

## AMBIENT SURGERY AND TANGENTIAL HOMOTOPY QUATERNIONIC PROJECTIVE SPACES

BY  
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**Introduction.** In this paper the word manifold will always mean oriented compact  $C^\infty$ -manifold. Unless otherwise specified, all homology and cohomology is taken with integral coefficients, and for  $M^n$  an  $n$ -manifold,  $[M] \in H_n(M, \partial M)$  will denote the orientation class of  $M$ . A map  $f: M \rightarrow N$  between  $n$ -manifolds is of degree  $+1$  if  $f_*([M]) = [N]$ .

We denote the quaternions by  $\mathcal{Q}$ , and quaternionic projective  $n$ -space, which may be described as the collection of quaternionic lines in  $\mathcal{Q}^{n+1}$ , by  $QP_n$ . The underlying set of  $QP_n$  can also be described as

$$\left\{ (x_1, \dots, x_{n+1}) \mid x_i \in \mathcal{Q} \text{ for } i = 1, \dots, n+1; \sum_{i=1}^{n+1} |x_i|^2 = 1 \right\}$$

modulo the equivalence relation  $(x_1, \dots, x_{n+1}) \sim (y_1, \dots, y_{n+1})$  if and only if for some  $s \in \mathcal{Q}$ ,  $|s| = 1$ ,  $(x_1, \dots, x_{n+1}) = s \cdot (y_1, \dots, y_{n+1})$ . By  $QP_{n-1} \subset QP_n$  we will always mean the copy of  $QP_{n-1}$  in  $QP_n$  defined by: the class of  $(x_1, \dots, x_{n+1})$  is in  $QP_{n-1} \subset QP_n$  if and only if  $x_{n+1} = 0$ .  $QP_n$  is given the structure of a manifold as in [14, §20.3].

$\mathcal{S}^4$  will denote the unit 4-disc bundle over  $QP_n$  defined by the standard vector 4-bundle over  $QP_n$ , whose total space is  $\{(l, v) \mid l \text{ a quaternionic line in } \mathcal{Q}^{n+1}, v \text{ an element of } l\}$ , and whose projection takes  $(l, v)$  to  $l$ .  $\mathcal{S}_0^3$  will denote the 3-sphere bundle over  $QP_n$  associated with  $\mathcal{S}^4$ . The group of  $\mathcal{S}_0^3$  may be reduced to  $S^3 =$  the multiplicative group of unit length quaternions, and, since the total space of  $\mathcal{S}_0^3$  over  $QP_n$  is  $S^{4n+3}$ , we see  $\mathcal{S}_0^3$  is the universal bundle with fiber  $S^3$  and group the multiplicative group of unit length quaternions for dimensions less than or equal to  $4n+2$ , see [14, §19].

Given a manifold  $M$  with submanifold  $N$ , we denote the normal disc bundle of  $N$  in  $M$  by  $\nu(N \subset M)$ , and we identify a tubular neighborhood of  $N$  in  $M$  with the total space of  $\nu(N \subset M)$ .  $\tau(M)$  will denote the tangent bundle of  $M$ , and  $0_M^n$  the trivial vector  $n$ -bundle over  $M$ . A map between manifolds  $f: R \rightarrow S$  is called tangential if for some integers  $k, l$ ,  $f^*(\tau(S)) \oplus 0_R^k \approx \tau(R) \oplus 0_R^l$ .

*Statement of results.* The first section of this paper is devoted to a discussion of surgery [6], [8], [12]. In particular, we show that in special cases one can do surgery

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to a submanifold  $N$  of a manifold  $M$  so that the modified manifold is also a submanifold of  $M$ . Using this technique we prove

**THEOREM 1.** *Let  $P^{m-1}$  be a simply connected submanifold of a simply connected manifold  $Q^m$ , where  $l \geq 3$ ,  $m-l \geq 5$ , and  $m-l$  congruent to 0, 1, or 3 modulo 4. Suppose  $P$ ,  $Q$ , and  $M^m$  are manifolds without boundary and that  $f: M^m \rightarrow Q^m$  is a degree +1 tangential homotopy equivalence. Then  $f$  is homotopic to a map  $f': M \rightarrow Q$  which is differentiable and transverse regular over  $P \subset Q$  and such that*

$$f'|_{f'^{-1}(P)}: f'^{-1}(P) \rightarrow P$$

*is a homotopy equivalence.*

We also prove a uniqueness theorem for the  $m-l \equiv 0$  modulo 4 case:

**THEOREM 2.** *Let  $P$  and  $Q$  be as in Theorem 1, with  $m-l \equiv 0 \pmod{4}$ . Assume  $M^{m+1}$  is an  $h$ -cobordism between  $M_1^m$  and  $M_2^m$ . Let  $f: M^{m+1} \rightarrow Q$  be a tangential homotopy equivalence which is differentiable and transverse regular over  $P \subset Q$ . Let  $N_i = f^{-1}(P) \cap M_i$  for  $i=1, 2$ . Assume further that  $f|_{N_i}: N_i \rightarrow P$  is a homotopy equivalence for  $i=1, 2$ . Then  $f$  is homotopic to a map  $f': M^{m+1} \rightarrow Q$  such that  $f'|_{M_1^m \cup M_2^m} = f|_{M_1^m \cup M_2^m}$ , and such that  $f'$  is differentiable and transverse regular over  $P \subset Q$  and  $f'|_{f'^{-1}(P)}: f'^{-1}(P) \rightarrow P$  is a homotopy equivalence so that  $f'^{-1}(P)$  is an  $h$ -cobordism between  $N_1$  and  $N_2$ .*

In §2 we apply the results of the first section to the study of manifolds of the same tangential homotopy type as  $QP_n$ .

**DEFINITION.** For  $n \geq 2$ ,  $\theta(QP_n)$  is the set of equivalence classes of pairs  $(M, f)$  where  $M$  is a  $4n$ -manifold without boundary and  $f: M \rightarrow QP_n$  is a tangential homotopy equivalence of degree +1 under the relation:  $(M, f) \sim (N, g)$  if and only if  $M$  and  $N$  are  $h$ -cobordant under a cobordism  $C$  such that there is a tangential homotopy equivalence  $h: C \rightarrow QP_n$  which restricts to  $f$  and  $g$  on the proper boundary components.

**DEFINITION.**  $\theta_n$  is the group of homotopy  $n$ -spheres under the equivalence relation of  $h$ -cobordism, as discussed in [6]. We will see that  $\theta_{4n}$  acts as a group on  $\theta(QP_n)$  by connected sum for  $n \geq 2$ .

Proofs that the following two definitions are independent of the choices made are given in §2. They depend on the theorems of §1.

**DEFINITION.** Given  $\alpha \in \theta(QP_n)$ , we will define  $r(\alpha) \in \theta(QP_{n-1})$  by choosing a representative element  $(M, f)$  for  $\alpha$  such that  $f: M \rightarrow QP_n$  is differentiable and transverse regular over  $QP_{n-1} \subset QP_n$  and such that  $f|_{f^{-1}(QP_{n-1})}: f^{-1}(QP_{n-1}) \rightarrow QP_{n-1}$  is a tangential homotopy equivalence, and setting

$$r(\alpha) = [(f^{-1}(QP_{n-1}), f|_{f^{-1}(QP_{n-1})})].$$

**DEFINITION.** Given  $\gamma \in \theta(QP_{n-1})$  we will define  $h(\gamma) \in \theta_{4n-1}$  as follows: choose a representative  $(N, g)$  for  $\gamma$  such that  $g: N \rightarrow QP_{n-1}$  is differentiable. Then  $g^*(\mathcal{S}_0^3)$  is a homotopy sphere. We define  $h(\gamma) = [g^*(\mathcal{S}_0^3)] \in \theta_{4n-1}$ .

If for  $\lambda \in \theta_{4n}$  we let  $c(\lambda) = \lambda \cdot [(QP_n, \text{identity})]$ , where the dot indicates the group action of  $\theta_{4n}$  in  $\theta(QP_n)$ , we obtain a sequence

$$\theta_{4n} \xrightarrow{c} \theta(QP_n) \xrightarrow{r} \theta(QP_{n-1}) \xrightarrow{h} \theta_{4n-1}.$$

In §2 we prove

**THEOREM 3.** (1) *image of  $r = \text{kernel of } h$ ,*

(2) *if  $r(\alpha) = r(\beta)$ , then  $\alpha = \lambda \cdot \beta$  for some  $\lambda \in \theta_{4n}$ .*

This theorem gives us an inductive geometric procedure whereby representatives for all elements of  $\theta(QP_n)$  may be constructed from elements of  $\theta(QP_{n-1})$ . We conclude §2 with this construction.

In §3 we investigate  $\theta(QP_2)$  as a starting point for inductions. We prove

**THEOREM 4.**  *$\theta(QP_2)$  contains at most two elements, with representatives given by  $(QP_2, \text{identity})$  and  $(QP_2 \# \Sigma^8, \text{id}')$ , where  $\Sigma^8$  is the nonstandard homotopy 8-sphere, and  $\text{id}'$  the obvious homeomorphism  $QP_2 \# \Sigma^8 \rightarrow QP_2$ .*

*Note.* This and Theorem 3 together imply that the number of elements in  $\theta(QP_n)$  is finite for all  $n \geq 2$ , since  $\theta_n$  is finite for all  $n \geq 8$  [6].

**DEFINITION.** A map between manifolds,  $h: M \rightarrow N$ , is *induced by a combinatorial equivalence* if for some  $C^\infty$  triangulations  $\tau_1: K_1 \rightarrow M$ ,  $\tau_2: K_2 \rightarrow N$ , and some combinatorial equivalence of simplicial complexes  $c: K_1 \rightarrow K_2$ ,  $h = \tau_2 \circ c \circ \tau_1^{-1}$ .

The final result of this paper is

**THEOREM 5.** *For  $n \geq 2$ , any representative  $(M, f)$  of an element of  $\theta(QP_n)$  has the property that  $f: M \rightarrow QP_n$  is homotopic to a map  $h: M \rightarrow QP_n$  induced by a combinatorial equivalence. Thus, for  $n \geq 2$ , all  $4n$ -manifolds of the tangential homotopy type of  $QP_n$  are combinatorially equivalent.*

*Note.* The restriction to manifolds of the tangential homotopy type of  $QP_n$  is essential. Hsiang [4] has shown that for any  $n \geq 2$  there exist infinitely many distinct manifolds of the homotopy type of  $QP_n$  all with different rational Pontrjagin classes. These manifolds are all combinatorially distinct.

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**1. Ambient surgery.** Corresponding to the assumptions of Theorems 1 and 2, there are two cases in which we will show that surgery can be done within an ambient manifold:

*Case (a).*  $P^{m-l}$  a submanifold of  $Q^m$  of codimension  $l$  where  $l \geq 3$ ,  $m-l \geq 5$ , and  $m-l \not\equiv 2 \pmod{4}$ . Suppose  $P$  and  $Q$  are both simply connected, and that  $P$ ,  $Q$ , and  $M^m$  are manifolds without boundary. Let  $f: M \rightarrow Q$  be a degree  $+1$  tangential homotopy equivalence. By the work of Thom [16] we can deform  $f$  by a homotopy

so that it becomes differentiable and transverse regular over  $P \subset Q$ . In this case we will do surgery to  $f^{-1}(P)$  to prove that  $f$  can be deformed by a series of homotopies to a map  $f': M \rightarrow Q$  such that  $f'$  is transverse regular over  $P \subset Q$  and

$$f'|_{f'^{-1}(P)}: f'^{-1}(P) \rightarrow P$$

is a homotopy equivalence.

Case (b). Assume  $P^{m-l}$  and  $Q^m$  are as in Case (a), but with  $m-l \equiv 0 \pmod 4$ . Suppose  $M^{m+1}$  is an  $h$ -cobordism between  $M_1^m$  and  $M_2^m$  and  $f: M^{m+1} \rightarrow Q$  is a tangential homotopy equivalence. Then  $f|M_i^m: M_i^m \rightarrow Q$  is a homotopy equivalence for  $i=1, 2$ . Again we may up to homotopy assume  $f$  differentiable and transverse regular over  $P \subset Q$ . We assume  $f$  is of this form, and further that if  $N_i = f^{-1}(P) \cap M_i$  then  $f|N_i: N_i \rightarrow P$  is a homotopy equivalence for  $i=1, 2$ . In this case we will do surgery to  $f^{-1}(P)$  to prove that  $f: M^{m+1} \rightarrow Q$  can be deformed by a series of homotopies, all of which are stationary on  $\partial M^{m+1}$ , to a map  $f': M^{m+1} \rightarrow Q$  which is differentiable and transverse regular over  $P \subset Q$  and such that  $f'^{-1}(P)$  is an  $h$ -cobordism between  $N_1$  and  $N_2$ .

Recall the following lemma from [16, p. 67].

LEMMA 1.1. *Given  $f: M^m \rightarrow Q$  differentiable and transverse regular over  $P \subset Q$  as in Case (a), if we put  $N = f^{-1}(P)$  then  $f|N: N \rightarrow P$  is a map of degree  $\pm 1$ , and thus  $N$  may be oriented so that  $f|N$  is a map of degree  $+1$ .*

In dealing with both Cases (a) and (b) from now on, we will assume  $f: M \rightarrow Q$  is differentiable and transverse regular over  $P \subset Q$ , and denote  $f^{-1}(P)$  by  $N$  and  $f|N: N \rightarrow P$  by  $g: N \rightarrow P$ .

We now make the inductive assumption that  $g: N \rightarrow P$  is such that  $g_*: H_r(N) \rightarrow H_r(P)$  is an isomorphism for all  $r < k$ , where  $0 \leq k \leq (m-l)/2$ . We will want to do surgery to  $N$  to modify  $H_k(N)$ . By the transversality of  $f$  to  $P \subset Q$ , we have  $\nu(N \subset M) \approx g^*(\nu(P \subset Q))$  and thus the assumption that  $f$  is tangential implies  $g: N \rightarrow P$  is tangential. In Case (a) Lemma 1.1 implies  $g$  is a map of degree  $+1$ , so by [12] we see  $g_*: H_k(N) \rightarrow H_k(P)$  is onto, and all classes in the kernel of this map are spherical.

For Case (b), the map  $g_*: H_i(N) \rightarrow H_i(P)$  is always onto, and since  $g_*: \pi_i(N) \rightarrow \pi_i(P)$  is an isomorphism for all  $i < k$ , we have that all classes in the kernel of  $g_*: H_k(N) \rightarrow H_k(P)$  are spherical see [12].

In either case, let  $\alpha \in H_k(N)$  be an element of kernel ( $g_*: H_k(N) \rightarrow H_k(P)$ ). Since  $\alpha$  is spherical and  $k \leq (m-l)/2$ ,  $m-l \geq 5$ , we may represent  $\alpha$  by an imbedding  $i: S^k \rightarrow N$ . In Case (b) it is clear we may choose  $i$  so that  $i(S^k) \cap \partial N = \emptyset$ . We will assume that  $\alpha$  is so represented, and further that  $\nu(i(S^k) \subset N)$  is trivial. It should be noted that  $i: S^k \rightarrow N$  is nullhomotopic in  $M$ , and thus

$$\tau(M)|_{i(S^k)} \approx 0^m \approx \nu(i(S^k) \subset N) \oplus \nu(N \subset M)|_{i(S^k)}.$$

Hence  $\nu(i(S^k) \subset N)$  is stably trivial, and this further condition is no restriction for  $k < (m-l)/2$ . In the middle dimension  $k = (m-l)/2$  we must show that kernel

( $g_*: H_k(N) \rightarrow H_k(P)$ ) can be reduced to zero by doing surgery only on classes  $\alpha \in H_k(N)$  which satisfy this additional condition.

The idea of the following work is to attach a  $(k+1)$ -disc to  $N$  in  $M$  via the map  $i$ , and, by thickening this disc, to get a spherical modification of  $N$  to a new submanifold  $N'$  of  $M$  and to deform the map  $f: M \rightarrow Q$  by a homotopy to a map  $f_1$  such that  $f_1^{-1}(P) = N'$ . The following lemma is readily seen, and its proof is omitted.

**LEMMA 1.2.** *The map  $f: M \rightarrow Q$  is homotopic to a map  $f_1: M \rightarrow Q$  such that  $f_1^{-1}(P) = N = f^{-1}(P)$ ,  $f_1$  is differentiable and transverse regular over  $P \subset Q$ , and  $f_1$  maps a neighborhood of  $i(S^k)$  in  $N$  to a point  $p \in P$ .*

Henceforth we will assume  $f: M \rightarrow Q$  has the properties described for  $f_1$  in Lemma 1.2.

