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FREE MODULES OVER FREE ALGEBRAS AND FREE GROUP ALGEBRAS: THE SCHREIER TECHNIQUE

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I. **Introduction.** It has become abundantly clear through the remarkable work of P. M. Cohn and G. Bergman that free ideal rings (rings over which submodules of free modules are again free of unique rank) form an extremely interesting class of rings. The main goal of this paper is to provide an alternative proof of the fact that free algebras (over a field) and free group algebras are free ideal rings (Cohn [2]–[6]). The method used here goes back to Schreier: we obtain free generators for a submodule of a free module from a special set of coset representatives (§III). The analogy with the Schreier generators of a subgroup of a free group is quite strict (see e.g. [10]) and theorems about free groups which only involve Schreier techniques have easy translations into theorems about free modules. Perhaps most striking is the fact that there is a "Schreier formula" relating the number of generators of a submodule to the dimension of the quotient module (§V).

If we restrict our attention to free algebras we obtain the following information on finitely presented modules; they are extensions of a free module by a finite dimensional module. Also, we find that the endomorphism algebra of a finitely presented module without free summands is finite dimensional. Thus the eigenring of an element in a free algebra is finite dimensional (§IV). This last fact can essentially be found in Cohn [3, Theorem 5.1].

In a somewhat different direction, it is not known whether a finitely generated subalgebra of a free algebra is finitely presented qua algebra. (Indeed it does not seem to be known whether a finitely generated subsemigroup of a free semigroup is finitely presented.)

We prove here that a subalgebra of finite codimension in a free algebra is finitely presented (§VII).

II. **Notation.** The notation is uniform. G is either the free monoid or the free group freely generated by the set $X=\{X_\alpha\}_{\alpha\in A}$, and F is the semigroup (group) algebra of G with coefficients in the field $\mathfrak{k}$. In the first case, F is the free algebra freely generated by X. N is the free right F-module with basis $E=\{e_\lambda\}_{\lambda\in\Lambda}$. If $\lambda=x_{\alpha_1}^{\varepsilon_1}\cdots x_{\alpha_n}^{\varepsilon_n}$, is an element of G (also called a monomial of F), an element $m=e_\lambda x$

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of N is called a monomial of N, and an element $\alpha e_\lambda x$ is called a monomial term of N. We define initial and terminal segments of monomials of G as follows: ${}^{(0)}x=1$, ${}^{(i)}x=x_{\alpha_1}^{\varepsilon_1}\cdots x_{\alpha_i}^{\varepsilon_i}$ for $i=1,\ldots,n$ and $x^{(i)}=x_{\alpha_{i+1}}^{\varepsilon_{i+1}}\cdots x_{\alpha_n}^{\varepsilon_n}$ for $i=0,\ldots,n-1$; $x^{(n)}=1$.

The length function on G induces as usual a degree function d on F. If we assign degree 0 to the e_λ's, then we also have, in the obvious way, a degree again called d on the module N (e.g. $d(2e_1x_2x_1+3e_2x_4)=2$). We also wish to well order N. This is done in the usual way by well ordering X, E and $\mathfrak{k}$ with 0 as the first element $x_\alpha < x_\alpha^{-1}$, ordering G by length and reverse lexicographical order on elements of the same length ($x_3x_1 < x_1x_2$), extending this order to monomial terms, and finally to sums of monomial terms in the obvious way. Thus $\alpha_1e_2x_1x_2 < \alpha_1e_2x_1x_2+e_3x_1$ and $e_3x_2 < e_5x_1$.

If S is a subset of N, then $\langle S\rangle$ is the $\mathfrak{k}$-space spanned by S and $\operatorname{Mod}(S)$ the F-submodule generated by S.

If M is a submodule of N, a transversal T for M in N is a complete irredundant set of coset representatives (including 0) for M in N, both being considered as abelian groups. The function ϕ which assigns to an element of N its representative in T is called the transversal function. If the transversal T is also a $\mathfrak{k}$-space (this will always be the case) then ϕ is $\mathfrak{k}$-linear. Further, if $a\in M$ and $r\in F$, then $(a-\phi(a))r\in M$ and hence

$$(1)\qquad \phi(ar) = \phi(\phi(a)r).$$

If $A<B$ are $\mathfrak{k}$-spaces and the dimension of B/A is n, we say that A has codimension n in B.

We will have occasion to use Cohn's notion of d-dependence. For a definition of this important concept we refer the reader to [2].

III. **Schreier transversals and Schreier generators.** We will only prove the main theorem for free modules over free group algebras. The proof for free algebras is similar (and easier). Thus G is now the free group, and M a submodule of the free F-module N.

