TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 145, November 1969

ON EXTENDING SEMIGROUPS OF CONTRACTIONS

BY
JAMES A. DEDDENS[1]

In this paper we prove the continuous analog of a theorem by Sz.-Nagy and Foiaş [7] on extending contractive operators. We also give a new proof of Cooper's Theorem [1], [5], [6]. We pattern our proof after that of Theorem 2.1 by Lax and Phillips [4], which is a special case of our Main Theorem (also p. 39 of [7]).

We will consider only complex Hilbert spaces and all operators will be linear. A subspace will be a closed linear manifold and if $\mathcal{M}$ is a subspace of a Hilbert space $\mathcal{H}$, then $P_{\mathcal{M}}$ will be the (orthogonal) projection of $\mathcal{H}$ onto $\mathcal{M}$. If $\mathcal{N}$ is a linear manifold in $\mathcal{H}$, $(\mathcal{N})^-$ will denote its closure. We say that $\{T(t), t\geqq 0\}$ is a strongly continuous semigroup of contractive operators on $\mathcal{H}$ if for all $t, s\geqq 0$, $T(t+s)=T(t)T(s)$, $\|T(t)\|\leqq 1$, $T(0)=I$ and for all $x\in\mathcal{H}$, $T(t)x\to T(s)x$ as $t\to s$. The semigroup $\{T(t), t\geqq 0\}$ on $\mathcal{H}$ is *unitarily equivalent* to $\{R(t), t\geqq 0\}$ on $\mathcal{K}$ if there exists a unitary U on $\mathcal{H}$ to $\mathcal{K}$, such that $UT(t)=R(t)U$ for all $t\geqq 0$, we denote this by $T(t)\simeq R(t)$.

A subspace $\mathcal{M}$ of $\mathcal{H}$ is said to be an *invariant subspace* for $\{T(t), t\geqq 0\}$ if $T(t)\mathcal{M}\subseteq\mathcal{M}$ for all $t\geqq 0$. If we also have $T(t)^*\mathcal{M}\subseteq\mathcal{M}$ for all $t\geqq 0$, then $\mathcal{M}$ is said to be a *reducing subspace* for $\{T(t), t\geqq 0\}$. An invariant subspace $\mathcal{M}$ of $\{T(t), t\geqq 0\}$ is said to be *full* if the smallest reducing subspace containing it is $\mathcal{H}$.

Let $\mathcal{K}$ be a Hilbert space, and let $L^2(\boldsymbol{R}^+, \mathcal{K})$ denote the Hilbert space of all strongly (Lebesgue) measurable functions f on $\boldsymbol{R}^+$ with values in $\mathcal{K}$ such that $\int_{\boldsymbol{R}^+} \|f(t)\|^2\,dt<\infty$. Define *backward translation* $\{B(t), t\geqq 0\}$ on $L^2(\boldsymbol{R}^+, \mathcal{K})$ by $(B(t)f)(s)=f(s+t)$. Then $\{B(t), t\geqq 0\}$ is a strongly continuous semigroup of coisometric operators.

MAIN THEOREM. *Let $\{T(t), t\geqq 0\}$ be a strongly continuous semigroup of contractive operators on a Hilbert space $\mathcal{H}$. Then there exists Hilbert spaces $\mathcal{K}_0$ and $\mathcal{K}_1$, a strongly continuous unitary semigroup $\{U(t)\}$ on $\mathcal{K}_1$ and a subspace $\mathcal{M}$ of $L^2(\boldsymbol{R}^+, \mathcal{K}_0)\oplus\mathcal{K}_1$ such that*

$$T(t)\simeq (B(t)\oplus U(t))|_{\mathcal{M}}$$

where $\{B(t)\}$ is backward translation on $L^2(\boldsymbol{R}^+, \mathcal{K}_0)$ and $\mathcal{M}$ is invariant for

$$\{B(t)\oplus U(t)\}.$$

If $\mathcal{M}$ is full then the extension is unique.

Received by the editors February 20, 1969.