By the transversality of  $f$  we know  $\nu(N \subset M) \approx g^*(\nu(P \subset Q))$ , so, by choosing a framing for  $\nu(P \subset Q)|_p$ , where  $p = f \circ i(S^k)$ , we get a framing  $\nu_1, \dots, \nu_l$  for  $\nu(N \subset M)|_{i(S^k)}$ . We identify  $\nu(N \subset M)$  with a tubular neighborhood of  $N$  in  $M$  and push  $i(S^k)$  out into  $M - N$  by a map  $i_1: S^k \times I \rightarrow M$  defined by  $i_1(x, \tau) = (i(x), \tau \cdot \nu_1)$ . Since  $f \circ i: S^k \rightarrow P$  is nullhomotopic and  $f: M \rightarrow Q$  is a homotopy equivalence, the class of  $i_1|(S^k \times (1))$  is null in  $\pi_k(M)$ . The following crucial proposition asserts this holds also in  $M - N$ .

**PROPOSITION 1.1.**  $i_1|(S^k \times (1))$  is a nullhomotopic imbedding of  $S^k$  in  $M - N$ .

**Proof.** The proof of this proposition will be in several parts.

(1) If  $k=0$ , we have  $S^k$  is just  $S^0=2$  points. Since  $N \cap i_1(S^0 \times (1)) = \emptyset$ , and  $M$  is path connected, we may choose a differentiable arc  $h: I \rightarrow M$  connecting the points of  $i_1(S^0 \times (1))$ , transverse regular over  $N$ . In Case (b),  $i_1(S^0 \times (1)) \cap \partial M$  can be assumed empty, and we may choose  $h: I \rightarrow M$  so that  $h(I) \cap \partial M = \emptyset$ . The condition that  $h$  is transverse to  $N$  implies  $h(I) \cap N = \emptyset$  since codimension of  $N$  in  $M$  is  $l > 1$ , so  $i_1(S^0 \times (1))$  is nullhomotopic in  $M - N$ .

(2) For  $k=1$  a similar proof applies. That is, we take a map  $h: D^2 \rightarrow M$  such that  $h|\partial D^2 = i_1|(S^1 \times (1))$  which is transverse to  $N$ . As above  $h$  can be chosen so that  $h(D^2) \cap \partial M = \emptyset$  in case (b). Transversality again implies  $h(D^2) \cap N = \emptyset$  since codimension of  $N$  in  $M$  is  $l > 2$ .

(3) For  $k \geq 2$ , the following lemma is applied to show the proposition. In applying the lemma, we note that  $f \circ i_1|(S^k \times (1))$  is clearly nullhomotopic in  $Q - P$ .

**LEMMA 1.3.** *Suppose  $k \geq 2$  and  $g_*: H_p(N) \rightarrow H_p(P)$  is an isomorphism for all  $p < k$ . Then if  $f' = f|_{M-N}$ ,  $f'_*: \pi_p(M-N) \rightarrow \pi_p(Q-P)$  is an isomorphism for  $p \leq k+l-2$ .*

**Proof.** We first prove this for Case (a).  $f$  is transverse regular over  $P \subset Q$ , so by [16] we may choose a tubular neighborhood  $T(P)$  of  $P$  in  $Q$  so that  $f^{-1}(T(P)) \equiv T(N)$

is a tubular neighborhood of  $N$  in  $M$ . We define  $M' = M - \text{interior of } T(N)$ ;  $Q' = Q - \text{interior of } T(P)$ . Then  $M'$  and  $Q'$  are manifolds with boundary, and  $H_p(M - N) \approx H_p(M')$ ,  $H_p(Q - P) \approx H_p(Q')$ . Consider the diagram:

$$\begin{array}{ccc}
 H_p(M - N) & \xrightarrow{f'_*} & H_p(Q - P) \\
 \uparrow p_1 & & \uparrow p_2 \\
 H_p(M') & \xrightarrow{h_*} & H_p(Q') \\
 \uparrow \cap [M'] & & \uparrow \cap [Q'] \\
 H^{m-p}(M', \partial M') & \xleftarrow{h_*} & H^{m-p}(Q', \partial Q') \\
 \uparrow E_1 & & \uparrow E_2 \\
 H^{m-p}(M, T(N)) & \xleftarrow{f_*} & H^{m-p}(Q, T(P)) \\
 \uparrow i_1 & & \uparrow i_2 \\
 H^{m-p}(M, N) & \xleftarrow{f_*} & H^{m-p}(Q, P)
 \end{array}$$

where the maps are the obvious inclusions or restrictions of  $f$ . All of the vertical maps are isomorphisms. By naturality of the cap product and the fact that  $h_*$  has degree  $+1$ ,  $h_* \circ (\cap [M']) \circ h^* = \cap [Q']$ . Hence  $f^*$  is a monomorphism, and  $f'_*$  is an epimorphism. Further, if  $f^*$  is an isomorphism, then so is  $f'_*$ .

Consider the commutative diagram:

$$\begin{array}{ccccccccc}
 H^{m-p-1}(Q) & \longrightarrow & H^{m-p-1}(P) & \longrightarrow & H^{m-p}(Q, P) & \longrightarrow & H^{m-p}(Q) & \longrightarrow & H^{m-p}(P) \\
 f^* \downarrow & & g^* \downarrow & & f^* \downarrow & & f^* \downarrow & & g^* \downarrow \\
 H^{m-p-1}(M) & \longrightarrow & H^{m-p-1}(N) & \longrightarrow & H^{m-p}(M, N) & \longrightarrow & H^{m-p}(M) & \longrightarrow & H^{m-p}(N)
 \end{array}$$

From this and the 5-lemma we conclude that  $f^*: H^{m-p}(Q, P) \rightarrow H^{m-p}(M, N)$  is an isomorphism whenever both  $g^*: H^{m-p-1}(P) \rightarrow H^{m-p-1}(N)$  and  $g^*: H^{m-p}(P) \rightarrow H^{m-p}(N)$  are.

The lemma of Novikov [7, p. 10] implies these conditions hold if

$$g_*: H_{p+1-l}(N) \rightarrow H_{p+1-l}(P) \quad \text{and} \quad g_*: H_{p-l}(N) \rightarrow H_{p-l}(P)$$

are isomorphisms. Thus  $f^*: H^{m-p}(Q, P) \rightarrow H^{m-p}(M, N)$ , and hence

$$f'_*: H_p(M - N) \rightarrow H_p(Q - P),$$

is an isomorphism for  $p + 1 - l < k$ , or  $p \leq k + l - 2$ .

Since  $k \geq 2$ ,  $\pi_1(N) \approx \pi_1(P) = 0$ , and, by assumption,  $\pi_1(M) \approx \pi_1(Q) = 0$ , so by Van Kampen's Theorem,  $\pi_1(M - N) \approx \pi_1(Q - P) = 0$ . We have shown

$$f'_*: H_p(M - N) \rightarrow H_p(Q - P)$$

is onto for all  $p$ , so by Whitehead's Theorem we now see  $f'_*: \pi_p(M-N) \rightarrow \pi_p(Q-P)$  is an isomorphism for all  $p \leq k+l-2$ . This concludes the proof of Lemma 1.3 and of Proposition 1.1 for Case (a).

Recall that in Case (b)  $M$  is assumed to be an  $h$ -cobordism between  $M_1$  and  $M_2$ ,  $N_1 = N \cap M_1$  and  $f|M_1: M_1 \rightarrow Q$  and  $f|M_2: M_2 \rightarrow Q$  are both assumed to be homotopy equivalences.

By the lemma for Case (a),  $f'_*: H_p(M_1 - N_1) \rightarrow H_p(Q-P)$  is an isomorphism for all  $p$ . Hence  $f'_*: H_p(M-N) \rightarrow H_p(Q-P)$  is onto for all  $p$ , and inclusion  $\star: H_p(M_1 - N_1) \rightarrow H_p(M-N)$  is one-to-one for all  $p$ . Thus  $f'_*: H_p(M-N) \rightarrow H_p(Q-P)$  is an isomorphism if inclusion  $\star: H_p(M_1 - N_1) \rightarrow H_p(M-N)$  is onto.

Choose disc bundle structures on tubular neighborhoods  $T(N_1)$  and  $T(N)$  of  $N_1$  in  $M_1$  and  $N$  in  $M$  respectively so that the inclusion gives a bundle map  $T(N_1) \rightarrow T(N)$ . Then the naturality of the Thom isomorphism implies that if

$$\text{incl}_\star: H_p(N_1) \rightarrow H_p(N)$$

is an isomorphism, then so is  $\text{incl}_\star: H_{p+i}(T(N_1), \partial T(N_1)) \rightarrow H_{p+i}(T(N), \partial T(N))$ .

By excision,  $H_{p+i}(T(N_1), \partial T(N_1)) \approx H_{p+i}(M_1, M_1 - N_1)$  and  $H_{p+i}(T(N), \partial T(N)) \approx H_{p+i}(M, M - N)$ . Also, if  $g_\star: H_p(N) \rightarrow H_p(P)$  is an isomorphism so is  $\text{incl}_\star: H_p(N_1) \rightarrow H_p(N)$ , so for  $p < k$  we have isomorphisms as indicated in the following diagram:

$$\begin{array}{ccccccccc} H_{p+i}(M_1) & \rightarrow & H_{p+i}(M_1, M_1 - N_1) & \rightarrow & H_{p+i-1}(M_1 - N_1) & \rightarrow & H_{p+i-1}(M_1) & \rightarrow & H_{p+i-1}(M_1, M_1 - N_1) \\ \downarrow \approx & & \downarrow \approx & & \downarrow & & \downarrow \approx & & \downarrow \approx \\ H_{p+i}(M) & \rightarrow & H_{p+i}(M, M - N) & \rightarrow & H_{p+i-1}(M - N) & \rightarrow & H_{p+i-1}(M) & \rightarrow & H_{p+i-1}(M, M - N) \end{array}$$

By the 5-lemma we conclude  $\text{incl}_\star: H_{p+i-1}(M_1 - N_1) \rightarrow H_{p+i-1}(M - N)$  is an isomorphism for  $p < k$ , and then  $\text{incl}_\star: H_i(M_1 - N_1) \rightarrow H_i(M - N)$  is an isomorphism for  $i \leq k+l-2$ . Thus we see  $f'_*: H_p(M-N) \rightarrow H_p(Q-P)$  is an isomorphism for  $p \leq k+l-2$ , and, as in the proof of Case (a),  $M-N$  and  $Q-P$  are simply connected, so we again apply Whitehead's Theorem to show  $f'_*: \pi_p(M-N) \rightarrow \pi_p(Q-P)$  is an isomorphism for  $p \leq k+l-2$ . This completes the proof of Lemma 1.3 and of Proposition 1.1.

We now identify  $\{x \in D^{k+1} \mid |x| \geq 1/2\}$  with  $S^k \times I$ , and define a map

$$h: \{x \in D^{k+1} \mid |x| \geq 1/2\} \rightarrow M$$

to be this identification composed with the above defined map  $i_1$ . Since  $f$  is transverse to  $P \subset Q$  and carries  $i_1(S^k \times (0))$  to  $p \in P$ ,  $f \circ h$  maps  $S_{1/2}^k = \{x \in D^{k+1} \mid |x| = 1/2\}$  to a single point  $x \in Q-P$ . Denote  $\{x \in D^{k+1} \mid |x| \leq 1/2\}$  by  $D_{1/2}^{k+1}$  so that

$$\partial D_{1/2}^{k+1} = S_{1/2}^k.$$

LEMMA 1.4. *There is a map  $h': D_{1/2}^{k+1} \rightarrow M-N$  such that  $h'|S_{1/2}^k = h|S_{1/2}^k$ , and  $f \circ h': (D_{1/2}^{k+1}, S_{1/2}^k) \rightarrow (Q-P, x)$  is nullhomotopic. Thus  $f: M \rightarrow Q$  can be deformed by a homotopy so that we may assume  $f \circ h'(D_{1/2}^{k+1}) = x$ .*

**Proof.** Consider the diagram:

$$\begin{array}{ccccc}
 \pi_{k+1}(M-N) & \xrightarrow{j_1} & \pi_{k+1}(M-N, h(S_{1/2}^k)) & \xrightarrow{\partial} & \pi_k(h(S_{1/2}^k)) \\
 \downarrow f_* & & \downarrow f_* & & \\
 0 & \longrightarrow & \pi_{k+1}(Q-P) & \xrightarrow{j_2} & \pi_{k+1}(Q-P, x) \longrightarrow 0.
 \end{array}$$

$h|_{S_{1/2}^k}: S_{1/2}^k \rightarrow M-N$  is nullhomotopic, so there is a map  $h_1: D_{1/2}^{k+1} \rightarrow M-N$  which extends  $h$ . Suppose  $h_1: (D_{1/2}^{k+1}, S_{1/2}^k) \rightarrow (M-N, h(S_{1/2}^k))$  represents an element  $\alpha \in \pi_{k+1}(M-N, h(S_{1/2}^k))$ . Let  $\gamma = j_1 \circ (j_2 f_*)^{-1} \circ f_*(\alpha)$ . Then a representative of the class  $\alpha + (-\gamma)$  defines a map  $h': D_{1/2}^{k+1} \rightarrow M-N$  which agrees with  $h$  on  $S_{1/2}^k$  such that  $f \circ h': (D_{1/2}^{k+1}, S_{1/2}^k) \rightarrow (Q-P, x)$  is nullhomotopic.

We may choose  $h'$  within its class to be an imbedding such that the map defined on  $D^{k+1}$  by  $h'$  on  $D_{1/2}^{k+1}$  and  $h$  on  $\{x \in D^{k+1} \mid |x| \geq 1/2\}$  is an imbedding [1, Theorem 4.1]. Also, we may now deform  $f$  by a homotopy in a tubular neighborhood of  $h'(D_{1/2}^{k+1})$  so that  $f \circ h'(D_{1/2}^{k+1}) = x \in Q-P$ .

We will henceforth denote the map  $D^{k+1} \rightarrow M$  constructed in Lemma 1.4 by the letter  $h$ , and will assume that  $f: M \rightarrow Q$  has been put into the form described in that lemma, and in Lemma 1.2.

We have assumed that  $\nu(i(S^k) \subset N)$  is trivial, so  $\nu(h(D^{k+1} \subset M)|h(S^k))$  is the sum of two bundles,  $\eta_1$  and  $\eta_2$ , where  $\eta_1$  is the  $(l-1)$ -dimensional trivial bundle given by  $\nu_2, \dots, \nu_l$ , that is the bundle which results when we split

$$\nu_1 = \nu(h(S^k) \subset h(D^{k+1}))$$

off  $\nu(N \subset M)|h(S^k)$ , and  $\eta_2$  is an  $(m-l-k)$ -dimensional trivial bundle in Case (a) and an  $(m+1-l-k)$ -dimensional trivial bundle in Case (b).  $\eta_1$  is, of course, already framed by our choice of  $\nu_2, \dots, \nu_l$ , pulled back by  $f$  from  $\nu(P \subset Q)|p$ .

**LEMMA 1.5.** *With notation as above, there is a framing of  $\nu(h(D^{k+1}) \subset M)$  such that restricted to  $h(S^k)$  it gives the required framing  $\nu_2, \dots, \nu_l$  of  $\eta_1$ , and some framing of  $\eta_2$ .*

**Proof.** Framings of  $\nu(h(D^{k+1}) \subset M)|h(S^k)$  are represented by elements of  $\pi_k(SO(m-k-1))$  in Case (a), and elements of  $\pi_k(SO(m-k))$  in Case (b), and extend over  $h(D^{k+1})$  if and only if the representative homotopy class is zero. Similarly, elements of  $\pi_k(SO(m-l-k))$  in Case (a) and  $\pi_k(SO(m+1-l-k))$  in Case (b) represent framings of  $\eta_2 = \nu(h(S^k) \subset N)$ . There is given a framing of  $\eta_1 = \nu(N \subset M)|h(S^k)$ . Thus the problem of framing  $\eta_2$  so that we get a framing of  $\nu(h(D^{k+1}) \subset M)|h(S^k)$  which extends over  $h(D^{k+1})$  depends only on the map  $\pi_k(SO(m-l-k)) \rightarrow \pi_k(SO(m-k-1))$  in Case (a) and on  $\pi_k(SO(m+1-l-k)) \rightarrow \pi_k(SO(m-k))$  in Case (b) induced by inclusion. If the map is onto then  $\eta_2$  may be framed as needed.

We see this by showing  $\pi_k(SO(p)) \rightarrow \pi_k(SO(p+1))$  is onto for all relevant values of  $p$ . By Steenrod [14, p. 117] this map is onto whenever  $k < p$ . Since we have by assumption  $k \leq (m-l)/2$ , or  $k < m-l-k+1$ , the proof for Case (b) is complete. Also, for Case (a), if  $m-l \equiv 1$  or  $3 \pmod 4$ , then  $k < (m-l)/2$ , and  $k < m-l-k$ , so the proof is also complete in these cases.