1. A partial Schreier basis for N mod M is a set B of *monomials* of N such that

(i) B is $\mathfrak{k}$-linearly independent modulo M,

(ii) if $b=e_\lambda x$ is in B, then all the initial segments $e_\lambda^{(i)}x$ are again in B.

A partial Schreier basis B is a Schreier basis if B also spans N modulo M. Following the proof of Lemma 1 of Dunwoody [7], there is no difficulty in establishing

LEMMA 1. *Any partial Schreier basis for N modulo M can be extended to a Schreier basis.*

From the definition of a Schreier basis B, it is clear that the space $\langle B\rangle$ is a transversal for M in N, with associated transversal function ϕ. We will have occasion to make use of a "minimal" Schreier basis, i.e. a basis B for which $d(\phi(y))\leqq d(y)$. Such a minimal Schreier basis always exists as the following argu-

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ment (borrowed from group theory) shows: Using the well order of §II, let ϕ be the function from N to N which assigns to an element of N the earliest element in its coset, and let B be the set of monomials for which $\phi(b)=b$. We claim that B is a minimal Schreier basis for $N \bmod M$, with transversal function ϕ. The minimality is of course clear. We first show that ϕ is $\mathfrak{k}$-linear. Thus let $\alpha \in \mathfrak{k}$, f and g be in N. Since $\phi(\alpha f+g)$ and $\alpha\phi(f)+\phi(g)$ lie in the same coset, we must show that $\alpha\phi(f)+\phi(g)$ is the least element in its coset. If not, for some $m \in M$, $m+\alpha\phi(f)+\phi(g)$ is earlier than $\alpha\phi(f)+\phi(g)$. This can only happen if the latest monomial of m is the same as some monomial of $\alpha\phi(f)+\phi(g)$, and hence only if the latest monomial of m is the same as some monomial of $\phi(a)$ or of $\phi(b)$. But then, modifying $\phi(a)$ or $\phi(b)$ by a suitable multiple of m yields an earlier element in either the coset of $\phi(a)$ or of $\phi(b)$, a contradiction. It is now clear that B is $\mathfrak{k}$-independent modulo M, for if $\sum \alpha_i b_i \in M$, then $0=\phi(\sum \alpha_i b_i)=\sum \alpha_i\phi(b_i)=\sum \alpha_i b_i$. To show that B spans $N \bmod M$, suppose that x is the least element of N which does not depend on $B \bmod M$. Then since $\phi(x)\leqq x$ we must have $\phi(x)=x$. Suppose $x=\sum \beta_i m_i$, with m_1 its latest monomial. Then, again by minimality, $\phi(\beta_1 m_1)=\beta_1 m_1$. Since $x-\beta_1 m_1<x$, there exist $b_i \in B$ with $x-\beta_1 m_1=\sum \alpha_i b_i \bmod M$. Since $m_1 \in B$, this shows that x does depend on $B \bmod M$. Finally, suppose $b \in B$ and $b=b'x_\alpha^\varepsilon$, with no cancellation between b' and x_α^ε. Then, by (1) $b=\phi(b)=\phi(\phi(b')x_\alpha)=b'x_\alpha$. Thus by minimality, $\phi(b')x_\alpha \geqq b'x_\alpha$. Since there is no cancellation, it follows that $\phi(b')\geqq b'$, and hence $b'=\phi(b')$. B then has the Schreier property and is then a minimal Schreier basis.

2. PROPOSITION 2. *Let now B be an arbitrary Schreier basis for N* mod M, *with transversal function ϕ. (The Schreier property will actually not be used until later.) Then M is generated by the set $\mathscr{S} \cup \mathscr{E}$, with*

$$\mathscr{S} = \{bx_\alpha-\phi(bx_\alpha) \mid b \in B,\ bx_\alpha-\phi(bx_\alpha) \neq 0\},$$
$$\mathscr{E} = \{e_\lambda-\phi(e_\lambda) \mid e_\lambda-\phi(e_\lambda) \neq 0\}.$$

PROOF. Let m be a monomial, say $m=x_{\alpha_1}^{\varepsilon_1}\cdots x_{\alpha_n}^{\varepsilon_n}$ of F. We define a function $\sigma\colon N \to \operatorname{Mod}(\mathscr{S} \cup \mathscr{E})$ on monomials of N by

$$e_\lambda m\sigma = (e_\lambda-\phi(e_\lambda))m+\sum_{i=0}^{n-1} (\phi(e_\lambda^{(i)}m)x_{\alpha_{i+1}}^{\varepsilon_{i+1}}-\phi(e_\lambda^{(i+1)}m))m^{(i+1)}. \tag{2}$$