[1] This paper is part of the author's doctoral thesis, and is supported in part by a NSF Traineeship and NSF Grant GP 8178.

COOPER'S THEOREM. *Let $\{V(t), t \geqq 0\}$ be a strongly continuous semigroup of isometric operators on a Hilbert space $\mathscr{H}$. Then there exists a unique reducing subspace $\mathscr{N}$ in $\mathscr{H}$ such that $\{V(t)|_{\mathscr{N}}\}$ is a unitary semigroup and $\{V(t)|_{\mathscr{N}^\perp}\}$ is unitarily equivalent to forward translation* [*the adjoint of backward translation*] *on $L^2(\boldsymbol{R}^+, \mathscr{K})$ for some $\mathscr{K}$.*

COROLLARY. *Let $\{V(t), t \geqq 0\}$ be a strongly continuous semigroup of isometric operators on a Hilbert space $\mathscr{H}$. Then there exists a strongly continuous unitary semigroup $\{U(t), t \geqq 0\}$ on $\mathscr{K} \supseteq \mathscr{H}$ such that $\mathscr{H}$ is a full invariant subspace for $\{U(t)\}$ and*

$$V(t) = U(t)|_{\mathscr{H}}.$$

The proof of the corollary consists merely of extending the forward translation on $L^2(\boldsymbol{R}^+, \mathscr{K})$ in Cooper's Theorem to forward translation on $L^2(\boldsymbol{R}, \mathscr{K})$.

In the course of the proof of the two theorems we need the following facts:

PROPOSITION (SEE LAX AND PHILLIPS [4, *Appendix* 1]). *Let $\{T(t), t \geqq 0\}$ be a strongly continuous semigroup of contractive operators on $\mathscr{H}$, and*

$$B \equiv s \lim_{h \to 0+} \frac{T(h)-I}{h}$$

$$\mathscr{D} \equiv \left\{x \in \mathscr{H} : \lim_{h \to 0+} \frac{T(h)x-x}{h} \text{ exists}\right\}.$$

Then

(a) *$d/dt(T(t)x) = BT(t)x = T(t)Bx$ for $x \in \mathscr{D}$, $t > 0$.*

(b) *$\mathscr{D}$ is a dense linear manifold, which is invariant for $\{T(t)\}$.*

(c) *$d/dt\,\|T(t)x\|^2 = 2\mathscr{R}e(BT(t)x, T(t)x)$ for $x \in \mathscr{D}$.*

(d) *$(Bx, x) + (x, Bx) \leqq 0$ for $x \in \mathscr{D}$.*

LEMMA 1. *Let $\mathscr{L} \subseteq \mathscr{H}_0 \oplus \mathscr{H}_1$ be an invariant subspace for $\{B(t) \oplus U(t), t \geqq 0\}$ where $\{U(t)\}$ is a unitary semigroup, and for all $x \in \mathscr{H}_0$, $B(t)^*B(t)x \to 0$ as $t \to \infty$. Then $(P_{\mathscr{H}_0}\mathscr{L})^-$, $(P_{\mathscr{H}_1}\mathscr{L})^-$ full imply $\mathscr{L}$ is full.*

Proof. Suppose $\mathscr{N} \supseteq \mathscr{L}$ and reduces $\{B(t) \oplus U(t)\}$. Then $(P_{\mathscr{H}_0}\mathscr{N})^- = \mathscr{H}_0$ and $(P_{\mathscr{H}_1}\mathscr{N})^- = \mathscr{H}_1$ since they contain $(P_{\mathscr{H}_0}\mathscr{L})^-$ and $(P_{\mathscr{H}_1}\mathscr{L})^-$ and reduce $\{B(t)\}$ and $\{U(t)\}$. For $x_0 \oplus x_1 \in \mathscr{N}$, we have

$$0 \oplus x_1 = \lim_{t \to \infty} (B(t) \oplus U(t))^*(B(t) \oplus U(t))(x_0 \oplus x_1) \in \mathscr{N}.$$

Hence $\mathscr{H}_0, \mathscr{H}_1 \subseteq \mathscr{N}$, so that $\mathscr{L}$ is full.

