STACKS, COSTACKS AND AXIOMATIC HOMOLOGY

BY

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Let \( p: \mathcal{E} \rightarrow \mathcal{X} \) be a sheaf (espace étalé) of abelian groups. Applying singular functor \( S \) one obtains a simplicial map \( \pi: E \rightarrow X \) with \( E = S(\mathcal{E}), X = S(\mathcal{X}) \) and \( \pi = S(p) \). The fibers \( \pi^{-1}(x), x \in X \), form a "local system of groups" over \( X \) which will be called a costack of abelian groups over the simplicial set \( X \). In general, a costack is defined as a functor on \( X \), regarded as a category. This is a generalized dual of the notion of a stack defined by Spanier [5].

The main objects of this note are (1) to develop a general theory of stacks and costacks over simplicial sets, (2) to construct a semisimplicial homology theory with "variable" coefficients which is unique in the sense of Eilenberg-Steenrod. The coefficients of the homology are a costack in an abelian category. In particular, when the coefficient costack is a locally constant costack the homology becomes the usual homology with local coefficients.

There are three chapters in this note. Chapter I is devoted to a study of stacks and costacks. It is partially a preparatory chapter. In Chapter II we define homology of costacks via usual chain complexes and prove that the homology so defined can be computed by projective resolutions by introducing a generalized torsion product functor. Under the equivalence of costacks and modules, this generalized functor is essentially the genuine torsion product functor of modules. The rest of Chapter II is a preparation for Chapter III, in which a homology theory of pairs of simplicial sets over a fixed simplicial set \( K \) is defined. Results of Chapter II ensure the existence of such a theory. Chapter III concludes with a proof of the uniqueness of this homology theory. This is a generalization of Eilenberg-Steenrod uniqueness theorem [1].

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CHAPTER I. STACKS AND COSTACKS

1. Definitions and notations. \( X = \{X_\nu\} \) is a simplicial set (semisimplicial complex) regarded as a category with objects simplexes \( x, x', \ldots \) and morphisms \( \mu_x: x \rightarrow x' \) for incidence map \( \mu \) such that \( \mu(x) = x' \). The morphisms determined by face operators and degeneracy operators are denoted by \( d_x^i, s_x^i, \) or simply \( d, s \). A simplicial
map $f: X \to Y$ is thus a functor. If, as in [3], the simplicial set $X$ is defined as a contravariant functor, then the associated category can be viewed as the graph of $X$.

Let $\mathcal{A}$ denote a category which has a projective generator $P$ and satisfies the properties (1) $\mathcal{A}$ is abelian, (2) $\mathcal{A}$ is closed under the formation of products and coproducts (sums), and (3) the product and coproduct of a family of short exact sequences in $\mathcal{A}$ are short exact sequences in $\mathcal{A}$. E.g.: $\mathcal{R}$-modules $\mathcal{M}_R$, abelian groups $\mathcal{A}b$, and compact abelian groups $\mathcal{A}b^*$ (the dual of $\mathcal{A}b$) are such categories. Since the category $X$ is small (the class of objects is a set), the functor category $\mathcal{A}^X$ is well defined with morphisms natural maps of functors. $\mathcal{A}^X$ satisfies the three properties of $\mathcal{A}$ listed above.

An object $A \in \mathcal{A}^X$ is a functor $A: X \to \mathcal{A}$ which is called a precostack on $X$ with values in $\mathcal{A}$). If $A$ satisfies the condition that $A(s) = A(s_x)$ is an isomorphism for every $s^i$ and $x$ of $X$, then we say that $A$ is a costack. Dually, prestacks and stacks are contravariant functors on $X$ to $\mathcal{A}$.

2. The functors $f_\#$ and $f^\#$. A simplicial map $f: X \to Y$ induces functors $f_\#: \mathcal{A}^X \to \mathcal{A}^Y$ and $f^\#: \mathcal{A}^Y \to \mathcal{A}^X$ as follows: For every $A$ in $\mathcal{A}^X$, $f_\#A = B: Y \to \mathcal{A}$ is the functor defined on objects $y$ and morphisms $\mu_y$ of $Y$ as

\[
By = \bigsqcup A_x, \quad B_{\mu_y} = \bigsqcup A(\mu_x),
\]

sum over all $x$ such that $fx = y$ and over all $\mu_x$ such that $f(\mu_x) = \mu_y$. It is easy to check that $f_\#$ is a well defined functor. The functor $f^\#$ is defined by composition of functors as $f^\#B = Bf$ for $B$ in $\mathcal{A}^Y$. Both $f_\#$ and $f^\#$ are exact functors and