Now, $\phi(e_\lambda^{(i)}m)=\sum_j \beta_{ij}b_j$, with $b_j \in B$. Thus, using (1) and the linearity of ϕ,

$$\begin{aligned}\phi(e_\lambda^{(i)}m)x_{\alpha_i}^{\varepsilon_i}-\phi(e_\lambda^{(i+1)}m) &= \phi(e_\lambda^{(i)}m)x_{\alpha_i}^{\varepsilon_i}-\phi(\phi(e_\lambda^{(i)}m)x_{\alpha_i}^{\varepsilon_i})\\ &= \sum_j \beta_{ij}(b_jx_{\alpha_i}^{\varepsilon_i}-\phi(b_jx_{\alpha_i}^{\varepsilon_i})).\end{aligned} \tag{3}$$

Further,

$$\begin{aligned}b_jx_\alpha^{-1}-\phi(b_jx_\alpha^{-1}) &= -[\phi(b_jx_\alpha^{-1})x_\alpha-\phi(b_jx_\alpha^{-1}x_\alpha)]x_\alpha^{-1}\\ &= -[\phi(b_jx_\alpha^{-1})x_\alpha-\phi(\phi(b_jx_\alpha^{-1})x_\alpha)]x_\alpha^{-1}.\end{aligned} \tag{4}$$

Expanding $\phi(b_jx_\alpha^{-1})$ as in (3), we find that $e_\lambda m\sigma \in \operatorname{Mod}(\mathscr{S} \cup \mathscr{E})$.

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We now extend σ to N by linearity, and find that $\sigma: N \to \operatorname{Mod}(\mathscr{S} \cup \mathscr{E})$. Now, by the definition of σ and the linearity of ϕ, $g\sigma = g - \phi(g)$, for any element $g \in N$. Thus, if $g \in M$, $g\sigma = g$, and σ is the identity on M. Hence $M = \operatorname{Mod}(\mathscr{S} \cup \mathscr{E})$ as claimed.

COROLLARY (CF. [11]). *Let R be a finitely generated $\mathfrak{k}$ algebra, P a finitely generated R-module, and Q a submodule of finite codimension in P (over $\mathfrak{k}$). Then Q is again finitely generated.*

Proof. If both R and P are free then a Schreier basis for P mod Q is finite and Q is finitely generated by Proposition 2. Otherwise, let $0 \to I \to F \to R \to 0$ be a presentation of R, with F a finitely generated free algebra. Then P becomes a finitely generated F-module with a presentation $0 \to M \to N \to P \to 0$, where N is a finitely generated free F-module. The corollary follows upon considering the complete inverse image of Q in N.

We note that it is only when $\mathfrak{k}$ is a finite field that we can also show that a finitely generated module has only finitely many submodules of a given finite codimension.

3. We now want to show that, in the notation of Proposition 2, $\mathscr{S} \cup \mathscr{E}$ generates M freely.

If $bx_\alpha - \phi(bx_\alpha) \in \mathscr{S}$, define $u(b, x_\alpha) = bx_\alpha - \phi(bx_\alpha)$. Similarly, if $e_\lambda - \phi(e_\lambda) \in \mathscr{E}$, define $u(e_\lambda) = e_\lambda - \phi(e_\lambda)$. Let M^* be the free F-module generated by elements $u^*(b, x_\alpha)$ and $u^*(e_\lambda)$ in one-to-one correspondence with the $u(b, x_\alpha)$ and $u(e_\lambda)$. Given $g \in N$, by (2), (3), (4) and the linearity of σ there is a well-defined procedure for writing $g\sigma$ as

$$g\sigma = \sum_{i,\alpha} u(b_i, x_\alpha) g_{i,\alpha} + \sum_\lambda u(e_\lambda) g_\lambda.$$

Define $\bar{\tau}: N \to M^*$ by

$$g\bar{\tau} = \sum_{i,\alpha} u^*(b_i, x_\alpha) g_{i,\alpha} + \sum_\lambda u^*(e_\lambda) g_\lambda.$$

We show that the restriction $\tau = \bar{\tau}|_M$ is an F-homomorphism of M onto M^* which carries $u(b, x_\alpha)$ onto $u^*(b, x_\alpha)$ and $u(e_\lambda)$ onto $u^*(e_\lambda)$. This is clearly sufficient to insure that the $u(b, x_\alpha)$ and the $u(e_\lambda)$ freely generate M.

