_{iH}) \langle f_{iH}, z\rangle,$$

which shows that F lies in the image of $\bigwedge^r \boldsymbol{M}^*$. Hence the map is an isomorphism.

Similar reasoning, applied to the second identity of Lemma 11, shows that each linear functional on $\bigwedge^r \boldsymbol{M}^*$ is induced by a unique element of $\bigwedge^r \boldsymbol{M}$, so each module is the conjugate of the other.

Proof of (vi), **Theorem 3.** We are considering the pairing

$$\pi\colon \bigwedge^r \boldsymbol{M} \times \bigwedge^{p-r} \boldsymbol{M} \to \boldsymbol{R}$$

given by $z \wedge w = \pi(z, w)e$, which induces

$$\bigwedge^r \boldsymbol{M} \to (\bigwedge^{p-r} \boldsymbol{M})^* \to \bigwedge^{p-r} \boldsymbol{M}^*.$$

Statement (vi) asserts that this map is an isomorphism.

It is an injection. Suppose $z \to 0$ for some $z \in \bigwedge^r \boldsymbol{M}$, which means $z \wedge w = 0$ for all $w \in \bigwedge^{p-r} \boldsymbol{M}$. By Lemma 11,

$$s_i^r z = \sum_H \langle f_{iH}, z\rangle x_{iH}.$$

Hence in particular,

$$\sum_H \langle f_{iH}, z\rangle x_{iH} \wedge x_{iK'} = 0,$$

where $|K| = r$. Thus for each H, and each i,

$$\langle f_{iH}, z\rangle x_i = 0,$$

$$s_i \langle f_{iH}, z\rangle = 0.$$

To prove $z=0$, we shall show that the $s_i f_{iH}$ span $\bigwedge^r \boldsymbol{M}^*$ and appeal to conclusion (v) of the theorem. But the second formula of Lemma 11 implies that if $g \in \bigwedge^r \boldsymbol{M}^*$, then $s_i^{r+1} g$ is a linear combination of $s_i f_{iH}$. Since $(s_1^{r+1}, \ldots, s_n^{r+1}) = \boldsymbol{R}$, so is g.

Now let $F \in (\bigwedge^{p-r} \boldsymbol{M})^*$. If $w \in \bigwedge^{p-r} \boldsymbol{M}$, then (Lemma 11)

$$s_i^{p-r} w = \sum_K \langle f_{iK'}, w \rangle x_{iK'}.$$

Hence

$$s_i^{p-r} F(w) = \sum_H \langle f_{iH'}, w \rangle F(x_{iH'})$$

on the one hand, showing that the functionals $w \to s_i \langle f_{iH'}, w \rangle$ span $(\bigwedge^{p-r} \boldsymbol{M})^*$; and

$$s_i^{p-r} x_{iH} \wedge w = \varepsilon_{H,H'} \langle f_{iH'}, w \rangle s_i e,$$

or $s_i^{p-r} \pi(x_{iH}, w) = \varepsilon_{H,H'} s_i \langle f_{iH'}, w \rangle$ on the other hand. This latter relation shows that each $w \to s_i \langle f_{iH'}, w \rangle$ is the image of $\pm s_i^{p-r} x_{iH}$; the mapping is surjective.

Proof of (vii), **Theorem 3.** Suppose we have homomorphisms

$$\begin{array}{ccccc} & & \boldsymbol{M} & & \\ & & \downarrow{\scriptstyle \phi} & & \\ \boldsymbol{A} & \xrightarrow[\psi]{} & \boldsymbol{C} & \longrightarrow & \boldsymbol{0} \end{array}$$

where the row is exact. Select $u_{ij} \in \boldsymbol{A}$ so $\psi(u_{ij}) = \phi(x_{ij})$. Define $\lambda: \boldsymbol{M} \to \boldsymbol{A}$ by

$$\lambda(z) = \sum b_i \langle f_{ij}, z \rangle u_{ij}.$$

By Lemma 5, $\psi \circ \lambda = \phi$. Hence $\boldsymbol{M}$ is projective; similarly $\boldsymbol{M}^*$ is so.