Case (a),  $m-l \equiv 0 \pmod 4$  remains, since we may have here  $k = m-k-l$  or  $k = (m-l)/2$ . Then  $k$  is an even number, and in the fibration sequence

$$\rightarrow \pi_{k+1}(S^k) \rightarrow \pi_k(SO(k)) \rightarrow \pi_k(SO(k+1)) \rightarrow \pi_k(S^k) \rightarrow \pi_{k-1}(SO(k)) \rightarrow$$

it is known that  $\pi_k(S^k) \rightarrow \pi_{k-1}(SO(k))$  is a monomorphism, so  $\pi_k(SO(k) \rightarrow \pi_k(SO(k+1))$  is onto, and we are done.

*Note.* In the above lemma a framing of  $\eta_2$  which extends over  $h(D^{k+1})$  is found. In Case (a) for  $m-l \equiv 3 \pmod 4$ , we will later need the fact that there are many such framings from which we may choose. Lemma 1.5 shows the framing  $\nu_2, \dots, \nu_l$  of  $\eta_1$  is represented by the zero element of  $\pi_k(V_{m-k-1, l-1})$ . Different methods of extending this frame over  $h(D^{k+1})$  in  $\nu(h(D^{k+1}) \subset M)$  are in 1-1 correspondence with elements of  $\pi_{k+1}(V_{m-k-1, l-1})$ . In the fibration sequence

$$\begin{aligned} \pi_{k+1}(SO(m-k-1)) &\longrightarrow \pi_{k+1}(V_{m-k-1, l-1}) \xrightarrow{\partial} \pi_k(SO(m-l-k)) \\ &\longrightarrow \pi_k(SO(m-k-1)) \end{aligned}$$

we see  $\pi_k(SO(m-k-1))$  is the stable group since  $k < m-k-l \leq m-k-3$ , so elements in the image of the map  $\partial$  are just the stably trivial elements of

$$\pi_k(SO(m-l-k)).$$

Consequently, given any framing of  $\eta_2$  which arises as in Lemma 1.5 for Case (a),  $m-l \equiv 3 \pmod 4$ , we may modify this frame by any stably trivial element of  $\pi_k(SO(m-l-k))$  and still have a frame which arises as in that lemma. The author is grateful to J. Levine for pointing out this possibility. See [5].

We have now attached a disc  $h(D^{k+1})$  to  $i(S^k)$  in  $M$ , and have framed

$$\nu(h(D^{k+1}) \subset M)$$

in such a way as to induce our given frame  $\nu_2, \dots, \nu_l$  on  $\nu(N \subset M)|h(S^k)$  and some frame on  $\nu(i(S^k) \subset N) = \eta_2$ . If we look at those frames on  $h(D^{k+1})$  which restrict to  $\eta_2$  on the boundary, and take a tubular neighborhood of these, we get an imbedding

$$\begin{aligned} \phi: D^{k+1} \times D^{m-l-k} &\rightarrow M^m \text{ in Case (a), or} \\ \phi: D^{k+1} \times D^{m+1-l-k} &\rightarrow M^{m+1} \text{ in Case (b)} \end{aligned}$$

where  $\phi|(S^k \times (0))$  represents the class  $\alpha \in H_k(N)$  we wish to kill by surgery. The modified manifold  $N'$  we seek is given after suitable rounding of corners by

$$N - \text{int } \phi(S^k \times D^{m-l-k}) \cup \phi(D^{k+1} \times S^{m-l-k-1}) \text{ in Case (a)}$$

and by

$$N - \text{int } \phi(S^k \times D^{m+1-l-k}) \cup \phi(D^{k+1} \times S^{m-l-k}) \text{ in Case (b).}$$

We now want to find a homotopy of  $f: M \rightarrow Q$  to a map  $f_1: M \rightarrow Q$  which is differentiable and transverse regular over  $P \subset Q$  and such that  $f_1^{-1}(P)$  is "isotopic" to the modified manifold  $N'$  described above. Since bundle maps can be carried along over homotopies, we will have that  $f_1$  is a tangential map, and so will have recovered the original conditions and be able to proceed inductively. Our method is similar to that of Haefliger [2].

Recall that  $h: D^{k+1} \rightarrow M$  starts along the vector field  $\nu_1$  in  $\nu(N \subset M)|i(S^k)$ . We extend this imbedding of  $D^{k+1}$  in  $M$  to an imbedding of  $D_2^{k+1}$  = the  $(k+1)$ -disc of radius 2 in  $M$  by mapping along the vector field  $-\nu_1$  in a tubular neighborhood of  $N$  in  $M$ . Denote the resulting map by  $h: D_2^{k+1} \rightarrow M$ . The framing we have chosen for  $\nu(h(D^{k+1}) \subset M)$  extends over  $\nu(h(D_2^{k+1}) \subset M)$ .

In the lemmas above, we have modified the map  $f$  so that  $f$  maps a tubular neighborhood of  $i(S^k)$  in  $N$  to  $p \in P$ , is transverse to  $P$ , and  $f \circ h|D_{1/2}^{k+1}$  has image a point  $x \in Q - P$ . It is now clear one can deform  $f$  by a homotopy and choose coordinates for a tubular neighborhood of  $h(D_2^{k+1})$  in  $M$  of the form  $(x, y, z) \in D_2^{k+1} \times D_2^r \times D^{l-1}$ , where  $r = m - l - k$  in Case (a) and  $r = m + 1 - l - k$  in Case (b), such that on this neighborhood the map  $f$  is defined by the following composition:

Let  $s_1: R \times D^{l-1} \rightarrow [-1, 1] \times D^{l-1}$  be identity on  $[-1, 1] \times D^{l-1}$ ,  $s_1(x, p) = (-1, p)$  for  $x \leq -1$ , and  $s_1(x, p) = (1, p)$  for  $x \geq 1$ .

$s_2: [-1, 1] \times D^{l-1} \rightarrow D^l$  be a combinatorial equivalence defined by radial shrinking.

$s_3: D^l \approx$  fiber over  $p \in P$  in a tubular neighborhood of  $P$  in  $Q$ .

Then  $f$  can be described as  $(x, y, z) \mapsto s_3 \circ s_2 \circ s_1(-2|x| + 2, z)$ , where  $|x|$  denotes the length of  $x \in D_2^{k+1}$ .

LEMMA 1.6. *There is a map  $r$  from the square  $0 \leq x, y \leq 2$  in the plane to  $R$  which is differentiable and transverse regular to  $0 \in R$  and such that  $r(2, y) = -2$  for all  $y$ ,  $r(x, 2) = -2x + 2$ , and such that  $r^{-1}(0)$  is a connected 1-dimensional submanifold of the square containing the segments  $\{0 \leq x \leq 1/4, y = 1/4\}$ ,  $\{x = 1, 1 \leq y \leq 2\}$  and such that*

$$r^{-1}(0) \cap (\{1 \leq y \leq 2\} \cup \{1 \leq x \leq 2\} \cup \{0 \leq y \leq 1/4\})$$

*is just these segments. Further, one can choose such a function  $r$  so that  $r(x, y) = -2$  for  $y \leq 1/10$ , and  $r(x, y) \geq 1.1$  for  $(x, y) \in \{x \leq 1/10, y \geq 1.8\}$ , and so that  $r$  is a function of  $y$  only on the set  $\{x \leq 1/10, y \leq 1.8\}$ .*

**Proof.** Define a function in polar coordinates in this region of the plane by  $g(r, \theta) = r^2 \cdot (\cos^2(\theta - \pi/4) - \sin^2(\theta - \pi/4)) + c(r^2)$  where  $c: [0, 4] \rightarrow R$  is a  $C^\infty$  decreasing map with  $c(0) = 1$ ,  $c(1/2) = 1/2$ , and  $c(\tau) = 0$  for  $\tau \geq 3/4$ . Let  $h_1(x, y)$  be this same function in ordinary orthogonal coordinates.

Let  $h_2(x, y) = -h_1(x - 1, y - 1/4)$ , and  $h_3(x, y) = p(x) \cdot h_2(x, y) + p(x) - 1$ , where  $p(x)$  is  $C^\infty$ , decreasing, and  $p(x) = 1$  for  $x \leq 1.1$ ,  $p(x) = 0$  for  $x \geq 1.25$ .

Let  $h_4(x, y) = h_3(x, y) + q(x)$  where  $q(x)$  is  $C^\infty$ , decreasing, and  $q(x) = 0$  for

$x \leq 1.5$ ,  $q(2) = -1$ , and then define  $r(x, y)$  to be  $r(x, y) = s(y) \cdot (h_4(x, y)) + (1 - s(y)) \cdot (-2x + 2)$ , where  $s$  is a  $C^\infty$  decreasing map,  $s(y) = 1$  for  $y \leq 1.5$ ,  $s(y) = 0$  for  $y \geq 1.75$ . This function  $r$  satisfies the conditions of the first sentence of the lemma, and clearly may be further modified to satisfy the conditions of the second sentence also.

Now define a homotopy

$$F: D_2^{k+1} \times D_2^l \times D^{l-1} \times [0, 1] \rightarrow R \times D^{l-1}$$

by

$$F(x, y, z, \tau) = ((1 - \tau) \cdot (-2|x| + 2) + \tau \cdot r(|x|, |y|), z).$$

Then  $s_3 \circ s_2 \circ s_1 \circ F$  defines a homotopy of  $f$  restricted to the tubular neighborhood of  $h(D_2^{k+1})$  in our coordinates, which is stationary on the boundary of this tubular neighborhood. Thus we may think of  $s_3 \circ s_2 \circ s_1 \circ F$  as defining a homotopy of  $f: M \rightarrow Q$  to a new map which we denote  $f_1: M \rightarrow Q$ , since

$$s_3 \circ s_2 \circ s_1 \circ F(x, y, z, 0) = s_3 \circ s_2 \circ s_1(-2|x| + 2, z) = f(x, y, z).$$

Note that  $s_3 \circ s_2 \circ s_1 \circ F(x, y, z, 1) = s_3 \circ s_2 \circ s_1(r(|x|, |y|), z)$  is differentiable in a neighborhood of the universe image of  $p \in P$  and is transverse regular over  $p \in P$ , with the inverse image of  $p$  diffeomorphic to

$$\begin{aligned} D^{k+1} \times S^{r-1} &= D^{k+1} \times S^{m-l-k-1} && \text{in Case (a)} \\ &= D^{k+1} \times S^{m-l-k} && \text{in Case (b).} \end{aligned}$$

Thus we end with a pair  $(M, f_1)$ , where  $f_1: M \rightarrow Q$  is differentiable and transverse regular over  $P \subset Q$  and  $f_1^{-1}(P)$  is just the manifold  $N = f^{-1}(P)$  with a surgery done on the class of  $i(S^k)$ . As noted before, the map  $f_1$  is tangential, so we have regained our original assumptions and can proceed inductively.

Now consider the separate cases:

Case (a): It is clear from the work of Novikov [7], [12] that one can proceed inductively through a series of surgeries as above to arrive at a pair  $(M, \bar{f})$ , where  $\bar{f}$  is differentiable and transverse to  $P \subset Q$  and  $\bar{f}|_{\bar{f}^{-1}(P)}: \bar{f}^{-1}(P) \rightarrow P$  induces isomorphisms of homology for all dimensions less than or equal to  $(m-l)/2 - 1$ , and  $\bar{f}^{-1}(P)$  simply connected. That is, surgeries as above can be used to kill the kernel of  $H_k(\bar{f}^{-1}(P)) \rightarrow H_k(P)$  for  $k \leq (m-l)/2 - 1$ . The cases  $m-l \equiv 0, 1, 3 \pmod 4$  are distinct.

If  $m-l \equiv 0 \pmod 4$ , then ordinary surgery may be done to the pair  $(\bar{f}^{-1}(P), \bar{f}|_{\bar{f}^{-1}(P)})$  until homology isomorphism results in the middle dimension also. (A proof of this may be found in [7, pp. 26–28], or [12, pp. 288–290]. One must choose the homology classes in  $\ker f_*$  to be killed with care.) It is further clear that these surgeries can be done ambiently as above, since the only obstruction to this is that we make a specific choice of framing of  $\nu(i(S^k) \subset N)$  in Lemma 1.5, and the proof that surgery can be used to kill the kernel of  $f_*$  in dimension  $(m-l)/2$  is independent of the specific framing used in performing the surgery. Hence, for  $m-l \equiv 0 \pmod 4$ , surgery may be completed in the middle dimension, and we end with a map

$f': M \rightarrow Q$  transverse to  $P \subset Q$  such that  $f'|f'^{-1}(P): f'^{-1}(P) \rightarrow P$  is a tangential homotopy equivalence as required in Theorem 1.

For  $m-l \equiv 1 \pmod 4$  the same argument applies. Ordinary surgery may be done to  $(\bar{f}^{-1}(P), \bar{f}|\bar{f}^{-1}(P))$  until homology isomorphism results in dimension  $(m-l-1)/2$ , as proven in [12, pp. 294–302]. Again one sees these surgeries can all be done ambiently in  $M$  as above, since the proof is independent of the choice of frame on  $\nu(i(S^k) \subset N)$ . Thus we again end with a map  $f': M \rightarrow Q$  as required by Theorem 1.

For  $m-l \equiv 3 \pmod 4$ , we must use the note following Lemma 1.5. Novikov proves that surgery can modify  $(\bar{f}^{-1}(P), \bar{f}|\bar{f}^{-1}(P))$  so that homology isomorphisms in dimension  $(m-l-1)/2$  result, but this proof requires use of specific framings on  $\nu(i(S^{(m-l-1)/2}) \subset N)$  in performing the surgeries. The freedom to vary a given framing by any stably trivial element of  $\pi_{(m-l-1)/2}(SO(m-l+1)/2)$  is, however, all that is needed. (See [12, pp. 298–302].) Since we may use these frames in doing ambient surgery, we again end with a map  $f': M \rightarrow Q$  as required in Theorem 1, and Theorem 1 is proven.

Case (b). Recall in this case we have  $f: M^{m+1} \rightarrow Q$  a tangential homotopy equivalence, where  $M^{m+1}$  is an  $h$ -cobordism between  $M_1^m$  and  $M_2^m$ , and we are doing surgery to  $N=f^{-1}(P)$ . Note first that since we are doing surgeries on classes represented by imbeddings  $i: S^k \rightarrow N$  where  $i(S^k) \cap \partial M = \emptyset$ , the homotopies of  $f$  we define may be assumed stationary on  $\partial M$ .

As in Novikov [12], [7, Proposition 4, p. 19] we get a sequence of modifications as above such that we arrive at a pair  $(M^{m+1}, f_1)$  where  $f_1$  is differentiable and transverse to  $P$ , and  $f_{1*}: H_*(f_1^{-1}(P)) \rightarrow H_*(P)$  is an isomorphism in dimensions less than or equal to  $(m-l)/2-1$ , and  $f_1^{-1}(P)$  simply connected.  $f_1^{-1}(P)$  has dimension  $m+1-l$ , where  $m-l \equiv 0 \pmod 4$ , so we may refer to [7, pp. 28–35], [12, p. 304] for a proof that ordinary surgery may be done to the pair  $(f_1^{-1}(P), f_1|f_1^{-1}(P))$  until we get a pair with homology isomorphism in dimension  $(m-l)/2$  as well, and thus get a homotopy equivalence.

Once again, the only obstruction to performing these surgeries ambiently as above is the choice of framing of  $\nu(i(S^{(m-l)/2}) \subset f_1^{-1}(P))$  made in Lemma 1.5. Since the proof that surgeries can be chosen to kill the kernel of  $f_{1*}: H_{(m-l)/2}(f_1^{-1}(P)) \rightarrow H_{(m-l)/2}(P)$  is independent of such choices, the surgeries may all be done in  $M^{m+1}$  as above.

Thus we can modify  $(M^{m+1}, f)$  to a pair  $(M^{m+1}, f')$  by a series of homotopies of  $f$  so that  $f': M \rightarrow Q$  is a tangential homotopy equivalence transverse to  $P \subset Q$ ,  $f|\partial M = f'|\partial M$ , and if  $N=f'^{-1}(P)$ , then  $f'|N: N \rightarrow P$  is a homotopy equivalence, so that  $N$  is an  $h$ -cobordism between  $N_1$  and  $N_2$ . Thus Theorem 2 is proven.