It is clear from the linearity of σ that τ is $\mathfrak{k}$-linear. Thus we must show that if $g \in M$ and $f \in F$, then $(gf)\tau = (g\tau)f$. By the linearity of τ we need only consider the case where f is a monomial. By an obvious reduction we may further assume that $f = x_\alpha^\varepsilon$. Suppose then that $g = \sum_{\lambda,j} \alpha_{\lambda j} e_\lambda m_{\lambda j}$, with the $m_{\lambda j}$'s monomials of F. Then

$$(g\tau)x_\alpha^\varepsilon = \sum_{\lambda,j} \alpha_{\lambda j}(e_\lambda m_{\lambda j})\bar{\tau} x_\alpha^\varepsilon.$$

We segregate the $e_\lambda m_{\lambda j}$'s which end in $x_\alpha^{-\varepsilon}$ and write

$$g = \sum{}' \alpha_{\lambda j} e_\lambda m_{\lambda j} + \sum{}'' \alpha_{\lambda j} e_\lambda \bar{m}_{\lambda j} x_\alpha^{-\varepsilon} \tag{5}$$

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where no $m_{\lambda j}$ in the first sum ends with $x_\alpha^{-\varepsilon}$. Let $k_{\lambda j}$ be the degree of $m_{\lambda j}$. Then, for $e_\lambda m_{\lambda j}$ in the first sum of (5), we have

$$
\begin{aligned}
(e_\lambda m_{\lambda j} x_\alpha^\varepsilon)\sigma &= (e_\lambda - \phi(e_\lambda))m_{\lambda j}x_\alpha^\varepsilon + \sum_{i=0}^{k_{\lambda j}-1} (\phi(e_\lambda^{(i)} m_{\lambda j})x_{\alpha_{i+1}}^{\varepsilon_{i+1}} - \phi(e_\lambda^{(i+1)} m_{\lambda j}))m_{\lambda j}^{(i+1)} x_\alpha^\varepsilon \\
&\quad + (\phi(e_\lambda m_{\lambda j})x_\alpha^\varepsilon - \phi(e_\lambda m_{\lambda j} x_\alpha^\varepsilon)) \\
&= (e_\lambda m_{\lambda j})\sigma x_\alpha^\varepsilon + (\phi(e_\lambda m_{\lambda j})x_\alpha^\varepsilon - \phi(e_\lambda m_{\lambda j} x_\alpha^\varepsilon)).
\end{aligned} \tag{6}
$$

For $e_\lambda m_{\lambda j}$ in the second sum of (5), we have

$$
\begin{aligned}
(e_\lambda m_{\lambda j} x_\alpha^\varepsilon)\sigma &= (e_\lambda \overline{m}_{\lambda j})\sigma = (e_\lambda - \phi(e_\lambda))\overline{m}_{\lambda j} \\
&\quad + \sum_{i=0}^{k_{\lambda j}-2} (\phi(e_\lambda^{(i)} \overline{m}_{\lambda j})x_{\alpha_{i+1}}^{\varepsilon_{i+1}} - \phi(e_\lambda^{(i+1)} \overline{m}_{\lambda j}))\overline{m}_{\lambda j}^{(i+1)} \\
&= (e_\lambda - \phi(e_\lambda))m_{\lambda j}x_\alpha^\varepsilon + \sum_{i=0}^{k_{\lambda j}-2} (\phi(e_\lambda^{(i)} m_{\lambda j})x_{\alpha_{i+1}}^{\varepsilon_{i+1}} - \phi(e_\lambda^{(i+1)} m_{\lambda j}))m_{\lambda j}^{(i+1)} x_\alpha^\varepsilon,
\end{aligned} \tag{7}
$$

the last equality following since the initial segments of $\overline{m}_{\lambda j}$ are the same as the initial segments of $m_{\lambda j}$. Thus, adding and subtracting the missing term in (7), we find that

$$
\begin{aligned}
(e_\lambda m_{\lambda j} x_\alpha^\varepsilon)\sigma &= (e_\lambda m_{\lambda j})\sigma x_\alpha^\varepsilon - [\phi(e_\lambda \overline{m}_{\lambda j})x_\alpha^{-\varepsilon} - \phi(e_\lambda m_{\lambda j})]x_\alpha^\varepsilon \\
&= (e_\lambda m_{\lambda j})\sigma x_\alpha^\varepsilon + [\phi(e_\lambda m_{\lambda j})x_\alpha^\varepsilon - \phi(e_\lambda m_{\lambda j} x_\alpha^\varepsilon)].
\end{aligned} \tag{8}
$$

Putting (6) and (8) together, we find that

$$(gx_\alpha^\varepsilon)\sigma = (g\sigma)x_\alpha^\varepsilon + \sum_{\lambda,j} \alpha_{\lambda j}(\phi(e_\lambda m_{\lambda j})x_\alpha^\varepsilon - \phi(e_\lambda m_{\lambda j} x_\alpha^\varepsilon)).$$