Remark. The isomorphism

$$\bigwedge^r \boldsymbol{M} \to \bigwedge^{p-r} \boldsymbol{M}^*$$

can be made explicit by use of interior products. Indeed, as Professor Osborn has pointed out, this provides an alternate proof of (v). The products $\lrcorner$ and $\llcorner$ are defined by

$$\langle f \lrcorner z, w \rangle = \langle f, z \wedge w \rangle$$

for $z \in \bigwedge^r \boldsymbol{M}$, $w \in \bigwedge^{p-r} \boldsymbol{M}$,

$$\langle g, h \llcorner e \rangle = \langle h \wedge g, e \rangle$$

for $h \in \bigwedge^{p-r} \boldsymbol{M}^*$, $g \in \bigwedge^r \boldsymbol{M}^*$. Thus

$$f \lrcorner : \bigwedge^r \boldsymbol{M} \to \bigwedge^{p-r} \boldsymbol{M}^*,$$
$$\llcorner e : \bigwedge^{p-r} \boldsymbol{M}^* \to \bigwedge^r \boldsymbol{M}.$$

The basic relation is the Cauchy-Binet Formula:

$$\langle f \lrcorner z, g \llcorner e \rangle = \langle g, z \rangle$$

for $z \in \bigwedge^r \boldsymbol{M}$, $g \in \bigwedge^{p-r} \boldsymbol{M}^*$.

This is proved by calculation for the generators $x_{iH}, f_{iK'}$. An easy consequence is the following pair of formulas:

$$(f \lrcorner z) \llcorner e = (-1)^{r(p-r)} z$$
$$f \lrcorner (h \llcorner e) = (-1)^{r(p-r)} h$$

for $z \in \bigwedge^r M$, $h \in \bigwedge^{p-r} M^*$. Hence $z \to f \lrcorner z$, $h \to h \llcorner e$ are isomorphisms, inverses of each other up to sign.

5. **Other results.** The following special case of Theorem 3 has some interest.

THEOREM 4. *Suppose $\bigwedge^p M$ is free of rank one and $\bigwedge^p M = Re$ where e is a pure vector, $e = x_1 \wedge \cdots \wedge x_p$. Then M is a free module of rank p.*

Proof. Proceed as above, but with $n=1$, $s_1=1$. Thus $\langle f_i, x_j \rangle = \delta_{ij}$ so $x_1, \ldots, x_p$ are linearly independent. We know they span (Lemma 5), hence they form a basis of M.

The following result is due to Osborn, who used it to obtain part of Theorem 3 under stronger hypotheses.

THEOREM 5. *Suppose R is a local ring and $\bigwedge^p M$ is free of rank one. Then M is free.*

Proof. Use the notation of Theorem 3. Then $\bigwedge^p M$ has basis $e = \sum b_i x_i$, where x_i are pure p-vectors, and $x_i = s_i e$ with $\sum b_i s_i = 1$. Since R is a local ring and $(s_1, \ldots, s_n) = R$, some s_i is a unit, hence x_i is a basis and Theorem 4 applies.

Lemma 3 above can be generalized in a way which might be useful.

THEOREM 6. *Let $\bigwedge^p M$ be cyclic. Let*

$$F \in \operatorname{Hom}(\textstyle\bigwedge^r M, N_1) \quad \textit{and} \quad G \in \operatorname{Hom}(\textstyle\bigwedge^{p-r+1} M, N_2).$$

Let $z_0, \ldots, z_p \in M$. Then

$$\sum_H \varepsilon_{H,H'} F(z_H) \otimes G(z_{H'}) = 0,$$

where H runs over r element subsets of $\{0, 1, \ldots, p\}$,

$$H = \{i_1 < i_2 < \cdots < i_r\}, \qquad H' = \{j_0 < j_1 < \cdots < j_{p-r}\}$$

is the complement of H in $\{0, 1, \ldots, p\}$, *and*

$$z_H = z_{i_1} \wedge z_{i_2} \wedge \cdots \wedge z_{i_r}, \qquad z_{H'} = z_{j_0} \wedge \cdots \wedge z_{j_{p-r}}.$$

Proof. The map

$$(z_0, \ldots, z_p) \to \sum_H \varepsilon_{H,H'} F(z_H) \otimes G(z_{H'})$$

is alternating multilinear on

$$\times^{p+1} M \to (\textstyle\bigwedge^r N_1) \otimes (\textstyle\bigwedge^{p-r+1} N_2).$$

But $\bigwedge^{p+1} M = 0$ by Theorem 1, hence the map vanishes.