**Proposition 2.1.** $f^\#$ is the adjoint of $f_\#$, i.e. there is a natural isomorphism

\[
\mathcal{A}^X(A, f^\#B) \to \mathcal{A}^Y(f_\#A, B), \quad A \in \mathcal{A}^X, B \in \mathcal{A}^Y.
\]

**Proof.** Let $\varphi = \{\varphi_x \mid x \in X\}$ be in $\mathcal{A}^X(A, f^\#B)$, i.e. $\varphi$ is a natural map with $\varphi_x: A_x \to (f^\#B)_x$. Then, for $y \in Y$ and all $x \in X$ such that $fx = y$, the universal mapping diagram of $\bigsqcup A_x$

![Diagram](image)

shows that the correspondence $\varphi \to \psi$ with $\varphi_x = \psi_x i_x$ defines a natural isomorphism.

Since $f_\#$ and $f^\#$ are exact, we have

**Corollary 2.2.** $f_\#$ preserves projectives and $f^\#$ preserves injectives.

For composite simplicial map $gf$ we have $(gf)_\# = g_\# f_\#$ and $(gf)^\# = f^\# g^\#$. 

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3. Projectives and generators in $\mathscr{A}^X$. Let $\Delta^n$ denote the simplicial analogue of the unit affine $n$-simplex and let $\delta$ be its nondegenerate $n$-simplex. For every $x \in X_n$, the correspondence $\delta \to x$ determines uniquely a simplicial map $x^\delta: \Delta^n \to X$. We shall show that the induced functor $x^\delta#: \mathscr{A}^\Delta \to \mathscr{A}^X$ (here $\Delta$ stands for $\Delta^n$) supplies projectives of $\mathscr{A}^X$.

**Theorem 3.1.** Let $P^\Delta: \Delta^n \to \mathscr{A}$ be the constant functor with value $P$ (a projective generator of $\mathscr{A}$), then $P^\Delta$ is a projective of $\mathscr{A}^\Delta$.

**Proof.** For any $A \in \mathscr{A}^\Delta$,

\[
\mathscr{A}^\Delta(P^\Delta, F) \simeq \mathscr{A}(P, F\delta).
\]

For, let $\varphi = \{\varphi_\sigma | \sigma \in \Delta^n\}$ be in $\mathscr{A}^\Delta(P^\Delta, F)$, then the commutative diagram

\[
\begin{array}{ccc}
P & \xrightarrow{\varphi_\sigma} & F\delta \\
\downarrow{\varphi_\sigma} & & \downarrow{\varphi_\sigma*} \\
F\sigma & \xrightarrow{1} & F(\sigma*\delta),
\end{array}
\]

where $\sigma^*$ is the incidence map of $\Delta^n$ determined by $\sigma$, shows that $\varphi$ is completely determined by $\varphi_\sigma$ and vice versa. Thus the correspondence $\varphi \to \varphi_\sigma$ gives rise to the isomorphism 3.1. This and a routine computation show that $A^\Delta$ is projective.

**Theorem 3.2.** $U = U = \bigsqcup_{x \in X} (x^\delta P^\Delta)$ is a projective generator of $\mathscr{A}^X$.

**Proof.** $U$ is projective since $x^\delta$ preserves projective, and coproduct of projectives is a projective. Now, a simple computation shows that

\[
\mathscr{A}^X(U, A) \simeq \prod \mathscr{A}(P^\Delta, A^x) \simeq \prod \mathscr{A}(P, A^x).
\]

Thus $\mathscr{A}^X(U, A) \neq 0$ for any $A \neq 0$. $U$ is a generator.

We conclude that since $\mathscr{A}^X$ has projective generators, it has enough projectives. Thus one can do homology in $\mathscr{A}^X$ by projective resolutions.

4. Stacks and costacks. A costack (resp. stack) as defined in §1 is a normalized precostack (resp. prestack). Since $A(d_{sx})A(s_x) = A(d_{sx}s_x) = 1$ for all $x \in X$, a precostack is normalized if and only if $A(d_{sx})$ is an isomorphism for all $d_{sx}$. The same holds true for stacks. In the rest of this paper, we shall leave out the dual theory for stacks.