Note 1. The reader should notice that the following weaker analogue of the uniqueness theorem can be proven for the cases  $m-l \equiv 1$  or  $3 \pmod 4$ , using the analogue of Case (b) above:

Let  $P$  and  $Q$  be as in Theorem 1, with  $m-l \equiv 1$  or  $3 \pmod 4$ . Assume  $M^{m+1}$ ,  $f: M^{m+1} \rightarrow Q$ ,  $M_i$  and  $N_i$  for  $i=1, 2$  are all as in Theorem 2, with  $f|N_i: N_i \rightarrow P$  a

homotopy equivalence for  $i=1, 2$ . Then  $f$  is homotopic via a homotopy which is stationary on  $\partial M^{m+1}$  to a map  $f': M^{m+1} \rightarrow Q$  such that  $\pi_1(f'^{-1}(P))=0$  and  $f'_*: H_i(f'^{-1}(P)) \rightarrow H_i(P)$  is an isomorphism for  $i \leq (m-l+1)/2-1$ . As in [12, pp. 304-306],  $N_1$  is diffeomorphic to  $N_2 \# \Sigma^{m-l}$  for some homotopy sphere  $\Sigma^{m-l}$  which bounds a parallelizable manifold.

*Note 2.* Before proceeding we should note that the following stronger analogue of Theorem 1 holds for manifolds with boundary.

**THEOREM.** *Suppose  $Q^m$  and  $P^{m-l}$  and  $M^m$  are manifolds with boundary with  $P^{m-l}$  a submanifold of  $Q^m$  so that  $\partial P \subset \partial Q$ , where  $l \geq 3, m-l > 6, m-l \neq 14$ , and  $\pi_1(Q) = \pi_1(\partial Q) = \pi_1(P) = \pi_1(\partial P) = 0$ . Suppose  $f: (M^m, \partial M^m) \rightarrow (Q^m, \partial Q^m)$  is a degree +1 tangential homotopy equivalence. Then  $f$  is homotopic to a map  $f': (M^m, \partial M^m) \rightarrow (Q^m, \partial Q^m)$  which is differentiable and transverse regular over  $(P, \partial P) \subset (Q, \partial Q)$  and such that  $f'|_{f'^{-1}(P, \partial P)}: f'^{-1}(P, \partial P) \rightarrow (P, \partial P)$  is a homotopy equivalence.*

The proof of this is similar in spirit to the work above. One readily shows the analogues of Lemma 1.1 and Proposition 1.1. Ambient surgery is then carried out on  $f^{-1}(\partial P)$  and  $f^{-1}(P)$  as above up to the middle dimension. The obstructions to middle dimensional surgery can be made to vanish since we allow our homotopies to alter  $f$  on  $\partial M^m$ . The interested reader will find a discussion and proof of this, including the necessary algebra and the techniques needed to avoid middle dimension obstructions, in Wagoner [17] and Wall [18].

**2. Homotopy quaternionic projective spaces.** We refer the reader to the introduction for the definition of  $\theta(QP_n)$ , which is defined for  $n \geq 2$ .

**LEMMA 2.1.** *The equivalence relation defining  $\theta(QP_n)$  can be described as follows:  $(M, f) \sim (N, g)$  if and only if  $M$  and  $N$  are diffeomorphic by a map  $d: M \rightarrow N$  such that  $f$  is homotopic to  $g \circ d$ .*

**Proof.**  $(M, f) \sim (N, g)$  in  $\theta(QP_n)$  implies there is an  $h$ -cobordism  $C$  and a tangential homotopy equivalence  $h: C \rightarrow QP_n$ . Since dimension  $M, N$  is greater than or equal to 5, there is a diffeomorphism  $d: M \times I \rightarrow C$  with  $d(x, 0) = x$ . Hence  $d|(M \times (1))$  is a diffeomorphism  $M \rightarrow N$  such that  $h \circ d|(M \times (1))$  is homotopic to  $h \circ d|(M \times (0))$ . Since  $h$  restricts to  $f$  and  $g, f \circ d$  is homotopic to  $g \circ d$ , and so  $f$  is homotopic to  $g \circ d$ .

Conversely, if  $(N, g)$  represents a class in  $\theta(QP_n)$  then  $g \circ \text{projection}: N \times I \rightarrow QP_n$  is a tangential homotopy equivalence. Assume  $d: M \rightarrow N$  is a diffeomorphism such that  $g \circ d$  is homotopic to  $f: M \rightarrow QP_n$ . Identify  $N \times (1) \subset N \times I$  with  $M$  via  $d$  to make  $N \times I$  an  $h$ -cobordism of  $N$  and  $M$ . Then  $h = g \circ \text{projection}$  restricts to  $g$  on  $N \times (0)$  and to a map homotopic to  $f$  on  $M = N \times (1)$ , and may be modified in a tubular neighborhood of  $M$  in  $N \times I$  to restrict to  $f$ .

**LEMMA 2.2.** *Suppose  $f: M^{4n} \rightarrow QP_n$  is a tangential homotopy equivalence and  $\Sigma^{4n}$  represents an element of  $\theta_{4n}$ . Let  $i: D^{4n} \rightarrow M$  be an imbedding. Then for some*

diffeomorphism of boundaries  $d: \partial D^{4n} \rightarrow \partial(M - i(D^{\circ 4n}))$ , the connected sum  $M \# \Sigma^{4n}$  is diffeomorphic to  $(M - i(D^{\circ 4n})) \cup_d D^{4n}$ . Define  $\text{id}' : M \# \Sigma^{4n} \rightarrow M$  by  $\text{id}'(x) = x$  for  $x \in M - i(D^{\circ 4n})$ , and  $\text{id}'(s, \tau) = i(i^{-1}(d(s)), \tau)$  for  $(s, \tau) \in D^{4n}$ , where we view  $D^{4n}$  as  $\{(s, \tau) | s \in S^{4n-1}, \tau \in I\}$  modulo  $(s, 0) \sim (r, 0)$  for any  $s, r \in S^{4n-1}$ . Then  $\text{id}'$  is a tangential map, and thus  $f \circ \text{id}' : M \# \Sigma^{4n} \rightarrow QP_n$  is a tangential homotopy equivalence.

**Proof.** The existence of the required diffeomorphism  $d: \partial D^{4n} \rightarrow \partial(M - i(D^{\circ 4n}))$  is proven by Smale [13], so all that remains is to show  $\text{id}'$  is tangential.

$\text{id}'|(M - i(D^{\circ 4n}))$  is identity, so we may collapse  $(M - i(D^{\circ 4n}))$  to a point to get a diagram with stable bundles

$$\begin{array}{ccc} \tau(M \# \Sigma^{4n}) - \text{id}'^*(\tau(M)) & \xrightarrow{\approx} & \eta \\ \downarrow & & \downarrow \\ M \# \Sigma^{4n} & \longrightarrow & S^{4n} = D^{4n}/\partial D^{4n} \end{array}$$

where the diagram defines the stable bundle  $\eta$ . We need only show  $\eta$  is trivial to prove  $\text{id}'$  is tangential.

Since a stable bundle over  $S^{4n}$  is trivial if and only if its  $n$ th Pontrjagin class is zero, we investigate the  $n$ th Pontrjagin class of  $\tau(M \# \Sigma^{4n}) - \text{id}'^*(\tau(M))$  which is zero if and only if the  $n$ th Pontrjagin class of  $\eta$  is. In the Mayer-Vietoris cohomology sequence of the triad  $(M \# \Sigma^{4n}; M - i(D^{\circ 4n}), D^{4n})$  inclusion:  $M - i(D^{\circ 4n}) \rightarrow M \# \Sigma^{4n}$  induces isomorphism of cohomology for all dimensions less than  $4n - 2$ . Let  $p_1, \dots, p_n$  denote the Pontrjagin classes of  $M$ , and  $p'_1, \dots, p'_n$  the Pontrjagin classes of  $M \# \Sigma^{4n}$ . Since  $\text{id}'|(M - i(D^{\circ 4n}))$  is tangential,  $p'_1 = \text{id}'^*(p_1), \dots, p'_{n-1} = \text{id}'^*(p_{n-1})$ .  $\text{index}(M \# \Sigma^{4n}) = \text{index}(M)$ , so if  $L_n$  is the  $n$ th Hirzebruch polynomial, then

$$\begin{aligned} L_n(p'_1, \dots, p'_n)[M \# \Sigma^{4n}] &= L_n(p_1, \dots, p_n)[M] \\ &= L_n(\text{id}'^*(p_1), \dots, \text{id}'^*(p_n))[M \# \Sigma^{4n}] \\ &= L_n(p'_1, \dots, p'_{n-1}, \text{id}'^*(p_n))[M \# \Sigma^{4n}], \end{aligned}$$

where we orient  $M \# \Sigma^{4n}$  so that  $\text{id}'$  is a degree  $+1$  map.

If  $L_n(X_1, \dots, X_n)$  is the  $n$ th Hirzebruch polynomial, the coefficient of  $X_n$  is nonzero. Thus we must have  $p'_n = \text{id}'^*(p_n)$  above, and hence the  $n$ th Pontrjagin class of  $\tau(M \# \Sigma^{4n}) - \text{id}'^*(\tau(M))$  is zero and  $\eta$  is trivial as needed.

**DEFINITION.**  $\theta_{4n}$  acts on  $\theta(QP_n)$  in the following manner:  $[\Sigma^{4n}] \cdot [(M, f)] \equiv [(M \# \Sigma^{4n}, f \circ \text{id}')]$ . This action is well defined by Lemma 2.2 of Kervaire-Milnor [6].

**DEFINITION.** Define  $r: \theta(QP_n) \rightarrow \theta(QP_{n-1})$  for all  $n \geq 3$  as follows: Given  $\alpha \in \theta(QP_n)$ , we choose a representative element  $(M, f)$ .  $QP_n$  has dimension  $4n$ , and codimension of  $QP_{n-1}$  in  $QP_n$  is 4,  $QP_n$  is simply connected for all  $n$ , so by Theorem 1 and the fact that we are free within the class  $\alpha$  to deform  $f$  by arbitrary homotopies, we may assume  $(M, f)$  is chosen with  $f$  differentiable and transverse to

$QP_{n-1} \subset QP_n$ , and such that  $f|f^{-1}(QP_{n-1}): f^{-1}(QP_{n-1}) \rightarrow QP_{n-1}$  is a tangential homotopy equivalence. Define  $r(\alpha)$  to be the class of  $(f^{-1}(QP_{n-1}), f|f^{-1}(QP_{n-1}))$  in  $\theta(QP_{n-1})$ . To show that the image of  $\alpha$  depends only on  $\alpha$ , suppose  $(N, g) \in \alpha$  is such that  $g: N \rightarrow QP_n$  is transverse to  $QP_{n-1} \subset QP_n$ , and  $g|g^{-1}(QP_{n-1}): g^{-1}(QP_{n-1}) \rightarrow QP_{n-1}$  is a homotopy equivalence. Let  $(C, h)$  be a cobordism between  $(M, f)$  and  $(N, g)$  as in the definition of equivalence in  $\theta(QP_n)$ . By Theorem 2, we may assume  $h: C \rightarrow QP_n$  is transverse regular over  $QP_{n-1} \subset QP_n$ , and that  $h|h^{-1}(QP_{n-1}): h^{-1}(QP_{n-1}) \rightarrow QP_{n-1}$  is a tangential homotopy equivalence. Then  $(h^{-1}(QP_{n-1}), h|h^{-1}(QP_{n-1}))$  is a cobordism between  $(f^{-1}(QP_{n-1}), f|f^{-1}(QP_{n-1}))$  and  $(g^{-1}(QP_{n-1}), g|g^{-1}(QP_{n-1}))$ , and so these represent the same class in  $\theta(QP_{n-1})$ .

LEMMA 2.3. *Suppose  $(M, f_1)$  represents  $\alpha \in \theta(QP_n)$ , with  $n \geq 3$ . Then  $f_1$  is homotopic to  $f: M \rightarrow QP_n$  such that*

- (a)  *$f$  is differentiable and transverse regular over  $QP_{n-1} \subset QP_n$ , and*

$$(f^{-1}(QP_{n-1}), f|f^{-1}(QP_{n-1}))$$

*is a representative of  $r(\alpha)$ .*

- (b) *If we consider  $QP_n$  as the Thom space of  $\nu(QP_{n-1} \subset QP_n)$ , then  $f$  defines a bundle isomorphism  $\nu(f^{-1}(QP_{n-1}) \subset M) \approx \nu(QP_{n-1} \subset QP_n)$  on a tubular neighborhood  $\nu$  of  $f^{-1}(QP_{n-1})$ , and  $f$  maps the complement of  $\nu$  to the distinguished point in  $QP_n = T(\nu(QP_{n-1} \subset QP_n))$ .*

**Proof.** See Thom [16] and the definition of  $r(\alpha)$ .

LEMMA 2.4. *For any  $\alpha \in \theta(QP_n)$  with  $n \geq 3$ , if a representative pair  $(M, f)$  for  $\alpha$  is chosen as in Lemma 2.3 then the tubular neighborhood  $\nu$  of  $f^{-1}(QP_{n-1})$  in  $M$  is bounded by a sphere, and  $M$  is diffeomorphic to  $\nu \cup_i D^{4n}$  for some attaching diffeomorphism of boundaries  $i: \partial D^{4n} \rightarrow \partial\nu$ .*

**Proof.** There are disc bundle isomorphisms  $\nu \approx f^*(\nu(QP_{n-1} \subset QP_n)) \approx f^*(\mathcal{S}^4)$ , where  $\mathcal{S}^4$  is the 4-disc bundle over  $QP_{n-1}$  described in the introduction. Hence there is a 3-sphere bundle isomorphism  $\partial\nu \approx f^*(\mathcal{S}_0^3)$ , where  $\mathcal{S}_0^3$  has total space  $S^{4n-1}$ . Then naturality of the homotopy sequence of a bundle implies  $\partial\nu$  has the homotopy type of  $S^{4n-1}$ .

Let  $D = \partial\nu \cup$  (complement of  $\nu$  in  $M$ ), so that  $\nu \cup D = M$ ,  $\nu \cap D = \partial\nu$ . From the Mayer-Vietoris sequence of this decomposition of  $M$  we see inclusion induces homology isomorphisms  $H_i(\nu) \approx H_i(M)$  for  $i \leq 4n - 1$ , and so  $H_i(D) = 0$  for  $i \leq 4n - 2$ . By duality,  $H_{4n}(D) \approx H^0(D, \partial\nu) = 0$  and  $H_{4n-1}(D) \approx H^1(D, \partial\nu) = 0$ , so  $H_i(D) = 0$  for all  $i \leq 4n$ .

Applying Van Kampen's Theorem to  $M$  with subsets  $\nu^\circ =$ interior of  $\nu$ , and  $M - f^{-1}(QP_{n-1})$ , we see  $D$  is simply connected, since  $D$  has the homotopy type of  $M - f^{-1}(QP_{n-1})$ . Hence  $D$  is contractible, and  $\partial D = \partial\nu$  is diffeomorphic to  $S^{4n-1}$ , as needed. See [13].

DEFINITION. Define  $h: \theta(QP_{n-1}) \rightarrow \theta_{4n-1}$  for  $n \geq 3$  as follows. Given  $\beta \in \theta(QP_{n-1})$  choose a representative  $(P, t)$  in which  $t: P \rightarrow QP_{n-1}$  is differentiable. As in the

proof of Lemma 2.4,  $t^*(\mathcal{S}_0^3)$  has the homotopy type of  $S^{4n-1}$ . We define  $h(\beta)$  to be the class in  $\theta_{4n-1}$  represented by  $t^*(\mathcal{S}_0^3)$ , and check that  $h$  is now a well defined map.

If  $(P, t) \sim (R, s)$  in  $\theta(QP_{n-1})$ , where both  $s$  and  $t$  are differentiable maps, we may choose a cobordism between them  $(C', h')$  such that  $h': C' \rightarrow QP_{n-1}$  is differentiable. Then  $h'^*(\mathcal{S}_0^3)$  is a manifold of the homotopy type of  $S^{4n-1}$ , and is thus an  $h$ -cobordism between  $t^*(\mathcal{S}_0^3)$  and  $s^*(\mathcal{S}_0^3)$ .

We now proceed with the proof of Theorem 3. Given  $\alpha \in \theta(QP_n)$ , choose a representative  $(M, f)$  for  $\alpha$  as in Lemma 2.3 so that  $[(f^{-1}(QP_{n-1}), f|f^{-1}(QP_{n-1}))] = r(\alpha)$ , and  $f: M \rightarrow QP_n$  is transverse to  $QP_{n-1}$ . Then  $(f|f^{-1}(QP_{n-1}))^*(\mathcal{S}_0^3) = \partial(\nu(f^{-1}(QP_{n-1}) \subset M)) \approx S^{4n-1}$  by Lemma 2.4, and thus image of  $r$  is contained in the kernel of  $h$ .