LEMMA 2. *For $i = 1, 2$, let $\mathscr{K}_0^i$, $\mathscr{K}_1^i$ be Hilbert spaces, $\{U_i(t), t \geqq 0\}$ be unitary semigroups, $\mathscr{M}_i$ be full invariant subspaces for $\{B_i(t) \oplus U_i(t)\}$ on $L^2(\boldsymbol{R}^+, \mathscr{K}_0^i) \oplus \mathscr{K}_1^i$, and $T_i(t) = (B_i(t) \oplus U_i(t))|_{\mathscr{M}_i}$. If $T_1(t) \cong T_2(t)$, then there exists a unitary Γ carrying $L^2(\boldsymbol{R}^+, \mathscr{K}_0^1)$ onto $L^2(\boldsymbol{R}^+, \mathscr{K}_0^2)$, $\mathscr{K}_1^1$ onto $\mathscr{K}_1^2$, and $\mathscr{M}_1$ onto $\mathscr{M}_2$, which implements $B_1(t) \cong B_2(t)$ and $U_1(t) \cong U_2(t)$.*

Proof. Let γ be the unitary from $\mathscr{M}_1$ onto $\mathscr{M}_2$ which satisfies $\gamma T_1(t)=T_2(t)\gamma$. Since the closed linear span of elements of the form

$$(B_i^*(t) \oplus U_i^*(t))x_i, \qquad x_i \in \mathscr{M}_i,\ t \geqq 0,$$

reduces $\{B_i(t) \oplus U_i(t)\}$, and contains $\mathscr{M}_i$, it is all of $L^2(\boldsymbol{R}^+, \mathscr{K}_0^i) \oplus \mathscr{K}_1^i$.

For $t_0, \dots, t_n \geqq 0$ and $x_1^0, \dots, x_1^n \in \mathscr{M}_1$ we define

$$\Gamma\left(\sum_{k=0}^{n} (B_1^*(t_k) \oplus U_1^*(t_k))x_1^k\right) = \sum_{k=0}^{n} (B_2^*(t_k) \oplus U_2^*(t_k))\gamma x_1^k.$$

Since

$$\left\| \sum_{k=0}^{n} (B_1^*(t_k) \oplus U_1^*(t_k))x_1^k \right\|^2 = \left\| \sum_{k=0}^{n} (B_2^*(t_k) \oplus U_2^*(t_k))\gamma x_1^k \right\|^2,$$

Γ is well defined, and has a unique extension to an isometry from $L^2(\boldsymbol{R}^+, \mathscr{K}_0^1) \oplus \mathscr{K}_1^1$ onto $L^2(\boldsymbol{R}^+, \mathscr{K}_0^2) \oplus \mathscr{K}_1^2$. Γ satisfies all the required properties, because

$$\Gamma(B_1^*(t) \oplus U_1^*(t)) = (B_2^*(t) \oplus U_2^*(t))\Gamma$$

and if $\Gamma(x \oplus y)=u \oplus v$ then $\Gamma(0 \oplus y)=0 \oplus v$.

For the discrete analogs of these two lemmas see Douglas [2].

Lemma 3. *If V^* is a coisometry with $\mathscr{N}$ as an invariant subspace, then*

$$V^*|_{\mathscr{N}} \textit{ is coisometric iff } \mathscr{N} \textit{ reduces } V^*.$$

Proof. We only prove that $V^*|_{\mathscr{N}}$ coisometric implies that $\mathscr{N}$ is reducing for V^*. Let $x \in \mathscr{N}$. Then

$$\|x\|^2 = \|Vx\|^2 = \|P_{\mathscr{N}}Vx\|^2 + \|P_{\mathscr{N}^\perp}Vx\|^2 = \|x\|^2 + \|P_{\mathscr{N}^\perp}Vx\|^2.$$

So that $\|P_{\mathscr{N}^\perp}Vx\|^2=0$ or $Vx \in \mathscr{N}$. Thus $\mathscr{N}$ is also invariant for V, hence reduces V^*.