Costacks form an abelian category $\mathcal{F}^X$ which is an exact full subcategory of $\mathscr{A}^X$. It is easily shown that $\mathcal{F}^X$ is a Serre subcategory of $\mathscr{A}^X$ in the sense that it is closed under the formation of subobjects, quotient objects and extensions. Also, $\mathcal{F}^X$ is closed under the formation of products and coproducts. Thus, by a theorem of Freyd [2], we have

**Proposition 4.1.** $\mathcal{F}^X$ is reflective and coreflective.
$\mathcal{F}^X$ is coreflective in the sense that for each $A \in \mathcal{A}^X$, there is $N^*A \in \mathcal{F}^X$ and a map $r: A \to N^*A$ such that for any $\overline{A} \in \mathcal{F}^X$ and any map $\varphi: A \to \overline{A}$ there is a unique map $\psi: N^*A \to \overline{A}$ with $\psi r = \varphi$. Reflectivity is defined dually.

The coreflector $N^*: \mathcal{A}^X \to \mathcal{F}^X$ is the coadjoint of the inclusion functor $J: \mathcal{A}^X \to \mathcal{F}^X$ and so preserves colimits. Since $J$ is exact, $N^*$ also preserves projectives. Thus

**Theorem 4.2.** Let $N X$ be the set of nondegenerate simplexes of $X$, then $U^* = N^* \coprod_{x \in N X} (x_#^{P^A})$ is a projective generator of $\mathcal{F}^X$.

The reflection $\mathcal{A}$ of $A$ is a costack defined as $\mathcal{A}x = Ax$ for $x \in N X$ and $A(sx) \simeq Ax$ for all degeneracy operators $s$. The reflector $N_*$ is exact and so its coadjoint functor $J$ preserves projectives. Hence, a projective resolution of $\mathcal{A}$ in $\mathcal{F}^X$ is also a projective resolution of $\mathcal{A}$ in $\mathcal{A}^X$. Summarizing, we say that $\mathcal{F}^X$ is homologically closed in $\mathcal{A}^X$.

5. **Generalized torsion product functor.** For each $A \in \mathcal{A}^X$, let $CA$ be the chain complex of objects in $\mathcal{A}$ with $n$-chains $\coprod_{x \in X_n} Ax$ and differential $\partial = \{\partial_n\}$ defined as

\[
\partial_n = \coprod_{x \in X_n} \left( \sum_{i=0}^{n} (-1)^i A(d^x_i) \right).
\]

The homology of $CA$ is denoted by $H(A)$.

**Theorem 5.1.** On $\mathcal{A}^X$, $H$ is naturally isomorphic to $LH_0$, the left derived functor of $H_0$.

**Proof.** To show that for every projective $A$ of $\mathcal{A}^X$, $H_n(A) = 0$ for $n > 0$. Since a projective is a summand of a coproduct of copies of projective generator $U$, it suffices to show that $H_n(U) = 0$ for $n > 0$. This is true since $C P^\Delta = C(x_#^{P^A})$ is acyclic and so is the coproduct $U = \coprod_{x \in X} (x_#^{P^A})$.

When $X$ has finitely many nondegenerate simplexes then the category of costacks of abelian groups over $X$ has a small projective generator $U$ and may be identified with the category of right $R$ modules, $R$ is the endomorphism ring of $U$; $H_0$ then becomes $\text{Tor}^R (\cdot, H_0(U))$.

**Example.** If $X$ is a simplicial complex, then $R \approx \coprod_{\sigma \leq \tau} Z(\sigma, \tau)$, where $\sigma \leq \tau$ means $\sigma$ is a face of $\tau$, $Z(\sigma, \tau)$ is the infinite cyclic group generated by the symbol $(\sigma, \tau)$. Observe that the multiplication in $R$ is defined by

\[
(\sigma, \rho)(\rho, \tau) = (\sigma, \tau); \quad (\sigma, \rho)(\rho', \tau) = 0 \quad \text{if} \quad \rho \neq \rho'.
\]

Chapter II. Homology with Variable Coefficients

6. **Homology of simplicial pairs.** $(X, X')$ is a simplicial pair with inclusion map $i: X' \to X$. The induced functor $i#: \mathcal{A}^X \to \mathcal{A}^X$ maps $A': X' \to \mathcal{A}$ onto $i#A' = A: X \to \mathcal{A}$ with supports in $X'$. Precisely, $Ax = A'x$ for $x \in X'$ and $Ax = 0$ for $x \in X - X'$. $i#$ is an exact full embedding and $i#i#$ is the identity functor of $\mathcal{A}^X$. 