Suppose  $\beta \in \theta(QP_{n-1})$  and  $h(\beta) = [S^{4n-1}] \in \theta_{4n-1}$ . Take  $(R, s) \in \beta$  with  $s: R \rightarrow QP_{n-1}$  differentiable, and identify the boundary of  $s^*(\mathcal{S}^4)$  with  $S^{4n-1}$ . In this fashion we attach  $D^{4n}$  to  $s^*(\mathcal{S}^4)$  differentiably. Let  $R_1 = s^*(\mathcal{S}^4) \cup D^{4n}$  be the result of such an attachment. Considering  $QP_n$  as  $\nu(QP_{n-1} \subset QP_n) \cup D^{4n}$ , or  $\mathcal{S}^4 \cup D^{4n}$ , by identifying a tubular neighborhood of  $QP_{n-1}$  with  $\mathcal{S}^4$ , we get a homotopy equivalence  $s_1: R_1 \rightarrow QP_n$  by radial extension of the bundle map  $s^*(\mathcal{S}^4) \rightarrow \mathcal{S}^4$ . By a proof entirely similar to that of Lemma 2.2, one can show that  $s_1$  is a tangential map, and thus that  $(R_1, s_1)$  represents an element  $\gamma$  of  $\theta(QP_n)$ . Clearly  $r(\gamma) = [(R, s)] = \beta \in \theta(QP_{n-1})$ , and  $\beta \in \text{image}(r)$ . Thus we have shown  $\text{image}(r) = \text{kernel}(h)$ .

Suppose now  $\alpha, \beta \in \theta(QP_n)$  are such that  $r(\alpha) = r(\beta)$ . Choose representative elements  $(M, f)$  and  $(N, g)$  for  $\alpha$  and  $\beta$  respectively as in Lemma 2.3. Then there is a cobordism between  $(f^{-1}(QP_{n-1}), f|f^{-1}(QP_{n-1}))$  and  $(g^{-1}(QP_{n-1}), g|g^{-1}(QP_{n-1}))$  since  $r(\alpha) = r(\beta)$ , and, since  $f$  and  $g$  are each differentiable in the respective inverse images of  $QP_{n-1}$ , we may assume this cobordism is  $(C, h)$  with  $h: C \rightarrow QP_{n-1}$  differentiable.

Associated with the resulting manifold  $h^*(\mathcal{S}^4)$ , a differentiable disc bundle, is the boundary 3-sphere bundle  $h^*(\mathcal{S}_0^3)$ . As before, naturality of the homotopy sequence of a bundle shows that  $h^*(\mathcal{S}_0^3)$  has the homotopy type of  $S^{4n-1}$ , and thus that  $h^*(\mathcal{S}_0^3)$  is an  $h$ -cobordism between

$$(f|f^{-1}(QP_{n-1}))^*(\mathcal{S}_0^3) \quad \text{and} \quad (g|g^{-1}(QP_{n-1}))^*(\mathcal{S}_0^3).$$

Since  $(f^{-1}(QP_{n-1}), f|f^{-1}(QP_{n-1})) \in r(\alpha)$ , and  $\text{image}(r) = \text{kernel}(h)$ ,  $h^*(\mathcal{S}_0^3)$  is thus diffeomorphic to  $S^{4n-1} \times I$ .

Attach  $D^{4n} \times I$  to  $h^*(\mathcal{S}^4)$  by a diffeomorphism  $S^{4n-1} \times I \approx \partial h^*(\mathcal{S}^4)$ , thus creating a manifold  $C'$ , and a homotopy equivalence  $h': C' \rightarrow QP_n$ , the radial extension of the natural bundle map. Then  $C'$  is an  $h$ -cobordism between its boundary components, which are respectively  $M \# \Sigma^{4n}$  and  $M \# \Sigma'^{4n}$ , where  $\Sigma^{4n}$  and  $\Sigma'^{4n}$  are homotopy  $4n$ -spheres. We now check that  $h': C' \rightarrow QP_n$  is tangential, and thus that  $(C', h')$  is a cobordism between  $[\Sigma^{4n}] \cdot \alpha$  and  $[\Sigma'^{4n}] \cdot \beta$  in  $\theta(QP_n)$ .

The proof that  $h'$  is tangential is as in Lemma 2.2. If  $i: h^*(\mathcal{S}^4) \rightarrow C'$  is inclusion, then  $i^*(\tau(C')) \approx i^*h'^*(\tau(QP_n))$  and hence the first  $n-1$  Pontrjagin classes of  $\tau(C')$

and  $h^*(\tau(QP_n))$  are the same. Since  $\tau(C')$  and  $h^*(\tau(QP_n))$  are stably isomorphic over  $h^*(\mathcal{S}^4) \subset C'$ , we may collapse this subspace to a line, and by analogy with Lemma 2.2 define a bundle  $\eta$  over  $S^{4n} \times I$ . One can then show  $\eta|S^{4n} \times (0)$  is stably trivial by computing its  $n$ th Pontrjagin class, and thus prove that  $h': C' \rightarrow QP_n$  is tangential. Then  $[\Sigma^{4n}] \cdot \alpha$  and  $[\Sigma'^{4n}] \cdot \beta$  represent the same class in  $\theta(QP_n)$ , and  $\alpha$  and  $\beta$  differ by an action of  $\theta_{4n}$  in  $\theta(QP_n)$ .

Conversely, choosing representatives as in Lemma 2.3, it is clear that if  $\alpha$  and  $\beta$  differ by some action of  $\theta_{4n}$ , then  $r(\alpha) = r(\beta)$ . This completes the proof of Theorem 3.

*Construction.* Theorem 3 gives us an inductive procedure whereby we can construct representatives of all elements of  $\theta(QP_n)$  from elements of  $\theta(QP_{n-1})$  for  $n \geq 3$ .

Given any representative  $(M, f)$  of  $\alpha \in \theta(QP_n)$ , we may deform  $f$  by a homotopy and assume it is in the form prescribed by Lemma 2.3. Then  $r(\alpha)$  is represented by  $(f^{-1}(QP_{n-1}), f|f^{-1}(QP_{n-1}))$ , and  $(f|f^{-1}(QP_{n-1}))^*(\mathcal{S}_0^3)$  is diffeomorphic to  $S^{4n-1}$ . Attach  $D^{4n}$  to  $(f|f^{-1}(QP_{n-1}))^*(\mathcal{S}^4)$  by a diffeomorphism of boundaries, thus creating a new manifold  $M'$  and a homotopy equivalence  $f': M' \rightarrow QP_n$ , which we have seen in the proof of Theorem 3 is tangential. Then  $(M', f')$  represents an element of  $\theta(QP_n)$ , and  $r([(M', f')]) = r(\alpha)$ , and so by Theorem 3  $[(M, f)]$  and  $[(M', f')]$  differ by some action of  $\theta_{4n}$  in  $\theta(QP_n)$ . Thus we see all tangential homotopy  $QP_n$ 's arise from tangential  $QP_{n-1}$ 's by the operation of pulling back the bundle  $\mathcal{S}^4$ , and capping the result with a disc  $D^{4n}$ .

3.  $\theta(QP_2)$ . The existence of the inductive procedure for constructing tangential homotopy quaternionic projective spaces given in the last section leads one to investigate the starting point,  $\theta(QP_2)$ .

Suppose  $(M, f)$  represents an element of  $\theta(QP_2)$ . Let  $i: S^4 \rightarrow M^8$  be an imbedding representing a generator of  $\pi_4(M^8) \approx \pi_4(QP_2) \approx Z$ .

LEMMA 3.1. *Let  $\nu$  denote a tubular neighborhood of  $i(S^4)$  in  $M$ . Then  $D = M - \text{interior}(\nu)$  is diffeomorphic to the 8-disc,  $D^8$ . Consequently, the 3-sphere bundle which bounds  $\nu(i(S^4) \subset M)$  has total space diffeomorphic to  $S^7$ .*

**Proof.** Since  $M$  is simply connected and  $D$  is a deformation retract of  $M - i(S^4)$ ,  $D$  is also simply connected. For  $q < 8$ ,  $H^q(M) \approx H^q(i(S^4))$ , so  $H_j(D) \approx H^{8-j}(M, i(S^4)) = 0$  for  $j \neq 0$ . Hence  $D$  is contractible, and  $D \approx D^8$  by the  $h$ -cobordism theorem.

We now investigate  $\nu(i(S^4) \subset M) = \nu$ . Since  $f: M \rightarrow QP_2$  is tangential,  $\nu \oplus \tau(S^4) \approx i^*(\tau(M)) \approx i^*f^*(\nu(QP_1 \subset QP_2) \oplus \tau(S^4))$ , where the isomorphisms indicated are stable isomorphisms, and thus  $\nu$  is stably isomorphic to  $i^*f^*(\nu(QP_1 \subset QP_2))$ .

Represent 3-sphere bundles over  $S^4$  by elements of  $\pi_3(SO(4))$  as in Steenrod [14, §§18–22], using the isomorphism described by Milnor [9],  $\pi_3(SO(4)) \approx Z \oplus Z$  under the correspondence  $(h, j) \leftrightarrow f_{hj}$ , where  $f_{hj}: S^3 \rightarrow SO(4)$  is defined by  $f_{hj}(u) \cdot v = u^h \cdot v \cdot u^j$  for  $v \in R^4$ . (Quaternion multiplication is indicated on the right.) It is known that under this isomorphism  $\nu(QP_1 \subset QP_2)$  corresponds to  $(1, 0) \in Z \oplus Z$ . See [9].

LEMMA 3.2. *If  $\mathcal{S}_{hj}$  is the 3-sphere bundle over  $S^4$  corresponding to  $(h, j)$  in this representation, then the first Pontrjagin class of  $\mathcal{S}_{hj}$  is  $p_1(\mathcal{S}_{hj}) = \pm 2(h-j)i$ , where  $i \in H^4(S^4)$  is a generator.*

**Proof.** See Milnor [9].

LEMMA 3.3.  $\nu = \nu(i(S^4) \subset M)$  is represented by an element of the form

$$(n+1, n) \in Z \oplus Z.$$

**Proof.**  $p_1(\tau(S^4)) = 0$ , so by Lemma 3.2  $\tau(S^4)$  is represented by an element of  $Z \oplus Z$  of the form  $(s, s)$ . Kernel  $\pi_3(SO(4) \rightarrow \pi_3(SO)) = Z$ , and since the element corresponding to  $\tau(S^4)$  is in this kernel, the kernel is contained in the diagonal of  $Z \oplus Z$ . Thus  $\nu$  is represented by  $(1, 0) + (n, n) = (n+1, n)$  for some  $n \in Z$  since  $\nu(QP_1 \subset QP_2)$  is represented by  $(1, 0)$  and is stably isomorphic to  $\nu$ .

LEMMA 3.4.  $\nu = \nu(i(S^4) \subset M) \approx \nu(QP_1 \subset QP_2)$ .

**Proof.** We will show that among those 3-sphere bundles over  $S^4$  with representatives of the form  $(n+1, n)$  in  $\pi_3(SO(4)) \approx Z \oplus Z$ , only the one corresponding to  $(1, 0)$  has a total space of the homotopy type of  $S^7$ .

Let  $T$  be the total space of some 4-disc bundle over  $S^4$  with associated 3-sphere bundle denoted by  $\partial$ .  $\pi_1(\partial) = \pi_2(\partial) = 0$ , then, and the first possibly nonvanishing homology group of  $\partial$  is  $H_3(\partial)$ . Consider the cohomology exact sequence

$$\begin{array}{ccccccc} H^3(\partial) & \rightarrow & H^4(T, \partial) & \xrightarrow{j^*} & H^4(T) & \rightarrow & H^4(\partial) \rightarrow H^5(T, \partial) \\ & & \uparrow \phi & \nearrow & & & \uparrow \phi \\ & & H^0(S^4) & & & & H^1(S^4) = 0 \end{array}$$

where the maps denoted by  $\phi$  are the Thom isomorphisms. The image of a generator of  $H^0(S^4)$  in  $H^4(T)$  under  $j^* \circ \phi$  is a generator if and only if  $H^4(\partial) = 0$ , and it is known that the Euler class of the 4-disc bundle of which  $T$  is the total space is the image of a generator  $H^0(S^4)$  under  $j^* \circ \phi$ .

The process of taking Euler classes of corresponding bundles gives a homomorphism  $\pi_3(SO(4)) \rightarrow H^4(S^4)$  (see [8]), and the Euler class of  $\tau(S^4)$  is twice a generator of  $H^4(S^4)$ , so in the diagonal set  $\{(n, n) \mid n \in Z\}$  of  $Z \times Z$ , all bundles represented have nonzero Euler classes except for  $(0, 0)$ . The Euler class of the bundle corresponding to  $(1, 0)$  is a generator of  $H^4(S^4)$  since this bundle is just  $\partial\nu(QP_1 \subset QP_2)$  and has total space  $S^7$ . Thus  $(1, 0)$  is the only element of  $Z \oplus Z$  of the form  $(n+1, n)$  whose corresponding bundle has Euler class a generator of  $H^4(S^4)$ , and  $(1, 0)$  is the only element of the form  $(n+1, n)$  representing a 3-sphere bundle whose total space has the homotopy type of  $S^7$ . Since the total space of  $\nu$  is  $S^7$ , we see  $\nu \approx \nu(QP_1 \subset QP_2)$ .

So, any manifold  $M^8$  of the tangential homotopy type of  $QP_2$  is of the form  $M = T \cup D^8$ , where  $T$  is the total space of the differentiable disc bundle  $\nu(QP_1 \subset QP_2)$ , and  $D^8$  is attached to  $T$  by a diffeomorphism of boundaries. Thus we have proven:

**PROPOSITION 3.1.** *Any 8 manifold without boundary of the tangential homotopy type of  $QP_2$  is diffeomorphic to a connected sum  $QP_2 \# \Sigma^8$ , where  $\Sigma^8$  is a homotopy 8-sphere. Thus, there are at most two distinct such manifolds, and these are combinatorially equivalent [11].*

To finish the computation of  $\theta(QP_2)$ , we investigate the homotopy classes of maps  $f: M \rightarrow QP_2$  which contain tangential homotopy equivalences. If  $f: M \rightarrow QP_2$  and  $g: M \rightarrow QP_2$  are two such equivalences, and if  $f$  and  $g$  are not homotopic, then  $f \circ g^{-1}: QP_2 \rightarrow QP_2$  is a tangential homotopy equivalence which is not homotopic to identity.

The remainder of this section is devoted to a proof that there is only one homotopy class of maps from  $QP_2$  to  $QP_2$  which contains a tangential homotopy equivalence. This will finish the proof of Theorem 4.

Let  $f: QP_2 \rightarrow QP_2$  be a tangential homotopy equivalence. It is known [3] that  $p_1(QP_2)$  = 1st Pontrjagin class of  $QP_2$  is twice a generator in  $H^4(QP_2) \approx \mathbb{Z}$ , and thus tangentiality of  $f$  implies  $f_*: \pi_4(QP_2) \rightarrow \pi_4(QP_2)$  is identity. So, in the standard C-W complex structure of  $QP_2$  we may assume  $f$  is the identity map on  $S^4 = 4$ -skeleton of  $QP_2$ , and that  $h_t: S^4 \rightarrow S^4$ , the identity map for all  $t \in I$ , is a homotopy between  $f$  and identity on the 7-skeleton of  $QP_2$ , which is the same as the 4-skeleton.

The partial homotopy of  $f$  with identity given by  $h_t$  gives rise to an obstruction cohomology class  $\delta^8(f, \text{identity}, h_t) \in H^8(QP_2, \pi_8(QP_2)) \approx \mathbb{Z}_2$ , since the homotopy sequence of the bundle  $S^3 \rightarrow S^{11} \rightarrow QP_2$  shows  $\pi_8(QP_2) \approx \pi_7(S^3) \approx \mathbb{Z}_2$ . If this obstruction class is zero then the partial homotopy  $h_t$  may be extended over  $QP_2$  to give a homotopy of  $f$  with identity. Moreover, if  $g$  is another such map, and  $\delta^8(f, \text{id}, h_t) \neq 0$ ,  $\delta^8(g, \text{id}, h_t) \neq 0$ , then  $f$  and  $g$  are homotopic by the additivity of obstruction cochains. Thus, we may finish the proof of the theorem by constructing a map  $t: QP_2 \rightarrow QP_2$  which is identity on the 4-skeleton of  $QP_2$  and which is homotopic to identity:  $QP_2 \rightarrow QP_2$ , but whose obstruction class  $\delta^8(t, \text{id}, h_t)$  is not zero. (The homotopy of  $t$  to identity cannot then be stationary on the 4-skeleton of  $QP_2$ .)