Proof of the main theorems. We first construct an extension for contractive semigroups, which we apply to coisometric semigroups to prove Cooper's Theorem. We then use the extension and the corollary to Cooper's Theorem to prove the Main Theorem.

For all $x, y \in \mathscr{D}$, let

$$(x, y)_1 = -(Bx, y)-(x, By).$$

Then $(\ ,\ )_1$ is a symmetric, positive semidefinite (proposition (d)) bilinear functional on $\mathscr{D}$. Hence the Schwarz inequality holds, which implies

$$\mathscr{N} = \{x \in \mathscr{D} : (x, x)_1 = 0\}$$

is a linear manifold. Form

$$\mathscr{K} = \mathscr{D}/\mathscr{N}$$

and

$$\mathscr{K}_0 = \text{completion of } \mathscr{K} \text{ in } \|\ \|_1 \text{ norm}.$$

Then $\mathscr{K}_0$ is a Hilbert space with inner product $(\ ,\ )_1$.

Using proposition (c) we see that

$$\|T(t)x\|_1^2 = -d/dt\|T(t)x\|^2 \quad \text{for } x \in \mathscr{D},\ t > 0.$$

So that for $\alpha > 0$

$$\int_0^\alpha \|T(t)x\|_1^2\,dt = -\int_0^\alpha d/dt\,\|T(t)x\|^2\,dt = \|x\|^2 - \|T(\alpha)x\|^2.$$

Hence letting $\alpha \to \infty$, we have

$$\|x\|^2 = \int_0^\infty \|T(t)x\|_1^2\,dt + \lim_{\alpha\to\infty} \|T(\alpha)x\|^2. \tag{1}$$

We now define

$$P_T = s \lim_{t\to\infty} T(t)^*T(t), \qquad A_T = P_T^{1/2},$$

so that, for $x \in \mathscr{H}$

$$\|A_T x\|^2 = \lim_{\alpha\to\infty} \|T(\alpha)x\|^2.$$

Let $\mathscr{K}' = (A_T\mathscr{H})^-$ and define $\{V_T(t)\}$ on $A_T\mathscr{H}$ by

$$V_T(t)A_T x = A_T T(t)x,$$

then $\{V_T(t)\}$ is isometric on $A_T\mathscr{H}$, so we can extend it continuously to $\mathscr{K}' = (A_T\mathscr{H})^-$.

Now (1) allows us to define an isometry

$$\Sigma : \mathscr{D} \to L^2(\boldsymbol{R}^+, \mathscr{K}_0) \oplus \mathscr{K}'$$

by

$$\Sigma x = Wx \oplus A_T x$$

where $(Wx)(t) = T(t)x$ (henceforth we will identify an element of $\mathscr{D}$ with its equivalence class in $\mathscr{K}_0$). By proposition (b) $\mathscr{D}$ is dense in $\mathscr{H}$, so we can extend Σ continuously to all of $\mathscr{H}$. Denote this new isometry by Σ. Then

$$\begin{aligned}\Sigma T(t) &= WT(t) \oplus A_T T(t) = B(t)W \oplus V_T(t)A_T \\ &= (B(t) \oplus V_T(t))\Sigma,\end{aligned}$$

where $\{B(t)\}$ is backward translation on $L^2(\boldsymbol{R}^+, \mathscr{K}_0)$.

Using Σ^{-1} we have

$$T(t) \cong (B(t) \oplus V_T(t))|_{\Sigma\mathscr{H}}.$$

To prove the Main Theorem we extend the isometric semigroup $\{V_T(t)\}$ to a unitary semigroup, which will follow from the corollary to Cooper's Theorem.