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Observe that \( i_\# i_\# A \) is a subobject of \( A \) with supports in \( X' \) and \( i_\# i_\# \) is an exact reflector. If we identify \( \mathcal{A}^X \) with its image under \( i_\# \), then

**Proposition 6.1.** \( \mathcal{A}^X \) is (identified as) a reflective Serre subcategory of \( \mathcal{A}^X \).

For every \( A \in \mathcal{A}^X \), define \( qA \) by the exact sequence \( 0 \to i_\# i_\# A \to A \to qA \to 0 \). \( qA \) has supports in \( X - X' \). In fact, any object in \( \mathcal{A}^X \) with supports in \( X - X' \) is the quotient of some \( A \) by \( i_\# i_\# A \). Such objects of \( \mathcal{A}^X \) are called relative precostacks. They form a full subcategory \( q\mathcal{A}^X \) of \( \mathcal{A}^X \).

**Proposition 6.2.** \( q\mathcal{A}^X \) is an exact coreflective Serre subcategory of \( \mathcal{A}^X \). The coreflector \( q \) is exact.

**Corollary 6.3.** The functors \( i_\# \) and \( q \) preserve projective resolutions.

Similar statements are true for normalized categories \( \mathcal{A}^{X'} \), \( \mathcal{A}^X \) and \( q\mathcal{A}^X \).

Recall that for every \( A \in \mathcal{A}^X \) there associates a chain complex \( CA \), the homology of \( CA \) is denoted by \( H(A) \). For a simplicial pair \( (X, X') \), define its homology with coefficients in \( A \in \mathcal{A}^X \) as \( H(X, X'; A) = H(qA) \). In particular, \( H(X; A) = H(A) \).

**Proposition 7.4.** \( H \) is a functor in the sense that simplicial maps \( f: (X, X') \to (Y, Y') \) and \( g: (Y, Y') \to (K, K') \) give rise to a map

\[
(gf)_*: H(X, X'; f^# B) \to H(Y, Y'; E), \quad B \in \mathcal{A}^Y.
\]
by simplicial maps \( f \), and the functor \( i_# \) induced by an inclusion map preserve normalization.

Let \( A \) be a coefficient costack, then the exact sequence \( 0 \to i_#i_#A \to A \to qA \to 0 \) gives rise to

**Proposition 7.1 (Exactness).** To each simplicial pair \((X, X')\) is associated an exact homology sequence

\[
\cdots \to H_q(X'; i^#A) \to H_q(X; A) \to H_{q-1}(X'; i^#A) \to \cdots
\]

where \( i : X' \to X \) is the inclusion map. Moreover, if \( f : (Y, Y') \to (X, X') \) is a simplicial map of pairs, then \( f \) induces a map \( f_* \) of homology sequences of the pairs.

Let \((X; X', X'')\) be a triad with inclusions

\[
\begin{align*}
(X', X' \cap X'') & \to (X' \cup X'', X'') \to (X, X' \cup X'') \\
(X'', X' \cap X'') & \to (X' \cup X'', X'') \to (X, X' \cup X'').
\end{align*}
\]

It is easily shown that

**Proposition 7.2 (Excision).** The excision maps \( i \) and \( j \) induce isomorphisms

\[
i_* : H_*(X', X' \cap X''; i^#h^#A) \to H_*(X' \cup X'', X'; h^#A)
\]

\[
j_* : H_*(X'', X' \cap X''; j^#h^#A) \to H_*(X' \cup X'', X''; h^#A).
\]

The following additivity properties of \( H \) are also easy to show.

**Proposition 7.3 (Additivity).** Given a simplicial pair \((X, X')\) and a family \( \{X_\alpha\} \) of simplicial subsets of \( X \) with the property that \( X = X' \cup (\bigcup X_\alpha) \) and \( X_\alpha \cap X_\beta \subseteq X' \) if \( \alpha \neq \beta \). Let \( X'_\alpha = X_\alpha \cap X' \) and let \( h_\alpha : (X_\alpha, X'_\alpha) \to (X, X') \) be the inclusion map, then for any coefficient costack \( A \) we have

\[
H_*(X, X'; A) \approx \bigsqcup_\alpha H_*(X_\alpha, X'_\alpha; h_\alpha^#A).
\]

In particular, when \( X' \) is void, we have

**Corollary 7.4.** \( H \) is infinitely additive.

Now, for each nondegenerate simplex \( x \) of \( X \) let \( \Delta^x \) denote the simplicial subset of \( X \) determined by faces of \( x \) and let \( \tilde{\Delta}^x \) be its "boundary simplicial subset." If \( i_x : \Delta^x \to X \) denotes the inclusion map then for any costack \( A \) on \( X \) the normalized chain complex of \( q(i_x^#A) \) has zero in all dimensions \( n \) except possibly for \( n = \dim x \). Thus

**Proposition 7.5.** \( H_n(\Delta^x, \tilde{\Delta}^x; i_x^#A) = 0 \) for \( n \neq \dim x \).
8. Strong homotopy and deformation. For \( n = 0, 1, 2, \ldots \), let \( I_n = [n, n+1] \), the closed unit interval as simplicial set, and let \( W = \bigcup_{n=0}^\infty I_n \) be the "simplicial half line."