Now, by the definition of τ, $(g\tau)x_\alpha^\varepsilon$ and $(gx_\alpha^\varepsilon)\tau$ only differ by the effect of the term $r = \sum_{\lambda,j} \alpha_{\lambda j}(\phi(e_\lambda m_{\lambda j})x_\alpha^\varepsilon - \phi(e_\lambda m_{\lambda j} x_\alpha^\varepsilon))$. We show that this effect is null. For say that $\phi(e_\lambda m_{\lambda j}) = \sum_k \gamma_{\lambda jk} b_k$, with $b_k \in B$. If $\varepsilon = 1$, then $r = \sum_{\lambda,j,k} \alpha_{\lambda j}\gamma_{\lambda,j,k} u(b_k, x_\alpha)$ and if $\varepsilon = -1$, then $r = \sum_{\lambda,j,k} \alpha_{\lambda j}\gamma_{\lambda,j,k} u(\phi(b_k x_\alpha^{-1}), x_\alpha)$, with the sums running over the allowable k's. In the first case, $0 = \phi(g) = \sum_{\lambda,j,k} \alpha_{\lambda j}\gamma_{\lambda jk} b_k$, and since the b_k's are linearly independent, for fixed k, $\sum_{\lambda,j} \alpha_{\lambda j}\gamma_{\lambda j,k} = 0$. Thus, for allowable k's,

$$\sum_{\lambda,j} \alpha_{\lambda j}\gamma_{\lambda jk} u^*(b_k, x_\alpha) = 0$$

and, summing over the allowable k's,

$$\sum_k \sum_{\lambda,j} \alpha_{\lambda j}\gamma_{\lambda jk} u^*(b_k, x_\alpha) = 0.$$

The second case follows similarly and we have shown that τ is an F-homomorphism.

Finally, we use the Schreier property to note that if $b \in \langle B \rangle$, then $b\bar{\tau} = 0$. For if $b = e_\lambda$, then $b\bar{\tau} = e_\lambda - \phi(e_\lambda) = e_\lambda - e_\lambda = 0$, and if $b = e_\lambda m$ with $m = x_{\alpha_1}^{\varepsilon_1} \cdots x_{\alpha_n}^{\varepsilon_n}$, then

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$\phi(e_\lambda^{(i)}m)x_{\alpha_{i+1}}^{\varepsilon_{i+1}} - \phi(e_\lambda^{(i+1)}m) = 0$, thus

$$u(e_\lambda)\tau = (e_\lambda - \phi(e_\lambda))\tau = e_\lambda\bar{\tau} - \phi(e_\lambda)\bar{\tau} = u^*(e_\lambda)$$

and

$$u(b, x_\alpha)\tau = bx_\alpha\bar{\tau} - \phi(bx_\alpha)\bar{\tau} = (bx_\alpha)\bar{\tau}.$$

It is easily seen that the only term left in the expansion of $(bx_\alpha)\sigma$ is $\phi(b)x_\alpha - \phi(bx_\alpha)$ which is just $u(b, x_\alpha)$. Therefore $(bx_\alpha)\bar{\tau} = u^*(b, x_\alpha)$. To recapitulate, we have

THEOREM 1. *Let F be either the free algebra over $\mathfrak{k}$ freely generated by $X = \{x_\alpha; \alpha \in A\}$ or the group algebra over $\mathfrak{k}$ of the free group freely generated by $X = \{x_\alpha; \alpha \in A\}$. Let N be the free right F-module freely generated by $\{e_\lambda; \lambda \in \Lambda\}$. Let M be a submodule of N, and let B be a Schreier basis for N modulo M, with associated transversal function ϕ. Let*

$$\mathscr{S} = \{bx_\alpha - \phi(bx_\alpha) \mid b \in B,\ x_\alpha \in X,\ bx_\alpha - \phi(bx_\alpha) \neq 0\},$$
$$\mathscr{E} = \{e_\lambda - \phi(e_\lambda) \mid e_\lambda - \phi(e_\lambda) \neq 0\}$$

then M is the free module freely generated by $\mathscr{S} \cup \mathscr{E}$(2).

If F is the free algebra and B is a minimal Schreier basis for N mod *M, then the generators obtained for M are d-independent* (*in the sense of Cohn*).

Proof. Only the last statement remains to be proved. Thus if u_i are Schreier generators for M which come from B and $f_i \in F$, we must show that $d(\sum u_i f_i) = \max(d(u_i) + d(f_i))$. This is however easy to see, for if $\sum u_i f_i = \sum \alpha_{\lambda i} e_\lambda m_{\lambda i}$, with $m_{\lambda i}$'s monomials, then, by the foregoing, $\sum u_i^* f_i = (\sum \alpha_{\lambda i} e_\lambda m_{\lambda i})\tau$. Now, if $e_\lambda m_{\lambda i}\bar{\tau} = \sum \bar{u}_j^* \bar{f}_j$, then since B is a minimal Schreier basis, $d(e_\lambda m_{\lambda i}) \geqq d(\bar{u}_j) + d(\bar{f}_j)$. The theorem now follows.