But we also want to prove fullness in the Main Theorem, so we show that

$$(P_{L^2(\boldsymbol{R}^+,\mathscr{K}_0)}(\Sigma\mathscr{H}))^- \text{ is a full invariant subspace.} \tag{2}$$

Let $\mathscr{L}$ be any reducing subspace for $\{B(t)\}$ such that

$$(P_{L^2(\boldsymbol{R}^+,\mathscr{K}_0)}(\Sigma\mathscr{H}))^- \subseteq \mathscr{L} \subseteq L^2(\boldsymbol{R}^+, \mathscr{K}_0).$$

We must show that $\mathscr{L}=L^2(\boldsymbol{R}^+, \mathscr{K}_0)$. We do this by showing that all characteristic functions with values in $\mathscr{D}/\mathscr{N}$ belong to $\mathscr{L}$. Since $\mathscr{L}$ reduces and

$$\chi_{[a,b]} = B(a)^*\chi_{[0,b-a]}$$

we need only show that for all $x \in \mathscr{D}$, $\alpha > 0$

$$\begin{aligned} h_x(t) &= x \quad t \leqq \alpha \\ &= 0 \quad t > \alpha, \end{aligned}$$

belongs to $\mathscr{L}$.

For $a > 0$, we have that

$$\begin{aligned} h_{ax}(t) &= (Wx)(t) - (B^*(a)B(a)Wx)(t) \\ &= \begin{cases} T(t)x & t \leqq a \\ 0 & t > a \end{cases} \end{aligned}$$

belongs to $\mathscr{L}$, since $Wx \in \mathscr{L}$ and $\mathscr{L}$ reduces $\{B(t)\}$. For every $\varepsilon > 0$, pick $a = \alpha/N > 0$ such that

$$\|x - T(t)x\|_1^2 < \varepsilon/\alpha \quad \text{for all } t < a.$$

[This is possible, since for all $x \in \mathscr{D}$

$$\|x - T(t)x\|_1^2 \leqq 4\|Bx\| \; \|x - T(t)x\|$$

and the fact that $\{T(t)\}$ is strongly continuous.]

This being done, consider

$$k_{ax}(t) = h_{ax}(t) + (B^*(a)h_{ax})(t) + \cdots + (B^*((N-1)a)h_{ax})(t).$$

Then k_{ax} belongs to $\mathscr{L}$ and

$$\begin{aligned} \|h_x - k_{ax}\|_{L^2}^2 &= \int_0^\infty \|h_x(t) - k_{ax}(t)\|_1^2 \, dt \\ &= \int_0^\alpha \|h_x(t) - k_{ax}(t)\|_1^2 \, dt \\ &= \sum_{j=0}^{N-1} \int_{ja}^{(j+1)a} \|x - T(t-ja)x\|_1^2 \, dt \\ &= \sum_{j=0}^{N-1} \int_0^a \|x - T(t)x\|_1^2 \, dt \\ &< \sum_{j=0}^{N-1} \int_0^a \varepsilon/\alpha \, dt = Na\varepsilon/\alpha = \varepsilon. \end{aligned}$$

Since $\mathscr{L}$ is closed, $h_x \in \mathscr{L}$. So that $\mathscr{L} = L^2(\boldsymbol{R}^+, \mathscr{K}_0)$.

We also have that

(3) $\qquad (P_{\mathscr{K}'}(\Sigma\mathscr{H}))^-$ is a full invariant subspace,

since, in fact, it equals $\mathscr{K}'$.

We now pause for a moment to prove Cooper's Theorem. Suppose $\{T(t), t \geqq 0\}$ had been coisometric. If $y = A_T x$ then $z = A_T T(t)^* x$ satisfies

$$V_T(t)z = A_T T(t) T(t)^* x = A_T x = y.$$

Thus $V_T(t)$ is onto, hence $\{V_T(t)\}$ is unitary. We next prove that Σ is unitary. Using (2) and (3) where

$$T(t) \cong (B(t) \oplus V_T(t))|_{\Sigma\mathscr{H}}$$

with $\{V_T(t)\}$ unitary, we can apply Lemma 1 to conclude that $\Sigma\mathscr{H}$ is a full invariant subspace. But since $\{T(t)\}$ and $\{B(t) \oplus V_T(t)\}$ are coisometric, Lemma 3 says that $\Sigma\mathscr{H}$ also reduces. Being full and reducing, $\Sigma\mathscr{H} = L^2(\boldsymbol{R}^+, \mathscr{K}_0) \oplus \mathscr{K}'$. So that Σ is onto, hence unitary, and

$$T(t) \cong B(t) \oplus V_T(t).$$

Taking adjoints we obtain Cooper's Theorem.