We view  $QP_2$  as the Thom space of the 4-disc bundle  $\mathcal{S}^4$  over  $S^4 \approx QP_1$ , and represent  $\mathcal{S}^4$  as follows.  $\mathcal{S}^4$  can be thought of as  $\{(s, v) \mid s \text{ a quaternionic line in } Q^2, v \text{ a point in } s\}$ . Define a map from  $\{(a, b, q) \mid a, b, q \in Q; |a|^2 + |b|^2 = 1, |q| \leq 1\}$  to  $\mathcal{S}^4$  taking  $(a, b, q)$  to (the line containing  $(a, b), q \cdot (a, b)$ ). If  $(a, b, q)$  and  $(a', b', q')$  have the same image under this map then  $(a, b) = s \cdot (a', b')$  for some  $s \in Q, |s| = 1$ , and  $q \cdot s \cdot (a', b') = q' \cdot (a', b')$  so that  $q = q' \cdot s^{-1}$ . Thus we can represent  $\mathcal{S}^4$  over  $S^4$  as  $\{(a, b, q) \mid a, b, q \in Q, |a|^2 + |b|^2 = 1, |q| \leq 1\}$  modulo the equivalence relation  $(a, b, q) \sim (a', b', q')$  if and only if  $a = s \cdot a', b = s \cdot b'$  and  $q = q' \cdot s^{-1}$  for some  $s \in Q$ .  $\mathcal{S}_0^3$  is given in this description by the set of equivalence classes which have a

representative with third coordinate of length one. We view  $S^4$  as  $Q \cup \{\infty\}$ , and identify  $QP_1 = \{(a, b, q) \in \mathcal{S}^4 \mid q=0\}$  with  $S^4$  under the correspondence  $[(a, b, 0)] \Leftrightarrow a^{-1}b$ , noting that if  $(a, b, 0) = (q' \cdot c, q' \cdot d, 0)$  then  $a^{-1}b = c^{-1}q'^{-1}q'd = c^{-1}d$ .

Let  $\eta: S^4 \rightarrow S^3$  be the essential map in  $\pi_4(S^3) \approx Z_2$ , where we think of  $S^3$  as the unit length quaternions. Then, with the above identifications we define  $t: QP_2 \rightarrow QP_2$  by  $t([(a, b, q)]) = [(a, b, \eta(a^{-1}b) \cdot q)]$ . One checks easily that this map  $\mathcal{S}^4 \rightarrow \mathcal{S}^4$  is well defined on equivalence classes, and maps  $\mathcal{S}_0^3$  to  $\mathcal{S}_0^3$  and so does in fact give a map  $QP_2 \rightarrow QP_2$ . We wish to show that the obstruction to this map being homotopic to identity:  $QP_2 \rightarrow QP_2$  via a homotopy which is stationary on the 4-skeleton of  $QP_2$  is the nontrivial element in  $\pi_8(QP_2)$ .

Identify  $D^8$  with  $\{(a, b, c) \mid a, b \in Q, c \in R, |a|^2 + |b|^2 = 1, 0 \leq c \leq 1\}$  modulo the relation  $(a, b, 0) \sim (c, d, 0)$  for any  $a, b, c, d$ . Then define  $p: D^8 \rightarrow \mathcal{S}^4 / \mathcal{S}_0^3 = QP_2$  by  $p([(a, b, c)]) = [(a, b, 1 - c^2)]$ . Note that on the boundary of  $D^8$  the map  $p$  is just the Hopf map  $S^7 \rightarrow S^4$ .

The obstruction class we wish to compute in  $\pi_8(QP_2)$  is readily seen to be represented by the following map: View  $S^8$  as two copies of  $D^8$  with boundaries identified. On one hemisphere we define our map to be  $p: D^8 \rightarrow QP_2$  as described above. On the other hemisphere we define our map to be  $t \circ p: D^8 \rightarrow QP_2$  as above. Since  $t$  is identity on  $S^4 = QP_1 \subset QP_2$  and  $p$  maps  $\partial D^8$  to this  $S^4$ , this gives a continuous well-defined map  $S^8 \rightarrow QP_2$  which is the Hopf map on the "equator" of  $S^8$ .

To show this map represents the nonzero class in  $\pi_8(QP_2) \approx Z_2$ , we will show the pull back of  $\mathcal{S}^4$  by this map is a nontrivial bundle over  $S^8$ . Analogous to the above description of  $\mathcal{S}^4$  over  $S^4$ , one may describe  $\mathcal{S}^4$  over  $QP_2$  as

$$\{(a, b, c, q) \mid a, b, c, q \in Q, |a|^2 + |b|^2 + |c|^2 = 1, |q| \leq 1\}$$

modulo the equivalence relation  $(a, b, c, q) \sim (a', b', c', q')$  if and only if for some  $s \in Q, a = s \cdot a', b = s \cdot b', c = s \cdot c',$  and  $q = q' \cdot s^{-1}$ . The trivial 4-disc bundle over  $D^8$  can be described as  $\{(a, b, c, q) \mid a, b, q \in Q, |a|^2 + |b|^2 = 1, |q| \leq 1; c \in R, 0 \leq c \leq 1\}$  modulo the relation  $(a, b, 0, q) \sim (c, d, 0, q)$  for any  $a, b, c, d \in Q$ . We may now describe a bundle map from this bundle to  $\mathcal{S}^4$  over  $QP_2$  by  $p'([(a, b, c, q)]) = [(c \cdot a, c \cdot b, 1 - c^2, q)]$ , which is easily seen to be well defined on equivalence classes. This bundle map lies over the base space map  $p: D^8 \rightarrow QP_2$ , as may be seen by comparing the two descriptions of  $QP_2$  which are used: the Thom space of the bundle  $\mathcal{S}^4$  over  $QP_1$  and the zero section of  $\mathcal{S}^4$  over  $QP_2$ .

A second map from the trivial 4-disc bundle over  $D^8$  to  $\mathcal{S}^4$  over  $QP_2$  is given by

$$\begin{aligned} (t \circ p)'([(a, b, c, q)]) &= [(c \cdot a, c \cdot b, \eta(a^{-1}b)(1 - c^2), q \cdot \eta(a^{-1}b)^{-1})] \\ &= [(\eta(a^{-1}b)^{-1} \cdot c \cdot a, \eta(a^{-1}b)^{-1} \cdot c \cdot b, (1 - c^2), q)]. \end{aligned}$$

Note that this map can be thought of as the composition of  $p'$  and the bundle map over  $t: QP_2 \rightarrow QP_2$  which takes "q" over  $[(a, b, c)]$  to "q" over  $t([(a, b, c)])$ , and is thus a continuous bundle map over  $t \circ p$ .

Looking at our representation of  $\mathcal{S}^4$  over  $QP_2$  restricted to  $S^4 \subset QP_2$ , we see that the inverse image of the point  $[(a, b, 0, q)]$  under  $p'$  is the set

$$\{(q'a, q'b, 1, qq'^{-1}) \mid q' \in Q, |q'| = 1\},$$

and the inverse image of this same point under the map  $(t \circ p)'$  is the set

$$\{(q'a, q'b, 1, qq'^{-1}\eta(a^{-1}b)) \mid q' \in Q, |q'| = 1\}.$$

Thus, if we take two copies of the trivial 4-disc bundle over  $D^8$  and identify the point represented by  $[(a, b, 1, q)]$  in one with the point represented by

$$[(a, b, 1, q \cdot \eta(a^{-1}b))]$$

in the other, we get a bundle  $\rho$  over  $S^8 = D^8 \cup D^8$ , and the maps  $p'$  and  $(t \circ p)'$  give a bundle map from this bundle to  $\mathcal{S}^4$  over  $QP_2$  which lies over  $p$  on one hemisphere, and over  $t \circ p$  on the other. That is, there is a bundle map from  $\rho$  to  $\mathcal{S}^4$  which lies over the element of  $\pi_8(QP_2)$  we wish to compute.

The bundle  $\rho$  has coordinate transition map  $S^7 \rightarrow SO(4)$  given by a composition  $S^7 \rightarrow S^4 \rightarrow S^3 \rightarrow SO(4)$  which takes  $(a, b)$  to  $f_{(a,b)}$  where  $f_{(a,b)}(q) = q \cdot \eta(a^{-1}b)$  with quaternion multiplication indicated on the right. This map in  $\pi_7(SO(4))$  is non-trivial (Steenrod [14, §22]), and thus  $\rho$  is a nontrivial bundle, and the map  $S^8 \rightarrow QP_2$  is not nullhomotopic. Thus we have proven that the obstruction to finding a homotopy of  $t: QP_2 \rightarrow QP_2$  to identity which is stationary on  $S^4 \subset QP_2$  is nonzero.

The proof of Theorem 4 will now be completed by showing that  $t: QP_2 \rightarrow QP_2$  is in fact homotopic to identity, and thus that the class of identity is the only homotopy class of maps  $QP_2 \rightarrow QP_2$  which contains a tangential homotopy equivalence. Recall that  $t$  comes from a bundle map of  $\mathcal{S}^4$  over  $S^4 = QP_1$  to itself which lies over identity:  $S^4 \rightarrow S^4$ . The principal bundle associated with  $\mathcal{S}^4$  is  $\mathcal{S}_0^3$ , with fiber and group  $S^3$  = the group of unit length quaternions. Since  $\mathcal{S}_0^3$  over  $S^4$  has total space  $S^7$ ,  $\mathcal{S}^4$  is a universal bundle for disc bundles with group  $S^3$  over complexes of dimension less than or equal to six [14]. Thus, there is a homotopy of  $t$  to the identity,  $T: \mathcal{S}^4 \times I \rightarrow \mathcal{S}^4$ , a bundle map which is  $t$  on  $\mathcal{S}^4 \times (0)$  and identity on  $\mathcal{S}^4 \times (1)$ . Since  $T$  maps  $\mathcal{S}_0^3 \times I$  to  $\mathcal{S}_0^3$ , this induces a map  $QP_2 \times I \rightarrow QP_2$  which is the required homotopy of  $t$  to identity.

**4. Combinatorial equivalence.** In this section the results of the preceding two sections are applied to investigate  $\theta(QP_n)$  for all  $n \geq 2$ .

**LEMMA 4.1.** *Suppose  $(M, f)$  and  $(N, g)$  represent the same element in  $\theta(QP_n)$  for some  $n \geq 2$ , and that  $g: N \rightarrow QP_n$  is homotopic to a map  $h: N \rightarrow QP_n$  which is induced by a combinatorial equivalence. Then  $f: M \rightarrow QP_n$  is also homotopic to a map induced by a combinatorial equivalence.*

**Proof.** Suppose  $\tau_1: K_1 \rightarrow N$ ,  $\tau_2: K_2 \rightarrow QP_n$ , and  $c: K_1 \rightarrow K_2$  give a combinatorial equivalence which induces  $h$ . Recall from Lemma 2.1 that there is a diffeomorphism

$d: N \rightarrow M$  such that  $g$  is homotopic to  $f \circ d$ . Then  $d \circ \tau_1: K_1 \rightarrow M$  is a  $C^\infty$  triangulation of  $M$ . Let  $h' = h \circ d^{-1}: M \rightarrow QP_n$ . Then  $h'$  is induced by the combinatorial equivalence given by  $d \circ \tau_1, \tau_2$ , and  $c$ . Further,  $f \circ d$  homotopic to  $g$  implies  $f$  homotopic to  $h'$ .

**LEMMA 4.2.** *Let  $\tau_1: K_1 \rightarrow M$  and  $\tau_2: K_2 \rightarrow M$  be  $C^\infty$  triangulations of the manifold  $M$ . Then for any Riemannian metric on  $M$  and any  $\varepsilon > 0$ , there is a combinatorial equivalence  $c: K_1 \rightarrow K_2$  such that the induced map  $h = \tau_2 \circ c \circ \tau_1^{-1}$  is an  $\varepsilon$ -approximation to identity.*

**Proof.** Given  $\varepsilon > 0$ , choose  $\varepsilon/2$  approximations  $\tau'_1: K'_1 \rightarrow M$  and  $\tau'_2: K'_2 \rightarrow M$  to  $\tau_1$  and  $\tau_2$  respectively where  $K'_i$  is a subdivision of  $K_i$  for  $i = 1, 2$  as in [10, p. 101] such that for some combinatorial equivalence  $c': K'_1 \rightarrow K'_2, \tau'_2 \circ c' \circ \tau'^{-1}_1: M \rightarrow M$  is identity. Since  $K'_1$  and  $K'_2$  are subdivisions of  $K_1$  and  $K_2, c'$  induces a combinatorial equivalence  $c: K_1 \rightarrow K_2$ . Then  $h = \tau_2 \circ c \circ \tau_1^{-1}$  is an  $\varepsilon$ -approximation to identity, and by definition is induced by  $c$ .

**PROPOSITION 4.1.** *If  $(M, f)$  is any representative of an element of  $\theta(QP_2)$ , then  $f: M \rightarrow QP_2$  is homotopic to a map induced by a combinatorial equivalence.*

**Proof.** By Lemma 4.1, we need only prove the proposition for one representative in each equivalence class in  $\theta(QP_2)$ . The proposition is obvious for  $(QP_2, \text{identity})$ , so, by Theorem 4, we need only work with  $(QP_2 \# \Sigma^8, \text{id}')$  where  $\text{id}'$  is defined as in Lemma 2.2 by a map  $\phi: (QP_2 - i(D^{\circ 8})) \cup_a D^8 \rightarrow (QP_2 - i(D^{\circ 8})) \cup_i D^8$  such that  $\phi(x) = x$  for  $x \in QP_2 - i(D^{\circ 8})$ , and  $\phi(s, \tau) = (d^{-1}(s), \tau)$  for  $(s, \tau) \in D^8$ , for  $d: S^7 \rightarrow S^7$  a representative of the nonzero element of  $\Gamma_8$ .

We now refer to [11, p. 546] where for each  $n$  a  $C^\infty$ -triangulation  $g_n: K^n \rightarrow D^n$  is constructed having the property that if  $h: S^{n-1} \rightarrow S^{n-1}$  is any map which induces a combinatorial equivalence of  $\partial|K^n|$  with itself then the radial extension of  $h$  to a map  $h_r: D^n \rightarrow D^n$  induces a combinatorial equivalence of  $K^n$  with itself.

Proceed as in [11]. Consider the triangulations of  $\partial(QP_2 - i(D^{\circ 8}))$  given by restriction to the boundary of  $i \circ g_8: K^8 \rightarrow QP_2 - i(D^{\circ 8})$ , and of  $d \circ i \circ g_8: K^8 \rightarrow QP_2 - i(D^{\circ 8})$ . Each of these  $C^\infty$ -triangulations has an extension over all of  $QP_2 - i(D^{\circ 8})$  as in [10, p. 101], and, by Lemma 4.2, we may choose a combinatorial equivalence  $c$  of these triangulations such that the map  $h$  induced by this equivalence is homotopic to identity via a homotopy which maps  $\partial(QP_2 - i(D^{\circ 8})) \times I$  to  $\partial(QP_2 - i(D^{\circ 8}))$ . Restricted to  $\partial(QP_2 - i(D^{\circ 8}))$ ,  $h$  gives a combinatorial equivalence  $c|\partial|K_1|$ . Thus the radial extension of  $h$  to a map  $h': (QP_2 - i(D^{\circ 8})) \cup_a D^8 \rightarrow (QP_2 - i(D^{\circ 8})) \cup_i D^8$  gives a combinatorial equivalence, and extending the homotopy of  $h$  to identity radially gives a homotopy of  $h'$  to  $\text{id}'$  as needed.

**PROPOSITION 4.2.** *Suppose  $f: M \rightarrow QP_n$  for  $n \geq 2$  is differentiable so that there is a natural differentiable structure on the pullback  $f^*(\mathcal{S}^4)$ , and that  $p: f^*(\mathcal{S}^4) \rightarrow \mathcal{S}^4$  is a map induced by a combinatorial equivalence. Let  $i: D^{4n+4} \rightarrow QP_{n+1}$  be an imbedding such that  $i(D^{4n+4}) \cap (QP_n \subset QP_{n+1}) = \emptyset$ , and identify  $QP_{n+1}$  with*

$\mathcal{S}^4 \cup_i D^{4n+4}$ . If  $d: S^{4n+3} \rightarrow \partial f^*(\mathcal{S}^4)$  is a diffeomorphism, and we form the manifold  $f^*(\mathcal{S}^4) \cup_d D^{4n+4}$  and extend  $p$  to  $p_d: f^*(\mathcal{S}^4) \cup_d D^{4n+4} \rightarrow \mathcal{S}^4 \cup_i D^{4n+4}$  by  $p_d(x) = p(x)$  for  $x \in f^*(\mathcal{S}^4)$ ,  $p_d(s, \tau) = (i^{-1}(p(d(s))), \tau)$  for  $(s, \tau) \in D^{4n+4}$ , then  $p_d$  is homotopic to a map  $p'_d: f^*(\mathcal{S}^4) \cup_d D^{4n+4} \rightarrow \mathcal{S}^4 \cup_i D^{4n+4}$  which is induced by a combinatorial equivalence.