IV. **Finitely presented modules over free algebras.** In this section F is the free algebra. If B is a Schreier basis for a submodule M of a free F-module and $b \in B$, we say that b is exceptional if there is a generator x_α of F for which $bx_\alpha - \phi(bx_\alpha) \neq 0$. Thus b is exceptional if it contributes to the set of generators for M. We are now in a position to show that finitely presented F-modules have large free submodules.

THEOREM 2. *Let P be a finitely related F-module. Then P contains a free module of finite codimension.*

Proof. Let $0 \to M \to N \to P \to 0$ be a presentation for P, with N free and M finitely generated, and let B be a Schreier basis for N mod M. Then, every basis for M is finite, hence by Theorem 1, B only contains finitely many exceptional elements. Thus there are finite subsets $\Lambda' \subset \Lambda$ and $A' \subset A$ such that the exceptional elements only involve letters with subscripts from Λ' and A'. Further there is an

(2) Actually the proof above works if F is the monoid algebra of the free product of a free group and a free monoid.

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integer n such that the exceptional elements have degree at most n. Let $\mathscr{T}'$ be the set of monomials of B which either are of degree greater than n or involve a letter subscripted by an element of $\Lambda-\Lambda'$ or $A-A'$. Then $\mathscr{T}'$ generates a submodule T of N, and it is easily seen that some subset $\mathscr{T}$ of $\mathscr{T}'$ freely generates T (for since T is generated by monomials, the set of monomials of N which are not in T form a Schreier basis for $N \bmod T$). We claim that $\mathscr{T}$ actually freely generates a free module modulo M. For let $b \in \mathscr{T}$. Then b is not exceptional and thus $\phi(bx_\alpha)=bx_\alpha$ for all $\alpha \in A$, and thus $bx_\alpha \in B$. Since $bx_\alpha \in \mathscr{T}'$, bx_α is still not exceptional. Then, for any monomial $m \in F$, $bm \in B$. Suppose now that there is a relation $\sum_{b_i \in \mathscr{T}} b_i q_i = 0 \bmod M$, with $q_i \in F$. Then, writing $q_i = \sum \alpha_{ij} m_{ij}$, with the m_{ij} monomials of F, we have $\sum_{b_i \in \mathscr{T}} \alpha_{ij} b_i m_{ij} = 0 \bmod M$ and thus

$$0 = \phi\Big(\sum \alpha_{ij} b_i m_{ij}\Big) = \sum \alpha_{ij} \phi(b_i m_{ij}) = \sum \alpha_{ij} b_i m_{ij} = \sum b_i q_i.$$

Since the b_i's are a free set, we must have $q_i=0$. Now, $T+M$ is complemented in N by the finite dimensional $\mathfrak{k}$-space spanned by the monomials of B of degree at most n and involving only subscripts from Λ' and A'. Thus the image of T in P fulfills the requirements of the theorem.

It is amusing to compare Theorem 2 with Stallings' recent result that a torsion free finitely generated group with a free subgroup of finite index is again free [12]: For modules, having a large free submodule is the rule rather than the exception.

COROLLARY. *A finitely presented F-module P is residually finite dimensional, and hence Hopfian. In other words P is a submodule of a direct product of finite dimensional modules, and any endomorphism of P onto P is an automorphism.*

Proof. The proof follows by standard arguments (see e.g. [9, Corollary 3]) from Theorem 2 and the fact that the corollary is true for finitely generated free modules.

Let R be a right ideal of F and I its idealizer (the intersection of all subalgebras of F in which R is a two sided ideal). The algebra I/R is called the eigenring of R and is usually denoted by $E(R)$. As was known to Fitting, $E(R)$ is isomorphic to the algebra of F-endomorphisms of the module F/R. We show that if R is finitely generated and nonzero, then $E(R)$ has finite $\mathfrak{k}$-dimension. This fact can also be proved by Cohn's methods (see [3, Theorem 5.1]). More generally, we have

THEOREM 3. *Let P be a finitely presented bound F-module* (*i.e.* $\operatorname{Hom}(P, F)=0$). *Then* $\operatorname{Hom}(P, P)$ *has finite* $\mathfrak{k}$*-dimension.*

Proof. By Theorem 2 we have the exact sequence

$$0 \to N \to P \to Q \to 0$$

where N is free of finite rank and Q is finite dimensional. Hence we have the exact sequence

$$0 \to \operatorname{Hom}(P, N) \to \operatorname{Hom}(P, P) \to \operatorname{Hom}(P, Q).$$

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Since every submodule of N is again free, Hom $(P, N)=0$ and we have the exact sequence

$$0 \to \text{Hom}(P, P) \to \text{Hom}(P, Q).$$

Thus, since Hom $(P, Q) \subseteq$ Hom (N, Q), Hom (P, P) is a submodule of the direct sum of finitely many copies of Q and the theorem is proved.