**Lemma 8.1.** For any constant costack \( E_x \) on \( X \) with value \( E \in \mathcal{A} \), the projection \( p: (X \times W, X' \times W) \to (X, X') \) defined by \( p(x, \sigma) = x \) induces a chain equivalence

\[
C(p): C(qp#E_x) \to C(E_x).
\]

**Proof.** Let \( \otimes: \mathcal{A} \times \mathcal{A} \to \mathcal{A} \) be the tensor functor defined by Freyd [2, p. 86] and let \( C(X, X'; Z) \) be the usual free chain complex of \((X, X')\). Then \( C(E_x) \cong E \otimes C(X, X'; Z) \) and \( C(qp#E_x) \cong E \otimes C(X \times W, X' \times W; Z) \). It is well known that \( p \) induces a chain equivalence of the free chain complexes. This gives rise to the chain equivalence (8.1).

**Lemma 8.2.** Let \( NX_n \) denote the set of all nondegenerate \( n \)-simplexes of \( X \). Then

\[
H^*(X \times X' - 1) \cong \bigoplus_{x \in NX_n} H^*(\Delta^x, \Delta^x).
\]

This follows immediately from Proposition 7.3.

**Proposition 8.3 (Strong Homotopy).** \( p: X \times W \to X \) induces isomorphism

\[
p_*: H_*(X \times W; p^#A) \to H_*(X; A)
\]

for any coefficient costack \( A \).

**Proof.** First, we shall show by induction that

\[
H_*(X^n \times W; p^#A) \cong H_*(X^n; A)
\]

for any nonnegative integer \( n \). The crucial point is the fact that \((p^#A)(x, \sigma) = A p(x, \sigma) = A x \) for all \( \sigma \in W \) and then \( H_*(\Delta^x, \Delta^x) \) and \( H_*(\Delta^x \times W, \Delta^x \times W) \) have constant coefficients for any fixed \( x \in NX \).

For the case \( n = 0 \), \( H_*(X^0 \times W) = \bigoplus_{x \in X_0} H_*(\Delta^x) \) is isomorphic to

\[
\bigoplus_{x \in X_0} H_*(\Delta^x)
\]

since, by Lemma 8.1, each summand \( H_*(\Delta^x \times W) \) is isomorphic to \( H_*(\Delta^x) \). Hence we have \( H_*(X^0 \times W) \cong H_*(X^0) \).

Assume inductively that \( H_*(X^r \times W) \cong H_*(X^r) \) for \( r = 1, 2, \ldots, n-1 \), and consider the commutative diagram

\[
\begin{array}{cccccc}
\cdots \to H_*(X^{n-1} \times W) & \to & H_*(X^n \times W) & \to & H_*(X^n \times W, X^{n-1} \times W) & \to & H_*(X^{n-1} \times W) & \to & \cdots \\
\downarrow 2 & & \downarrow 3 & & \downarrow 4 & & \downarrow 5 & & \\
\cdots \to H_*(X^{n-1}) & \to & H_*(X^n) & \to & H_*(X^n, X^{n-1}) & \to & H_*(X_{n-1}, X^{n-2}) & \to & \cdots
\end{array}
\]
where the maps 2 and 5 are isomorphisms. Since

$$H_\bullet(X^n \times W, X^{n-1} \times W) \simeq \bigsqcup_{x \in X_n} H_\bullet(\Delta^x \times W, \Delta^x \times W)$$

and $H_\bullet(X^n, X^{n-1}) \simeq \bigsqcup_{x \in X_n} H_\bullet(\Delta^x, \Delta^x)$ by Lemma 8.2, it follows from Lemma 8.1 that the map 4 is an isomorphism. Hence, by the five lemma, the map 3 is an isomorphism. This proves (8.4) and, of course, the case when $X$ is finite dimensional.

Now, suppose that $X$ is infinite dimensional with $X^0 \subset X^1 \subset X^2 \subset \cdots \subset X$. Clearly, $H_q(X^n) = H_q(X^{n-1}) = \cdots = H_q(X)$ for $n > q + 1$. This and (8.4) prove (8.3).