Some of the obvious generalizations of Theorem 3, conclusion (i) are wrong. For example the $\boldsymbol{Z}$ module $\boldsymbol{M}=\boldsymbol{Q}/\boldsymbol{Z}$ is not finitely generated, but $\bigwedge^2 \boldsymbol{M}=\boldsymbol{0}$. Thus in general $\bigwedge^p \boldsymbol{M}$ finitely generated does not imply $\boldsymbol{M}$ finitely generated. Also $\bigwedge^p \boldsymbol{M}$ finitely generated (or even generated by one element) does not imply $\bigwedge^p \boldsymbol{M}^*$ finitely generated. For example, let $\boldsymbol{k}$ be a field, V a (countably) infinite dimensional $\boldsymbol{k}$-space. Set $\boldsymbol{R}=\boldsymbol{k}\oplus V$ with trivial multiplication in V. Set $\boldsymbol{M}=\boldsymbol{R}/V\approx\boldsymbol{k}$. Then $\boldsymbol{M}$ is a cyclic $\boldsymbol{R}$-module. But $\boldsymbol{M}^*=\operatorname{Hom}(\boldsymbol{M},\boldsymbol{R})\approx V$, so each $\bigwedge^p \boldsymbol{M}^*$ is infinite dimensional.

In Theorem 2 it was shown that $\bigwedge^p \boldsymbol{M}$ free of rank q implies $\bigwedge^{p+q}\boldsymbol{M}=\boldsymbol{0}$. Assuming, of course, $q\geqq 1$, does this imply $\bigwedge^p \boldsymbol{M}^*$ free of rank q and the duality situation of Theorem 3? This is difficult and will be postponed for later investigation.

6. **Examples.** Let $\boldsymbol{R}$ be the ring of real analytic functions on the p-sphere $\boldsymbol{S}^p$ in Euclidean $\boldsymbol{E}^{p+1}$ defined by $x_0^2+\cdots+x_p^2=1$. Let $\boldsymbol{M}$ be the module of real analytic differentiable one-forms on $\boldsymbol{S}^p$. The $\bigwedge^p \boldsymbol{M}$ is free of rank one with basis the element of area

$$\sigma=\sum(-1)^j x_j\,dx_0\wedge\cdots\wedge dx_{j-1}\wedge dx_{j+1}\wedge\cdots\wedge dx_p.$$

By Theorem 3, $\boldsymbol{M}$ is projective; but $\boldsymbol{M}$ is not free unless $\boldsymbol{S}^p$ is parallelizable, i.e., $p=1$, 3, or 7. Our work above shows that each real analytic one-form on $\boldsymbol{S}^p$ can be written $\sum f_i\,dx_i$ where $f_i\in\boldsymbol{R}$. (This is not obvious because the elements of $\boldsymbol{M}$ are cross sections of the bundle of one-forms at all points of $\boldsymbol{S}^p$.) Similar remarks apply to any real analytic manifold which is orientable and is a submanifold of euclidean space.

An algebraic model may be constructed as follows. Let $\boldsymbol{S}$ be a ring and $\boldsymbol{R}=\boldsymbol{S}[r,s,t]$ subject to $r^2+s^2+t^2=1$. Let $\boldsymbol{M}=\boldsymbol{R}x+\boldsymbol{R}y+\boldsymbol{R}z$ subject to $rx+sy+tz=0$. Then $\bigwedge^2\boldsymbol{M}$ has the basis $e=ry\wedge z+sz\wedge x+tx\wedge y$ so $\boldsymbol{M}$ is projective; but $\boldsymbol{M}$ is not free.

A similar example is obtained from the ring $\boldsymbol{R}=\boldsymbol{S}[a,b,c,r,s,t]$ subject to the single relation $ar+bs+ct=1$. The module is $\boldsymbol{M}=\boldsymbol{R}x+\boldsymbol{R}y+\boldsymbol{R}z$ with the generating relation $rx+sy+tz=0$. The element $e=ay\wedge z+bz\wedge x+cx\wedge y$ is a basis of $\bigwedge^2\boldsymbol{M}$, so $\boldsymbol{M}$ is projective; but $\boldsymbol{M}$ is not free.

To see that e is a basis, first observe by the defining relations that

$$x\wedge y=te,\qquad y\wedge z=re,\qquad z\wedge x=se,$$

hence e generates $\bigwedge^2\boldsymbol{M}$. Next the map $F\colon\times^2\boldsymbol{M}\to\boldsymbol{R}$ given by

$$F(\alpha_1x+\alpha_2y+\alpha_3z,\beta_1x+\beta_2y+\beta_3z)=\begin{vmatrix}\alpha_1&\alpha_2&\alpha_3\\ \beta_1&\beta_2&\beta_3\\ r&s&t\end{vmatrix}$$

is well defined and is alternating bilinear, hence defines $F\colon\bigwedge^2\boldsymbol{M}\to\boldsymbol{R}$. But $F(e)=1$, hence e is free.

The proof that $\boldsymbol{M}$ is not free is more complicated.

References

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