We now use the corollary to Cooper's Theorem to extend $\{V_T(t)\}$ to a unitary semigroup in the Main Theorem. Fullness follows from Lemma 1, (2), and fullness in the corollary. Lemma 2 is the statement that uniqueness follows from the fullness of $\mathscr{M}$. Hence the Main Theorem is proven.

REMARKS. (1) In Cooper's Theorem, $\mathscr{N}$ is

$$\mathscr{N} = \Big(s \lim_{t \to \infty} V(t) V^*(t)\Big)\mathscr{H} = \bigcap_{t \geqq 0} V(t)\mathscr{H}.$$

(2) The coefficient space $(\mathscr{D}/\mathscr{N})^-$ obtained in the two theorems can be seen to be isometric to $\mathscr{R}^\perp = R\mathscr{H}$ (see Masani [5] and Sz.-Nagy [6]). This is accomplished using the isometry

$$x \to \frac{1}{\sqrt{2}} R(B-I)x, \qquad x \in \mathscr{D}$$

where

$$B = s \lim_{h \to 0+} \frac{T(h)-I}{h}$$

$$T = (B+I)(B-I)^{-1}$$

$$R = (I - T^*T)^{1/2}.$$

In the case of Cooper's Theorem $\{T(t)\}$ coisometric implies T is coisometric, hence $R = I - T^*T$, and

$$\mathscr{R}^\perp = R\mathscr{H} = (T^*\mathscr{H})^\perp.$$

(3) If in the Main Theorem, for all $x \in \mathscr{H}$, $T(t)x \to 0$ as $t \to \infty$, then $A_T \equiv 0$, and the term $\{U(t)\}$ is absent. This is Theorem 2.1 of [4].

(4) The extra work needed to prove the fullness of the subspace $(P_{L^2}(\Sigma\mathscr{H}))^-$ can be justified, since it shows that the obtained extension is the unique one.

(5) The Main Theorem can also be proven using Cooper's Theorem, and the fact that every strongly continuous contractive semigroup has a unitary dilation

[7]. This is because having a unitary dilation is equivalent to having a coisometric extension. To see this suppose $T(t)=P_{\mathscr{H}}U(t)|_{\mathscr{H}}$. Let $\mathscr{M}=\bigvee_{t\geqq 0} U^*(t)\mathscr{H}$. $\mathscr{M}$ is invariant for $\{U^*(t)\}$, so set $V(t)=U^*(t)|_{\mathscr{M}}$. Then $\mathscr{H}$ is invariant for $\{V^*(t)\}$ and $T(t)=V^*(t)|_{\mathscr{H}}$. Since, for $x\in\mathscr{H}$, $V(t)x=T^*(t)x+(V(t)-T^*(t))x\in\mathscr{H}\oplus\mathscr{M}\ominus\mathscr{H}$, we have $P_{\mathscr{H}}^{\perp}V(t)x=(V(t)-T^*(t))x$. But

$$\mathscr{M}\ominus\mathscr{H} = P_{\mathscr{H}}^{\perp}\mathscr{M} = \bigvee_{t\geqq 0} (V(t)-T^*(t))\mathscr{H}$$

and

$$V(t)(V(s)-T^*(s)) = (V(t+s)-T^*(t+s))-(V(t)-T^*(t))T(s).$$

So that $\mathscr{M}\ominus\mathscr{H}$ is invariant for $\{V(t)\}$ or $\mathscr{H}$ is invariant for $\{V^*(t)\}$. The proof is the continuous analog of a proof in [7, p. 12].

I would like to thank my advisor Dr. Peter A. Fillmore for his guidance in the above work.

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Indiana University,
Bloomington, Indiana