**Corollary 8.4 (Homotopy).** Let $p: X \times I \to X$ be the simplicial map defined by $p(x, \alpha) = x$ for $x \in X$ and any $\alpha \in I$, then for any $A \in \mathcal{S}X$, $p_*: H_\bullet(X \times I; p#A) \to H_\bullet(X; A)$ is an isomorphism.

For, we have retractions $X \times W \to I \times [0] \to X \times [0]$ such that $r_*r'_* = (rr')_*$ is an isomorphism.

**Proposition 8.5 (Deformation).** The projection $p: \bigcup_n X^n \times I_n \to X$ defined by $p(x, \alpha) = x$, where $(x, \alpha) \in X^n \times I_n$, $n = 0, 1, 2, \ldots$, induces isomorphism

$$p_*: H_\bullet(L; p#A) \to H_\bullet(X; A), \quad L = \bigcup_n X^n \times I_n.$$  

**Proof.** Let $L^n = \bigcup_{n=0}^\infty X^n \times I$, and let $LX^n = L^n \cup (X^n \times [n + 1, \infty))$, then $L^n \subset LX^n \subset L$. Since $LX^n$ is a deformation retract of $X^n \times W$, $H_q(LX^n) \simeq H_q(X^n \times W)$. Hence, by Proposition 8.3, $H_q(LX^n) \simeq H_q(X^n) \simeq H_q(X)$ for $n > q + 1$. Thus for any $q \geq 0$, there is $n > q + 1$ such that

$$H_q(X) \simeq H_q(LX^n) \simeq H_q(LX^{n+1}) \simeq \cdots \simeq H_q(L).$$

The proof is complete.

**Chapter III. Homology Theory on $\mathcal{C}_K$**

9. $K$-pairs and axioms for homology. Let $K$ be a fixed simplicial set. A $K$-pair is a simplicial pair $(X, X')$ together with a simplicial map $\varphi: X \to K$. Such a $K$-pair is denoted by $(X, X')_\varphi$. $(K, K')_i$ is written $(K, K')$ and $(X, \phi)_o$ is written $X_\phi$. When $\varphi = \phi^\delta: \Delta^d \to K$, the subscript $\phi^\delta$ is abbreviated by $\phi$.

Given two $K$-pairs $(X, X')_\varphi$ and $(Y, Y')_\psi$, a $K$-map $f: (X, X')_\varphi \to (Y, Y')_\psi$ is, by definition, a simplicial map $f: (X, X') \to (Y, Y')$ such that $\varphi = \psi f$. In particular, an inclusion map $i: (Y, Y') \to (X, X')$ is a $K$-map $i: (Y, Y')_\varphi \to (X, X')_\varphi$ for any simplicial map $\varphi: X \to K$. $(Y, Y')_\varphi$ is called a $K$-subpair of $(X, X')_\varphi$. We shall omit the inclusion map in the notation of a $K$-subpair. E.g.: write $(Y, Y')_\phi$ for $(Y, Y')_{\phi^\delta}$, $X_\phi$ for $X_{\phi^\delta}$, $X^\delta_\phi$ for $X^\delta_{\phi^\delta}$, $(\Delta^d, \Delta^d)_\phi$ for $(\Delta^d, \Delta^d)_{\phi^\delta}$, etc.

$K$-pairs form a category, denoted by $\mathcal{C}_K$, with morphisms $K$-maps. Any $K$-pair of the form $(K, K')$ is a terminal object (right zero object).
A homology theory on $\mathcal{C}_K$ with values in the category $\mathcal{A}$ is a sequence of functors $H_\bullet: \mathcal{C}_K \to \mathcal{A}$ together with a family of natural transformations $d_q: H_q(X, X')_\sigma \to H_{q-1}X_\sigma$, $q > 0$, satisfying the following axioms:

**Axiom 1 (Exactness axiom).** For each $(X, X')_\sigma$ with inclusion maps $X'_\sigma \subset X_\sigma \supset (X, X')_\sigma$ there is an exact triangle of $(X, X')_\sigma$,

$$H_\bullet H'_\bullet \xrightarrow{i_*} H_\bullet H_\sigma \xrightarrow{\partial} H_{\bullet-1}(X, X')_\sigma \xrightarrow{j_*} H_\bullet(X, X')_\sigma,$$

where $i_* = H_{q0}$, $j_* = H_{qj}$.