**Proof.** Again we will use Munkres' triangulation  $g_{4n+4}: K^{4n+4} \rightarrow D^{4n+4}$  and proceed as in [11].  $d \circ g_{4n+4}$  and  $p \circ d \circ g_{4n+4}$  are respectively  $C^\infty$ -triangulations of  $\partial f^*(\mathcal{S}^4)$  and  $\partial \mathcal{S}^4$ . These extend to triangulations  $\tau_3: K_3 \rightarrow f^*(\mathcal{S}^4)$  and  $\tau_4: K_4 \rightarrow \mathcal{S}^4$  respectively, as in [10, p. 101]. By Lemma 4.2, there are combinatorial equivalences  $c_1$  and  $c_2$  such that in the following diagram  $i_1$  and  $i_2$  are approximations to the identity maps close enough so that  $i_1$  and  $i_2$  are each homotopic to identity via homotopies which always carry  $\partial f^*(\mathcal{S}^4)$  to  $\partial f^*(\mathcal{S}^4)$  and  $\partial \mathcal{S}^4$  to  $\partial \mathcal{S}^4$ . In the diagram,  $c$  is a combinatorial equivalence, and  $p = \tau_2 \circ c \circ \tau_1^{-1}$ , where  $\tau_1$  and  $\tau_2$  are  $C^\infty$ -triangulations.

$$\begin{array}{ccccccc}
 f^*(\mathcal{S}^4) & \xrightarrow{i_1} & f^*(\mathcal{S}^4) & \xrightarrow{p} & \mathcal{S}^4 & \xrightarrow{i_2} & \mathcal{S}^4 \\
 \tau_3 \downarrow & & \tau_1 \downarrow & & \tau_2 \downarrow & & \tau_4 \downarrow \\
 K_3 & \xrightarrow{c_1} & K_1 & \xrightarrow{c} & K_2 & \xrightarrow{c_2} & K_4
 \end{array}$$

Restricted to  $\partial f^*(\mathcal{S}^4)$ , the map  $i_2 \circ p \circ i_1$  induces the equivalence  $c_2 \circ c \circ c_1$  on  $\partial K^{4n+4}$ , and thus the radial extension of  $i_2 \circ p \circ i_1$  induces a combinatorial equivalence  $K^{4n+4} \rightarrow K^{4n+4}$ . So, triangulating  $D^{4n+4}$  by  $g_{4n+4}$ , we get

$$\begin{array}{ccc}
 f^*(\mathcal{S}^4) \cup_d D^{4n+4} & \xrightarrow{p'_d} & \mathcal{S}^4 \cup_i D^{4n+4} \equiv QP_{n+1} \\
 \tau'_3 \downarrow & & \tau'_4 \downarrow \\
 K'_3 & \xrightarrow{c'} & K'_4
 \end{array}$$

where  $c'$  is a combinatorial equivalence and  $\tau'_3, \tau'_4$  are  $C^\infty$ -triangulations and  $p'_d$  is the radial extension of  $i_2 \circ p \circ i_1$ . But  $p'_d$  is homotopic to the radial extension of  $p$  via radial extensions of the homotopies of  $i_1$  and  $i_2$  to identity maps, and the proposition is proven.

**LEMMA 4.3.** Suppose  $f: \sigma^p \rightarrow R^n$  is a map of a  $p$ -simplex into  $R^n$  such that for some subdivision of  $\partial\sigma$ ,  $f|_{\partial\sigma}$  is differentiable on each simplex of  $\partial\sigma$ . Then  $f$  is homotopic to a map  $f': \sigma^p \rightarrow R^n$  such that  $f'|_{\partial\sigma} = f|_{\partial\sigma}$ , and for some subdivision of  $\sigma$ ,  $f'$  is differentiable on each simplex in  $\sigma$ .

**Proof.** Assume  $\sigma^p \subset R^p$  such that  $b_\sigma =$  barycenter of  $\sigma$  is at the origin of  $R^p$ . Let  $l: R^p \rightarrow R^p$  be defined by  $l(x) = x/2$ , and let  $N = \sigma - l(\sigma)$ . Subdivide  $\sigma$  and its faces so that  $f|_{\partial\sigma}$  is differentiable on each simplex of  $\partial\sigma$  and triangulate  $l(\partial\sigma)$  by the composition of this triangulation of  $\partial\sigma$  and  $l$ . Sets of the form  $\{x \in N \mid \tau \cdot x \in \gamma$

for some  $\tau, 1 \leq \tau \leq 2$  as  $\gamma$  ranges over the simplices of  $\partial\sigma$ , the simplices of  $\partial\sigma$ , and the images of the simplices of  $\partial\sigma$  under  $l$  give a cellular decomposition of  $N$  [10, p. 70].

As in [10], this cell complex may be divided into a simplicial complex without subdividing any of the cells of  $\partial N = \partial\sigma \cup l(\partial\sigma)$ , since these cells are already simplices. Define  $f': \partial N \rightarrow R^n$  by  $f'|_{\partial\sigma} = f|_{\partial\sigma}$ , and  $f'|_{l(\partial\sigma)}$  is that map which is linear on each simplex of  $l(\partial\sigma)$ , and such that  $f$  and  $f'$  agree on the vertices of  $l(\partial\sigma)$ . Given  $x \in N$ , let  $\tau(x)$  be the real number between 1 and 2 such that  $\tau(x) \cdot x \in \partial\sigma$ . Then define  $f'$  on the interior of  $N$  by

$$f'(x) = (2 - \tau(x)) \cdot f(\tau(x) \cdot x) + (\tau(x) - 1) \cdot f'(\tau(x) \cdot x/2).$$

Then  $f'$  is differentiable on each simplex of the triangulation of  $N$ , and  $f'$  is linear on each simplex in  $\partial(l(\sigma))$ . We triangulate  $l(\sigma)$  by taking the join of  $b_\sigma$  with the simplices in  $\partial(l(\sigma))$ , and extend  $f'$  linearly over the simplices in  $l(\sigma)$ , letting  $f'(b_\sigma) = f(b_\sigma)$ , to get the required map  $f': \sigma \rightarrow R^n$ .

**LEMMA 4.4.** *Let  $\sigma^p$  be a simplex in Euclidean space. Then any  $C^\infty$ -triangulation of  $\partial(\sigma) \times D^k$  has a subdivision which can be extended to a  $C^\infty$ -triangulation of  $\sigma \times D^k$ .*

**Proof.** The proof is in two parts; first extend a subdivision to a triangulation of  $\partial(\sigma \times D^k)$ , and then extend a subdivision of this to the needed triangulation of  $\sigma \times D^k$ .

Suppose  $\tau: K \rightarrow \partial(\sigma) \times D^k$  is a  $C^\infty$ -triangulation.  $\partial(\sigma \times D^k) = \partial(\sigma) \times D^k \cup \sigma \times \partial D^k$ , where the intersection  $\partial\sigma \times \partial D^k$  already has a  $C^\infty$ -triangulation then. Subdivide  $K$  so that in this triangulation of  $\partial\sigma \times \partial D^k$  every simplex  $\gamma$  of the subdivided complex which is mapped by  $\tau$  into  $\partial\sigma \times \partial D^k$  is mapped into a face of  $\sigma$  by (projection  $\circ \tau$ ):  $K \rightarrow \partial\sigma$ .

Let  $r: \sigma^p \rightarrow D^p$  be a map as in [11, p. 546] which is a  $C^\infty$ -triangulation of  $D^p$  if we take the first barycentric subdivision of  $\sigma$ , and define a map  $(\partial\sigma) \times (\partial D^k) \rightarrow S^{p-1} \times S^{k-1}$  by  $(x, v) \rightarrow (r(x), v)$ . This induces a  $C^\infty$ -triangulation of  $S^{p-1} \times S^{k-1}$  which can be extended [10] to a  $C^\infty$ -triangulation of  $D^p \times S^{k-1}$ . We further subdivide this extension so that it pulls back to a  $C^\infty$ -triangulation of  $\partial(\sigma \times D^k)$  which extends a subdivision of  $K$ . Denote this triangulation of  $\partial(\sigma \times D^k)$  by

$$\tau': K' \rightarrow \partial(\sigma \times D^k).$$

Let  $b_\sigma$  denote the barycenter of  $\sigma$ , and 0 the center of  $D^k$ . For some integer  $s \geq 2$ , the distance from  $b_\sigma$  to  $\partial\sigma$  is greater than  $2/s$ . For  $(x, y) \in \sigma \times D^k - (b_\sigma, 0)$ , let  $p(x, y)$  denote the unique point of  $\partial(\sigma \times D^k)$  which lies on the ray from  $(b_\sigma, 0)$  through  $(x, y)$ , and  $d(x, y)$  the distance from  $(x, y)$  to  $p(x, y)$ . Let

$$N = \{(x, y) \in \sigma \times D^k \mid d(x, y) \leq 1/s\},$$

and define  $p_1: N \rightarrow \partial(\sigma \times D^k) \times [0, 1/s]$  by  $p_1(x, y) = (p(x, y), d(x, y))$ . Then  $p_1$  is a homeomorphism, and

- (1) for  $\gamma$  any face of  $\sigma$ ,  $p_1|_{p_1^{-1}(\gamma \times D^k)}$  is a  $C^\infty$ -map of rank equal to  $\dim \gamma + k$ ,
- (2)  $p_1|_{p_1^{-1}(\sigma \times \partial D^k)}$  is a  $C^\infty$ -map of rank  $p + k - 1$ .

Proceed as in [10, p. 102]. Subdivide  $K'$  to a complex  $K^2$  such that the image of each simplex of  $K^2$  lies in a subset of  $\partial(\sigma \times D^k)$  of the form  $\gamma \times D^k$  for  $\gamma$  a face of  $\sigma$ , or in  $\sigma \times \partial D^k$ . Then triangulate  $\partial(\sigma \times D^k) \times [0, 1/s]$  by first triangulating  $\partial(\sigma \times D^k) \times I$  as in [10, p. 102] and then mapping  $\partial(\sigma \times D^k) \times I \rightarrow \partial(\sigma \times D^k) \times [0, 1/s]$  by  $(x, \tau) \rightarrow (x, \tau/s)$ . Then  $p_1^{-1}$  composed with this triangulation of  $\partial(\sigma \times D^k) \times [0, 1/s]$  gives a  $C^\infty$ -triangulation of  $N$  which extends  $\tau': K^2 \rightarrow \partial(\sigma \times D^k)$ . As in [10], we fit this triangulation of  $N$  together with a  $C^\infty$ -triangulation of interior  $(\sigma \times D^k)$  to get the needed triangulation of  $\sigma \times D^k$ .

**PROPOSITION 4.3.** *Assume  $f: M \rightarrow QP_n$  is homotopic to a map  $h$  induced by a combinatorial equivalence. Then  $f$  is homotopic to a differentiable map  $f_1: M \rightarrow QP_n$  such that the natural map  $\tilde{f}_1: f_1^*(\mathcal{S}^4) \rightarrow \mathcal{S}^4$  is homotopic to a map induced by a combinatorial equivalence, and the homotopy may be taken so that it always maps  $\partial(f_1^*(\mathcal{S}^4))$  to  $\partial(\mathcal{S}^4)$ .*

**Proof.** We may choose trivialization of  $\mathcal{S}^4$  over a finite number, say  $m$ , of coordinate neighborhoods in  $QP_n$  such that

(1) the trivializations are given by maps  $\tau_i: D^{\circ 4n} \times D^4 \rightarrow \mathcal{S}^4$  for  $1 \leq i \leq m$ , where  $D^{\circ 4n} = \{x \in D^{4n} \mid |x| < 1\}$ , and the induced base space maps  $\tau_i|_{D^{\circ 4n} \times (0)}$  are diffeomorphisms onto their images which we denote by  $0_i \subset QP_n$ .

(2)  $QP_n \subset \bigcup_{i=1}^m \tau_i(D_{1-\varepsilon}^{4n} \times (0))$  for some  $\varepsilon > 0$  where  $D_{1-\varepsilon}^{4n} = \{x \in D^{4n} \mid |x| \leq 1 - \varepsilon\}$ .

(3) the coordinate bundle transition maps in  $\mathcal{S}^4$  induced by the  $\tau_i, g_{ij}: 0_i \cap 0_j \rightarrow SO(4)$  are  $C^\infty$ -maps for all  $i, j, 1 \leq i, j \leq m$ .

Assume  $\tau_1: K_1 \rightarrow M$  and  $\tau_2: K_2 \rightarrow QP_n$  are  $C^\infty$ -triangulations, and  $c: K_1 \rightarrow K_2$  is a combinatorial equivalence such that  $h = \tau_2 \circ c \circ \tau_1^{-1}$ . Since  $f$  is homotopic to  $h$ ,  $f$  is homotopic to a differentiable map  $f_1$  which is an  $\varepsilon/5$  approximation to  $h$  in some Riemannian metric on  $QP_n$  for which the trivialization maps  $\tau_i$  all induce distance increasing maps on base spaces. Assume  $K_1$  and  $K_2$  are subdivided so that  $c: K_1 \rightarrow K_2$  is a simplicial isomorphism, and thus  $h \circ \tau_1: K_1 \rightarrow QP_n$  is a  $C^\infty$ -triangulation.

For any  $i, 1 \leq i \leq m, h^{-1}(0_i) \cap f_1^{-1}(0_i)$  contains  $h^{-1}(\tau_i(D_{1-\varepsilon/2}^{4n} \times (0)))$ , and hence sets of the form  $h^{-1}(0_i) \cap f_1^{-1}(0_i)$  form an open cover of  $M$ . We assume  $K_1$  has been subdivided so that for each simplex  $\sigma$  of  $K_1$  there is an  $i$  such that

$$\tau_1(\sigma) \subset h^{-1}(0_i) \cap f_1^{-1}(0_i).$$

We pull the trivialization we have selected of  $\mathcal{S}^4$  over  $0_i$  back to a trivialization of  $h^*(\mathcal{S}^4)$  over  $h^{-1}(0_i)$  and of  $f_1^*(\mathcal{S}^4)$  over  $f_1^{-1}(0_i)$  for each  $i$ . Given any  $4n$ -simplex  $\sigma$  of  $K_1$ , trivialize  $h^*(\mathcal{S}^4)|_{\tau_1(\sigma)}$  and  $f_1^*(\mathcal{S}^4)|_{\tau_1(\sigma)}$  by fixing some choice of  $i$  such that  $\tau_1(\sigma) \subset f_1^{-1}(0_i) \cap h^{-1}(0_i)$  and using the corresponding trivialization over  $\tau_1(\sigma)$ . This induces a trivialization of  $h^*(\mathcal{S}^4)|_{\tau_1(\gamma)}$  and of  $f_1^*(\mathcal{S}^4)|_{\tau_1(\gamma)}$  for any face  $\gamma$  of  $\sigma$ , and in this fashion fixes trivializations of  $h^*(\mathcal{S}^4)$  and  $f_1^*(\mathcal{S}^4)$  over each of the simplices of the triangulation of  $M$  given by  $\tau_1: K_1 \rightarrow M$ .

Let  $H: M \times I \rightarrow QP_n$  be a homotopy with  $H|M \times (0) = h, H|M \times (1) = f_1$ , and

identify  $H^*(\mathcal{S}^4)|_{M \times (0)}$  with  $h^*(\mathcal{S}^4)$  and  $H^*(\mathcal{S}^4)|_{M \times (1)}$  with  $f_1^*(\mathcal{S}^4)$ . Then a bundle isomorphism  $H^*(\mathcal{S}^4) \approx h^*(\mathcal{S}^4) \times I$  gives us a bundle isomorphism

$$b_1: h^*(\mathcal{S}^4) \rightarrow f_1^*(\mathcal{S}^4)$$

over identity:  $M \rightarrow M$  such that if  $\bar{h}: h^*(\mathcal{S}^4) \rightarrow \mathcal{S}^4$  and  $\bar{f}_1: f_1^*(\mathcal{S}^4) \rightarrow \mathcal{S}^4$  are the natural maps then  $\bar{h}b_1^{-1}$  is homotopic as a bundle map to  $\bar{f}_1$ .