The above proof is due to P. M. Cohn. It replaces my cumbersome proof of a weaker theorem.

Using Proposition 3.11 of [6], we have

COROLLARY. *If* $\mathfrak{k}$ *is algebraically closed, and* $x \in F$ *is an atom* (x *is multiplicatively indecomposable*), *then* $E(xF)=\mathfrak{k}$.

If R is a two sided ideal of F, then $E(R)=F/R$. Thus we have the

COROLLARY. *If R is a finitely generated right ideal of F, and R is also a two sided ideal of F, then F/R has finite* $\mathfrak{k}$*-dimension.*

V. **The Schreier formula.** If R is a free ideal ring and P is an R-module with presentation $0 \to M \to N \to P \to 0$, then the Euler characteristic $\chi_R(P)$ is defined to be $\chi_R(P)=\text{rank } N-\text{rank } M$. If R is a field, then $\chi_R(P)=\dim_R(P)$. In analogy to Schreier's formula for subgroups of a free group, the following theorem relates the rank of a submodule of a free F-module to its codimension.

THEOREM 4. *Suppose that G is free on r generators and that M is a submodule of codimension n in the free F-module N of* rank k. *Then*

$$\text{rank } M = n(r-1)+k.$$

Equivalently

$$-\chi_F(N/M) = (r-1)\chi_{\mathfrak{k}}(N/M).$$

Proof. The proof is of course much the same as in group theory. The slightly harder case is when G is the free group, and we assume that F is the free group algebra. As usual B is a Schreier basis for N modulo M, with transversal function ϕ. Suppose $b \in B$ has positive degree. Then $b=b'x_\alpha^\varepsilon$ with $b' \in B$, with no cancellation between b' and x_α^ε. If $\varepsilon=1$, then $b'x_\alpha-\phi(b'x_\alpha)=0$ and if $\varepsilon=-1$, then $\phi(b'x_\alpha^{-1})x_\alpha - \phi(b'x_\alpha^{-1}x_\alpha)=0$. In this fashion we have a one-to-one correspondence between the elements of B of positive degree and the $bx_\alpha-\phi(bx_\alpha)$ which are zero. If $n_e=$ Card $\{e_\lambda \mid e_\lambda-\phi(e_\lambda)=0\}$, then there are $n-n_e$ b's of positive degree, and hence $nr-(n-n_e)$ elements in $\mathscr{S}$. On the other hand there are obviously $k-n_e$ elements in $\mathscr{E}$. Thus there are

$$nr-(n-n_e)+(k-n_e) = n(r-1)+k$$

elements in $\mathscr{S} \cup \mathscr{E}$, as claimed.

In the special case that N is itself the free algebra, P. M. Cohn (private communication) has a much shorter proof of this theorem. The argument involves

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examining the coefficients of a power series of the type used by Golod and Šafarevič.

Using arguments like Karrass and Solitar's [8], it is not difficult to extend the second corollary to Theorem 3 to the free group algebra. Since all our results are also true for left modules, we have

COROLLARY. *Suppose that I is a two sided ideal in F. Then the rank of I as a right F-module equals its rank as a left F-module.*

Here again Cohn has pointed out that the corollary can be proved by an elementary argument which involves what Bergman [1] calls w-bases.

VI. **Dunwoody's theorem.** If V is a vector space and V_i a decreasing sequence of subspaces of V with intersection U, a finite set T which is dependent modulo every V_i is also dependent modulo U. Bearing this elementary fact in mind, there is no difficulty in adapting Dunwoody's methods [7] to show

THEOREM 5. *Let F be the free algebra or the free group algebra and let N be a free F-module. Let further $M_1 \supset M_2 \supset \cdots$ be a decreasing sequence of submodules of N with intersection M. Then any finitely generated direct summand of M is a direct summand of all but finitely many of the M_i's.*

COROLLARY. *If R is a right ideal of F, then*

$$\left(\bigcap M_i\right)R = \bigcap (M_iR).$$

VII. **Subalgebras of finite codimension.** We show in this last section that a subalgebra T of finite codimension in a finitely generated free algebra is, if not again free, at least a finitely presented algebra. Fortunately, we may restrict our attention to right ideals (with 1 adjoined). For an argument along the lines of the proof of Lemma 1 of [11] shows that if T has finite codimension in F, then T contains a right ideal R of F which is again of finite codimension and it is a simple matter to show that if R (with 1 adjoined) is finitely presented, so is T. The result will follow once we have shown that a right ideal of F is also a free *left* module over a suitable free algebra. We use both Schreier bases (the Schreier property is however irrelevant here) and Cohn independence.