Let $j_0, j_1: (X, X') \to (X \times I, X' \times I)$ and $p: (X \times I, X' \times I) \to (X, X')$ be simplicial maps defined by $j_0x = (x, 0)$, $j_1x = (x, 1)$, and $p(x, \sigma) = x$, respectively, where $x \in X$, $\sigma \in I$. Then for any simplicial map $\varphi: X \to K$, $j_0$, $j_1$, and $p$ are $K$-maps as shown in the commutative diagram

$$X \xrightarrow{j_0} X \times I \xrightarrow{p} X \xrightarrow{\varphi} K \xleftarrow{\varphi p} X \xrightarrow{\varphi} X,$$

Two $K$-maps $f, g: (X, X')_\sigma \to (Y, Y')_\sigma$ are $K$-homotopic if there is a $K$-map $h: (X \times I, X' \times I)_\sigma \to (Y, Y')_\sigma$, called a $K$-homotopy of $f$ and $g$, such that $f = h j_0$, $g = h j_1$.

**Axiom 2 (Homotopy axiom).** $p_*$ induced by the $K$-projection $p: (X \times I, X' \times I)_\sigma \to (X, X')_\sigma$ is an isomorphism, or equivalently, if $f$ and $g$ are $K$-homotopic then $f_* = g_*$. 

**Axiom 3 (Excision axiom).** The excision maps $i$ and $j$ of §7 regarded as $K$-maps induce isomorphisms $i_*$ and $j_*$. 

For the dimension axiom we need the following argument: In analogy to ordinary simplicial homology theory, let $C^{q-1}$ be the closed star of a vertex in $\Delta^q$ [1, p. 78], then $(\Delta^q; \Delta^{q-1}, C^{q-1})_\sigma$ is a proper triad with respect to $H_\sigma$. This and the exactness axiom give rise to the diagram

$$H_q(\Delta^q, \Delta^{q-1})_\sigma \xrightarrow{\partial} H_{q-1}(\Delta^{q-1})_\sigma \xrightarrow{h_*} H_{q-1}(\Delta^q, C^{q-1})_\sigma \xrightarrow{j_*^{-1}} H_{q-1}(\Delta^{q-1}, \Delta^{q-1})_\sigma,$$
where $d_\sigma = \tau$ is the $i$th face of $\sigma \in k$, $h$ is an inclusion map, $j$ is an excision map, and $F^i = j^{-1}_* h_* \partial$.

**Axiom 4 (Dimension axiom).** For any $x \in FX$ with $x = \sigma$, $x_\sigma^4 : H_\sigma(\Delta^\sigma, \hat{\Delta}^\sigma) \rightarrow H_\sigma(\hat{\Delta}^\sigma, \hat{\Delta}^\sigma)$ is an isomorphism and $H_\sigma(\Delta^\sigma, \hat{\Delta}^\sigma) = 0$ for $n \neq q$. If $\sigma = s^i \tau$, then $F^i$ defined by (9.3) is an isomorphism.

**Axiom 5 (Additivity axiom).** Let $(X_a, X'_a)_a$ be $K$-subpairs of $(X, X')_a$ defined as in Proposition 7.3, then

$$H_*(X, X')_a \cong \bigoplus_{\alpha} H_*(X_a, X'_a)_a.$$  

**Axiom 6 (Deformation axiom).** $p_* = H_*(p)$, where $p$ is the $K$-map $p : L_\partial \rightarrow X_\sigma$ defined as in Proposition 8.5, is an isomorphism.

**Remark.** These axioms are of course modelled on those of Eilenberg-Steenrod [1] supplemented by Milnor’s additivity axiom [4]. If $\mathcal{A}$ satisfies AB5 (exactness of directed colimits) they could be somewhat abbreviated by supposing that directed colimits were preserved. We must avoid this supposition if we are to have a selfdual theory: it is false even for group-valued cohomology, i.e. homology with values in $\mathcal{A}h^*$.

10. Existence theorem, coefficient costacks. Let $A$ be a costack on $K$ with values in $\mathcal{A}$. For each $(X, X')_a \in \mathcal{C}'_K$, let

$$(10.1) ~ H_*(X, X')_a ; A) = H_*(X, X'; q^\# A),$$

the right-hand side is the homology of the simplicial pair $(X, X')$ with coefficients in $q^\# A$ as defined in the previous chapter.