Given an arbitrary simplex  $\sigma$  of  $K_1$ , it is possible that  $\sigma$  is a face of several  $4n$ -simplices, and thus that  $h^*(\mathcal{S}^4)$  and  $f_1^*(\mathcal{S}^4)$  are each trivialized above in several ways over  $\tau_1(\sigma)$ . If specific trivializations of  $h^*(\mathcal{S}^4)$  and  $f_1^*(\mathcal{S}^4)$  are fixed over  $\tau_1(\sigma)$ ,  $b_1$  induces a map  $\sigma \times D^4 \rightarrow \sigma \times D^4$  which is represented by a map  $\sigma \rightarrow SO(4)$ . If trivializations pulled back from the trivialization  $\tau_i$  of  $\mathcal{S}^4$  over  $0_i$  are used, we will denote this induced map by  $b_\sigma^i: \sigma \rightarrow SO(4)$ . If trivializations pulled back from  $\tau_j$  over  $0_j$  are used, we get  $b_\sigma^j$  where  $b_\sigma^j = (g_{ij} \circ f_1 \circ \tau_1) \cdot b_\sigma^i \cdot (g_{ji} \circ h \circ \tau_1)$  where the dots indicate composition of elements in  $SO(4)$ . Thus if  $b_\sigma^i$  is a  $C^\infty$ -map, then  $b_\sigma^j$  is also  $C^\infty$ .

Subdivide  $K_1$  so that for all simplices  $\gamma \in K_1$  and all  $i$ ,  $b_\gamma^i(\gamma)$  is contained in a coordinate neighborhood  $U_\gamma^i$  in the differentiable structure of  $SO(4)$ . For each  $\gamma \in K_1$ , and all  $i$ , choose a diffeomorphism  $d_\gamma^i: U_\gamma^i \rightarrow R^6$ . We smooth the map  $b_1$  as follows:

Suppose  $p \geq 1$  and  $b_1$  is such that all of the induced maps  $b_\sigma^i$  are  $C^\infty$  for all simplices  $\sigma$  of  $K_1$  of dimension less than or equal to  $p-1$ . Let  $\gamma$  be a  $p$ -simplex of  $K_1$ . Choose some  $i$ ,  $1 \leq i \leq m$  such that  $\gamma$  is a  $p$ -face of a  $4n$ -simplex  $\sigma$  where  $\tau_1(\sigma) \subset h^{-1}(0_i) \cap f_1^{-1}(0_i)$ , and thus fix a choice of one of our trivializations of  $h^*(\mathcal{S}^4)$  and  $f_1^*(\mathcal{S}^4)$  over  $\tau_1(\gamma)$ . For any face  $\alpha$  of  $\gamma$ ,  $b_\gamma^i|_\alpha: \alpha \rightarrow SO(4)$  is  $C^\infty$  by assumption, and so  $d_\gamma^i \circ b_\gamma^i: \alpha \rightarrow R^6$  is  $C^\infty$  for all faces  $\alpha$  of  $\gamma$ . By Lemma 4.3, we may choose a map  $c_\gamma^i: \gamma \rightarrow R^6$  homotopic to  $d_\gamma^i \circ b_\gamma^i$  such that  $c_\gamma^i|_{\partial\gamma} = d_\gamma^i \circ b_\gamma^i|_{\partial\gamma}$  and  $c_\gamma^i$  is a  $C^\infty$ -map on each simplex of some subdivision of  $\gamma$ . Then  $d_\gamma^{i-1} \circ c_\gamma^i: \gamma \rightarrow SO(4)$  is equal to  $b_\gamma^i$  on  $\partial\gamma$  and differentiable on each simplex of some subdivision of  $\gamma$ . Thus  $b_1$  can be deformed by a bundle homotopy which is stationary over the image under  $\tau_1$  of the  $p$ -skeleton of  $K_1$  minus  $\gamma$  so that the induced map of the resulting bundle map over  $\tau_1(\gamma)$  in the trivializations of  $h^*(\mathcal{S}^4)$  and  $f_1^*(\mathcal{S}^4)$  over  $\tau_1(\gamma)$  corresponding to  $i$  is  $d_\gamma^{i-1} \circ c_\gamma^i$ . Do this for all  $p$ -simplices  $\gamma$  of  $K_1$  to get a bundle map which we again denote by  $b_1$  such that the induced maps  $b_\gamma^i$  are all  $C^\infty$ -maps on some subdivision of  $\gamma$  for all  $\gamma$  of dimension less than or equal to  $p$ . Since all maps  $b_\sigma^i$  are in fact  $C^\infty$  for  $\sigma$  of dimension 0, we see by induction that we may subdivide  $K_1$  to a complex  $K$  and deform  $b_1$  by a series of bundle homotopies to a map  $b$  such that all induced maps  $b_\sigma^i: \sigma \rightarrow SO(4)$  are  $C^\infty$  for all  $\sigma \in K$ ,  $1 \leq i \leq m$ .

Since  $K$  is a subdivision of  $K_1$ , each  $4n$ -simplex of  $K$  is contained in a unique  $4n$ -simplex of  $K_1$ , and so our choices of trivializations of  $h^*(\mathcal{S}^4)$  and  $f_1^*(\mathcal{S}^4)$  over the simplices  $\tau_1(\sigma)$  for  $\sigma \in K$ , induce choices of trivializations over the simplices  $\tau_1(\gamma)$  for  $\gamma \in K$ . Let  $\gamma$  be a simplex of  $K$ . We will need the following facts:

- (1) If  $\gamma$  is a face of two different  $4n$ -simplices  $\sigma_1$  and  $\sigma_2$  of  $K$ , the bundle

$h^*(\mathcal{S}^4)|_{\tau_1(\gamma)}$  is trivialized above in two possibly distinct ways. The map induced by this  $\gamma \times D^4 \rightarrow \gamma \times D^4$  is a diffeomorphism.

**Proof.** This map is a diffeomorphism if and only if the map induced  $\gamma \rightarrow SO(4)$  is a  $C^\infty$ -map. If coordinates over  $\tau_1(\sigma_1)$  come from the trivializing map  $\tau_1$  and coordinates over  $\tau_1(\sigma_2)$  come from  $\tau_j$ , the map  $\gamma \rightarrow SO(4)$  is  $g_{ij} \circ h \circ \tau_1: \gamma \rightarrow SO(4)$ . Since  $h \circ \tau_1: K \rightarrow QP_n$  is a  $C^\infty$ -triangulation, we may extend  $g_{ij} \circ h \circ \tau_1$  to a  $C^\infty$ -map on a neighborhood  $N(\gamma)$  of  $\gamma$  in Euclidean space, and the map  $\gamma \times D^4 \rightarrow \gamma \times D^4$  is a diffeomorphism.

(2) Using one of our choices of trivialization of  $h^*(\mathcal{S}^4)|_{\tau_1(\gamma)}$  we map  $\gamma \times D^4$  to  $h^*(\mathcal{S}^4)|_{\tau_1(\gamma)}$ . Then the composition

$$\gamma \times D^4 \longrightarrow h^*(\mathcal{S}^4)|_{\tau_1(\gamma)} \xrightarrow{\bar{h}} \mathcal{S}^4$$

is a diffeomorphism onto its image where  $\bar{h}$  is the natural map.

**Proof.** Since  $h \circ \tau_1|_\gamma: \gamma \rightarrow QP_n$  is a  $C^\infty$ -map of rank equal to the dimension of  $\gamma$ , and we have chosen  $\tau_i$  so that the map  $0_i \times D^4 \rightarrow \mathcal{S}^4$  is a diffeomorphism onto its image, this is clear.

(3) Suppose for some simplex  $\sigma$  and one of our choices of trivialization of  $h^*(\mathcal{S}^4)|_{\tau_1(\gamma)}$ , we have a map  $\sigma \rightarrow h^*(\mathcal{S}^4)|_{\tau_1(\gamma)}$  such that the composition  $\sigma \rightarrow \gamma \times D^4$  is a  $C^\infty$ -map of rank equal to the dimension of  $\sigma$ . Then, by (1) above, for any of the other choices of trivialization of  $h^*(\mathcal{S}^4)|_{\tau_1(\gamma)}$  fixed above, the composition  $\sigma \rightarrow \gamma \times D^4$  is a  $C^\infty$ -map of rank equal to  $\dim \sigma$ .

We will call a triangulation of  $h^*(\mathcal{S}^4)$  differentiable if the image of each simplex  $\sigma$  of the triangulation is contained in  $h^*(\mathcal{S}^4)|_{\tau_1(\gamma)}$  for some  $\gamma \in K$ , and the triangulation is such that the induced map  $\sigma \rightarrow \gamma \times D^4$  for at least one, and thus by (3) all, of our choices of trivialization of  $h^*(\mathcal{S}^4)|_{\tau_1(\gamma)}$  is a  $C^\infty$ -map of rank equal to  $\dim \sigma$ . We now construct such a triangulation of  $h^*(\mathcal{S}^4)$  inductively.

Assume  $h^*(\mathcal{S}^4)|_{\tau_1(K^p)}$ , where  $K^p$  = the  $p$ -skeleton of  $K$ , is differentially triangulated in this sense by  $\tau'_p: K'^p \rightarrow h^*(\mathcal{S}^4)|_{\tau_1(K^p)}$ . (Note then  $\dim K'^p = p + 4$ .) Let  $\sigma^{p+1}$  be any  $(p + 1)$ -simplex of  $K$ , and choose one of the trivializations we have fixed for  $h^*(\mathcal{S}^4)|_{\tau_1(\sigma)}$ . Then  $h^*(\mathcal{S}^4)|_{\tau_1(\partial\sigma)}$  is already triangulated by the restriction of  $\tau'_p$  to a subcomplex  $K'_p$  of  $K'^p$ . Using the trivialization map  $h^*(\mathcal{S}^4)|_{\tau_1(\sigma)} \approx \sigma \times D^4$ , we get a  $C^\infty$ -triangulation  $K'_p \rightarrow (\partial\sigma) \times D^4$ . By Lemma 4.4 for  $k = 4$ , there is a differentiable triangulation of  $\sigma \times D^4$  which extends a subdivision of  $\tau'_p: K'_p \rightarrow (\partial\sigma) \times D^4$ . Thus there is a complex  $\bar{K}_p$  which contains a subdivision of  $K'_p$  and a map  $\bar{\tau}_p: \bar{K}_p \rightarrow h^*(\mathcal{S}^4)|_{\tau_1(\sigma)}$  which extends  $\tau'_p$  and which is a differentiable triangulation of  $h^*(\mathcal{S}^4)|_{\tau_1(\sigma)}$  in the above defined sense. We adjoin  $\bar{K}_p$  to a subdivision of  $K'^p$  by identifying subdivisions of  $K'_p$  so that the resulting complex gives a differentiable triangulation of  $h^*(\mathcal{S}^4)|_{\tau_1(K^p \cup \sigma)}$ .

Proceeding in this fashion for each  $(p + 1)$ -simplex  $\sigma$  of  $K$ , we obtain a complex  $K'^{p+1}$  which contains a subdivision of  $K'^p$ , and a map

$$\tau'_{p+1}: K'^{p+1} \rightarrow h^*(\mathcal{S}^4)|_{\tau_1(K^{p+1})}$$

which extends  $\tau'_p$  and which is a differentiable triangulation of  $h^*(\mathcal{S}^4)|_{\tau_1(K^{p+1})}$ . Since  $h^*(\mathcal{S}^4)|_{\tau_1(K^0)}$  may easily be triangulated differentiably, we see by induction that there is a differentiable triangulation  $\tau': K' \rightarrow h^*(\mathcal{S}^4)$ .

For any simplex  $\sigma'$  of  $K'$ , then, there is some  $\sigma \in K$  such that  $\tau'(\sigma') \subset h^*(\mathcal{S}^4)|_{\tau_1(\sigma)}$ , and for any of the trivializations of  $h^*(\mathcal{S}^4)|_{\tau_1(\sigma)}$ , we have fixed, the induced map  $\sigma' \rightarrow \sigma \times D^4$  is a  $C^\infty$ -map of rank equal to  $\dim \sigma'$ . By property (2) above of the trivialization maps,  $\bar{h} \circ \tau'|_{\sigma'}$  is a  $C^\infty$ -map of rank equal to  $\dim \sigma'$  also, and hence  $\bar{h} \circ \tau': K' \rightarrow \mathcal{S}^4$  is a  $C^\infty$ -triangulation of  $\mathcal{S}^4$ .

Again, for  $\sigma'$  a simplex of  $K'$ , we choose  $\sigma \in K$  such that  $\tau'(\sigma') \subset h^*(\mathcal{S}^4)|_{\tau_1(\sigma)}$ . If trivializations in  $h^*(\mathcal{S}^4)$  and  $f_1^*(\mathcal{S}^4)$  induced from  $\tau_1$  in  $\mathcal{S}^4$  over  $0_i$  are used, then the induced maps  $\sigma' \rightarrow h^*(\mathcal{S}^4) \rightarrow \sigma \times D^4$  and thus

$$\sigma' \rightarrow \sigma \times D^4 \rightarrow \sigma \times D^4 \rightarrow f_1^*(\mathcal{S}^4)$$

are  $C^\infty$  and of rank equal to  $\dim \sigma'$ , where the map  $\sigma \times D^4 \rightarrow \sigma \times D^4$  is given by  $(x, v) \rightarrow (x, b'_\sigma(x) \cdot v)$  and  $\sigma \times D^4 \rightarrow f_1^*(\mathcal{S}^4)$  is the trivialization map. But this is just  $b \circ \tau'|_{\sigma'}$ , so  $b \circ \tau': K' \rightarrow f_1^*(\mathcal{S}^4)$  is a  $C^\infty$ -triangulation.

Thus  $\bar{h} \circ \tau': K' \rightarrow \mathcal{S}^4$  and  $b \circ \tau': K' \rightarrow f_1^*(\mathcal{S}^4)$  are  $C^\infty$ -triangulations, and  $\bar{h}b^{-1}: f_1^*(\mathcal{S}^4) \rightarrow \mathcal{S}^4$  is a map induced by a combinatorial equivalence, identity:  $K' \rightarrow K'$ .  $\bar{h}b^{-1}$  is homotopic to  $\bar{h}b_1^{-1}$  which is homotopic to  $\bar{f}_1$ , so the proposition is proven.

The proof of Theorem 5 can now be given: Proposition 4.1 is just Theorem 5 for the case  $n=2$ . We proceed by induction on  $n$ .

Assume Theorem 5 is known true for all  $m$  satisfying  $2 \leq m \leq r-1$ , and that  $(M, f)$  represents an element of  $\theta(QP_r)$ . By the results of §2,  $(M, f) \sim (N, g)$  in  $\theta(QP_r)$  where  $g: N \rightarrow QP_r$  is differentiable and transverse regular over  $QP_{r-1} \subset QP_r$  and  $[(g^{-1}(QP_{r-1}), g | g^{-1}(QP_{r-1}))] = r[(M, f)]$ . Then

$$(S, \tau) \equiv (g^{-1}(QP_{r-1}), g | g^{-1}(QP_{r-1}))$$

satisfies the theorem, and by Proposition 4.3,  $\tau$  is homotopic to a differentiable map  $\tau_1: S \rightarrow QP_{r-1}$  such that  $\bar{\tau}_1: \tau_1^*(\mathcal{S}^4) \rightarrow \mathcal{S}^4$  is homotopic as a bundle map to a map induced by a combinatorial equivalence.  $[(S, \tau)] = [(S, \tau_1)] \in \text{image}(r)$  implies  $\partial(\tau_1^*(\mathcal{S}^4)) = \tau_1^*(\mathcal{S}^3)$  is diffeomorphic to  $S^{4r-1}$ , and if we choose a diffeomorphism  $d$  and attach  $D^{4r}$  to  $\tau_1^*(\mathcal{S}^4)$  using  $d$  then  $\bar{\tau}_1$  extends radially to a tangential homotopy equivalence  $\bar{\tau}: \tau_1^*(\mathcal{S}^4) \cup_d D^{4r} \rightarrow QP_r$ . By Proposition 4.2,  $\bar{\tau}$  is homotopic to a map induced by a combinatorial equivalence.

Since  $r[(\tau_1^*(\mathcal{S}^4) \cup_d D^{4r}, \bar{\tau})] = r[(N, g)]$ , there is some choice of attaching diffeomorphism  $d$  such that  $[(N, g)] = [(\tau_1^*(\mathcal{S}^4) \cup_d D^{4r}, \bar{\tau})]$  in  $\theta(QP_r)$ . Since  $(M, f) \sim (N, g)$ , Lemma 4.1 shows  $f: M \rightarrow QP_r$  is homotopic to a map induced by a combinatorial equivalence. Hence Theorem 5 is proven for all integers  $m$  satisfying  $2 \leq m \leq r$ , and by induction the theorem is proven.

*Note.* It seems likely that the more sophisticated techniques of Sullivan [15] could be applied to prove Theorem 5.

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