THEOREM 6. *Let F be a free algebra and R a right ideal of F. Let B be a minimal Schreier basis for F modulo R, with transversal function ϕ. Let $\mathscr{T}$ be a right d-independent set of right module generators for R, and let $T = \mathrm{Alg}\,(\mathscr{T})$. Let finally $g_{jk} = t_j b_k$ with $t_j \in \mathscr{T}$ and $b_k \in B$. Then R is the free left T-module freely generated by the g_{jk}.*

Proof. Let M be the left T-module generated by the g_{jk}. We first show that $M = R$. It is clear that $M \subset R$. Since every element $r \in R$ can be written as $r = \sum t_i q_i$, $q_i \in F$, we need only show that $tm \in M$ for $t \in T$ and m a monomial. Suppose then that m is a monomial of least degree for which $tm \notin M$ for some $t \in T$. Then, if $\phi(m) = \sum \alpha_i b_i$

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the minimality of B insures that $d(b_i) \leqq d(m)$. Now $m-\phi(m) \in R$, so there exist $q_j \in R$ and $t_j \in T$ with $m = \sum \alpha_i b_i + \sum t_j q_j$. By the right independence of $\mathscr{T}$, it follows that $d(t_j q_j) \leqq d(m)$ and hence, since $d(t_j) > 0$, that $d(q_j) < d(m)$. By the assumption on the degree of m, $\sum t_j q_j \in M$ and

$$tm = \sum \alpha_i b_i + t \sum t_j q_j$$

is also in M. This contradiction shows that $R = M$.

It remains to show that M is free. Note that T is free since $\mathscr{T}$ is right d-independent. Suppose that there is a relation

$$\sum \pi_{jk} g_{jk} = 0 \qquad \pi_{jk} \in T, \tag{9}$$

and write

$$\pi_{jk} = \sum_l t_l \pi_{jkl} + \alpha_{jk} \qquad t_l \in \mathscr{T},\ \pi_{jkl} \in T.$$

Then

$$\sum_{l,k} \alpha_{lk} g_{lk} + \sum_{j,k} \left(\sum_l t_l \pi_{jkl} \right) g_{jk} = 0.$$

Thus

$$\sum_l \sum_k t_l \alpha_{lk} b_k + \sum_l \sum_{j,k} t_l \pi_{jkl} g_{jk} = 0,$$

and

$$\sum_l t_l \left(\sum_k \alpha_{lk} b_k - \sum_{j,k} \pi_{jkl} g_{jk} \right) = 0.$$

By the freedom of the t_l, we have

$$\sum_k \alpha_{lk} - \sum_{j,k} \pi_{jkl} g_{jk} = 0,$$

and thus $\sum \alpha_{lk} b_k \in R$. Since the b_k's are independent modulo R, $\alpha_{lk} = 0$ for all l and k. Thus each π_{jk} has order at least 1. We may then assume that (9) cannot occur nontrivially when any π_{jk} has order at most n. If the minimal order of the π_{jk} is $n+1$, then

$$\pi_{jk} = \sum t_l \pi_{jkl} \qquad \pi_{jkl} \in T,\ t_l \in \mathscr{T},$$

and

$$\sum_{j,k} \sum_l t_l \pi_{jkl} g_{jk} = \sum_l t_l \sum_{j,k} \pi_{jkl} g_{jk} = 0.$$

Thus, again by the freedom of $\mathscr{T}$,

$$\sum_{j,k} \pi_{jkl} g_{jk} = 0.$$

Since some π_{jkl} has order n it follows that (10) is a trivial relation. Thus $\pi_{jk} = 0$ for all j and k and the theorem is proved.

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COROLLARY. *If R_1 is a subalgebra of finite codimension in the finitely generated free algebra F, then R_1 is finitely presented.*

Proof. As we remarked, we may assume that $R_1 = \text{Alg}(R)$ with R a right ideal of finite codimension. Then, in the notation of the theorem, both B and $\mathcal{T}$ are finite. Since B must contain an element of degree 0, we may assume that $1 \in B$. It then follows from the theorem that R is generated by the finite set $\{g_{jk}\}$. Then, since R is a free T module on the g_{jk} and T is a free algebra the multiplication table

$$g_{j_1k_1}g_{j_2k_2} = \sum_{l,m} \pi_{j_1,j_2,k_1,k_2,l_m} g_{lm}$$

with $\pi_{j_1,j_2,k_1,k_2,l_m} \in T$ is clearly a complete set of defining relations for R_1.

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