If $f : (X, X')_a \rightarrow (Y, Y')_\psi$ is a $K$-map, then $f = q^\#$ and so $f^\# q^\# = q^\#$. We then have $H_*(X, X')_a ; A) = H_*(X, X'; f^\# q^\# A)$. The map $f_* : H_*(X, X'; f^\# q^\# A) \rightarrow H_*(Y, Y'; q^\# A)$ is the induced map $H_*(f : H_*(X, X')_a; A) \rightarrow H_*(Y, Y')_a; A)$. The results of Chapter II show that

**Theorem 10.1 (Existence Theorem).** For every costack $A$ on $K$ there is a homology theory $H_*$ on $\mathcal{C}'_K$ defined by the chain homology functor as

$$H_*(X, X')_a ; A) = H_*(q q^\# A).$$

Now, let $H_*$ be any homology theory on $\mathcal{C}'_K$. The coefficient costack $A$ of $H_*$ is, by definition, the costack on $K$ with $A_\sigma = H_*(\Delta^\sigma, \hat{\Delta}^\sigma)_a$ for $\sigma \in K$ and with $A(d^i) = F^i$, $A(s^i) = (F^i)^{-1}$. We observe that the coefficient costack of the homology theory $H_*$ in the theorem is just that $A$.

If $K$ is a point, a $K$-pair is just a pair of simplicial sets and the theory $H_*$ in the theorem is the ordinary simplicial homology with local coefficients.

11. Uniqueness theorem. We shall show that the $H_*$ in Theorem 10.1 is essentially the only homology theory on $\mathcal{C}'_K$. 
Theorem 11.1 (Uniqueness Theorem). Let $h_\bullet$ be any homology theory on $\mathscr{C}_X$. There is a natural isomorphism

$$h_\bullet(X, X')_\circ \cong H_\bullet((X, X')_\circ; A),$$

where $A$ is the coefficient costack of the theory $h_\bullet$.

Proof. First we show (11.1) for finite dimensional case. Let $\phi = X^{-1} \subset X^1 \subset X^2 \subset \cdots \subset X' = X_\circ$ (the subscripts $\varphi$ in $X_\varphi$ are omitted) be the increasing filtration of $X_\circ$ by skeletons. It is an easy consequence of the dimension axiom that the associated spectral sequence collapses and that $h_\bullet(X_\circ)$ is naturally isomorphic to the homology of the chain complex $C^h$ with

$$C^h_q = H_q(X^{s-1}, X^s) \cong \bigoplus_x H_q(A^\varphi(x), A), \quad x \in N X_q.$$

It follows from the dimension axiom and the definition of $A$ that

$$C^h_q \cong \bigoplus_x H_q(A^\varphi(x), A), \quad x \in N X_q.$$

Thus $C^h_q \cong \bigoplus_x (\varphi#A)_x = C_q(\varphi#A)$. From the constructions of $A$ and $C^h$ we observe that $C^h \cong C(\varphi#A)$ as chain complexes. Hence $h_\bullet(X_\circ) \cong H_\bullet(X_\circ; A)$. Therefore (11.1) follows from the exactness axiom and the five lemma.

Next, suppose that $X$ is infinite dimensional. We have seen that it suffices to prove the isomorphism for the absolute case. From the first part of this proof, we see that for a fixed integer $q \geq 0$ and any integer $n > q$ there is a canonical isomorphism $h_\bullet(X^n) \cong H_\bullet(X^n; A)$. But

$$H_\bullet(X^n; A) \cong H_\bullet(X^{n+1}; A) \cong \cdots H_\bullet(X_\circ; A),$$

we have a direct system

$$h_\bullet(X^0) \xrightarrow{i^0_\bullet} h_\bullet(X^1) \xrightarrow{i^1_\bullet} h_\bullet(X^2) \xrightarrow{i^2_\bullet} \cdots$$

with isomorphisms $i^n_\bullet$ for $n > q + 1$.

Now, use Axioms 1, 2, 3, 5, and 6 and proceed as in [4], we get a Mayer-Vietoris sequence

$$\lim_{n=0} h_\bullet(X^n) \xrightarrow{\partial'} \lim_{n=0} h_\bullet(X^n) \xrightarrow{q} h_\bullet(X_\circ)$$

with Coker $f_\circ = \lim h_\bullet(X^n)$. Dual to the Lemma 2 of [4], denote the kernel of $f_\circ$ by $\mathcal{L}''(h_{q-1}(X^n))$ and call $\mathcal{L}''$ the derived functor of $\lim$, then there is an exact sequence

$$0 \rightarrow \lim h_\bullet(X^n) \rightarrow h_\bullet(X) \rightarrow \mathcal{L}''(h_{q-1}(X^n)) \rightarrow 0.$$
and a similar one for $H_q$. Apply (11.2) and (11.3), we have $L'(h_{q-1}(X^*)) = 0$ and $h_q(X_\infty) \approx H_q(X_\infty; A)$.

REFERENCES


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