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LIE ISOMORPHISMS OF DERIVED RINGS OF SIMPLE RINGS(1)

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1. **Introduction.** A Lie subring L of an associative ring R is an additive subgroup of R such that $[x, y]=xy-yx\in L$, whenever x and y are in L. Clearly $[R, R]$, the additive subgroup of R generated by all commutators $[x, y]$, is such a Lie subring of R. If L_1 is a Lie subring of R and L_2 is a Lie subring of S, then a Lie isomorphism ϕ of L_1 onto L_2 is a one-one additive mapping of L_1 onto L_2 which preserves commutators, i.e.

$$\phi(x+y) = \phi(x)+\phi(y)$$
$$\phi(xy-yx) = \phi(x)\phi(y)-\phi(y)\phi(x)$$

for all $x, y\in L_1$. In this paper, we will assume that $L_1=[R, R]$ and $L_2=[S, S]$ where R and S are simple rings with identity. We shall also assume that the characteristic of R is different from 2 and 3, and that R contains three nonzero orthogonal idempotents whose sum is the identity. We will then show that ϕ may be extended to either an isomorphism of R onto S, or to the negative of an anti-isomorphism of R onto S. This result generalizes a theorem of Martindale [4, p. 916, Theorem 5].

2. **Lie isomorphisms and the Peirce decomposition.** Let e_1, e_2, and e_3 be the orthogonal idempotents of R, i.e.

$$e_i^2 = e_i \neq 0; \quad \sum_{i=1}^{3} e_i = 1, \; e_ie_j = 0 \text{ for } i \neq j.$$

It is well known that we can obtain the Peirce decomposition

$$R = \bigoplus \sum_{i,j=1}^{3} R_{ij} \quad \text{where } R_{ij} = e_iRe_j.$$

We will denote an element in R_{ij} by x_{ij}. The proof of the theorem requires a careful analysis of those properties of the Peirce decomposition, which are invariant under Lie isomorphisms.

Let S be a simple ring with identity. Let S_r and S_l denote the right and left multiplications respectively of S, and denote the center of S by Z.

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2.1. LEMMA. $S^* \otimes_Z S \cong S_l S_r$.

Proof. Let $\eta: S^* \otimes_Z S \to S_l S_r$ be given by

$$\left(\sum_{i=1}^{n} a_i^* \otimes b_i\right)\eta = \sum_{i=1}^{n} a_{il} b_{ir}.$$

Since $(1^* \otimes 1)\eta = 1$, we know $\eta \neq 0$. Since $S^* \otimes_Z S$ is simple, η is an isomorphism, and η is clearly a surjection.

The following lemma illustrates how one can solve certain "generalized polynomial identities" using the tensor product.

2.2. LEMMA. *Let S be a simple ring with identity of characteristic not 2 or 3, such that $[S, S]^- = S$, where $[S, S]$ denotes the subring generated by $[S, S]^-$. Suppose $[[[x, a], a], a] = 0$ for all $x \in [S, S]$. Then there is $z \in Z$ such that $(a+z)^2 = 0$.*

Proof. Since $[[[x, a], a], a] = 0$ for all $x \in [S, S]$, we may choose $x = [y, a]$ where y is arbitrary in S. Hence $[[[[y, a], a], a], a] = 0$ for all $y \in S$. In terms of mappings, this gives that $(a_r - a_l)^4 = 0$. Since $[a_r, a_l] = 0$, we can expand the previous relation to obtain

$$a_r^4 - 4a_r^3 a_l + 6a_r^2 a_l^2 - 4a_r a_l^3 + a_l^4 = 0.$$

By 2.1 we may replace this equation by:

$$(1) \qquad 1 \otimes a^4 - 4a \otimes a^3 + 6a^2 \otimes a^2 - 4a^3 \otimes a + a^4 \otimes 1 = 0.$$

Since $1 \neq 0$, the set $\{a^4, a^3, a^2, a, 1\}$ is a dependent set over Z. We may assume that $\{a^3, a^2, a, 1\}$ is a dependent set. Otherwise

$$a^4 = \alpha a^3 + \beta a^2 + \gamma a + \delta, \qquad \alpha, \beta, \gamma, \delta \in Z.$$

Substituting this in (1), we obtain:

$$(\alpha - 4a) \otimes a^3 + (\beta + 6a^2) \otimes a^2 + (\gamma - 4a^3) \otimes a + (\alpha a^3 + \beta a^2 + \gamma a + \delta) \otimes 1 = 0.$$

The independence of $\{a^3, a^2, a, 1\}$ gives that $\alpha - 4a = 0$. But then $a \in Z$ and $z = -a$ satisfies the theorem. We now claim that the set $\{a^2, a, 1\}$ is a dependent set. If this is not the case, then we have:

$$a^3 = \alpha a^2 + \beta a + \gamma \qquad \alpha, \beta, \gamma \in Z$$

whence

$$a^4 = (\alpha^2 + \beta)a^2 + (\alpha\beta + \gamma)a + \alpha\gamma.$$

These relations, when substituted into (1), give

$$(-6a^2 - 4\alpha a + \alpha^2 + \beta) \otimes a^2 + (\alpha a^2 - 3\beta a + 2\gamma + \alpha\beta) \otimes a$$
$$+ [(a^2 + \beta)a^2 + (\alpha\beta - 3\gamma)a + 2\alpha\gamma] \otimes 1 = 0.$$

The assumed independence of $\{a^2, a, 1\}$ gives that $-6a^2 - 4\alpha a + \alpha^2 + \beta = 0$ which contradicts the independence of $\{a^2, a, 1\}$. Thus $\{a^2, a, 1\}$ is a dependent set as claimed. Furthermore, if $\{a, 1\}$ is dependent, then $a \in Z$ and $z = -a$ satisfies the theorem. If $\{a, 1\}$ is independent, then we have that:

$$a^2 = \alpha a + \beta \tag{2}$$

whence

$$a^3 = (\alpha^2 + \beta)a + \beta\alpha. \tag{3}$$

But $[[[x, a], a], a] = 0$ for all $x \in [S, S]$, so

$$xa^3 - 3axa^2 + 3a^2xa - a^3x = 0 \quad \text{for all } x \in [S, S].$$

Substituting the relations (2) and (3) this equation becomes after simplification

$$(\alpha^2 + 4\beta)[x, a] = 0 \quad \text{for all } x \in [S, S]. \tag{4}$$

Now, if $[x, a] = 0$ for all $x \in [S, S]$, then, since $[S, S]^- = S$, $a \in Z$ and we are done as before. If $[x, a] \neq 0$ for some $x \in [S, S]$, then, since Z is a field, $\alpha^2 + 4\beta = 0$. Let $z = -\alpha/2$. Now $(a - \alpha/2)^2 = a^2 - \alpha a + \alpha^2/4 = a^2 - \alpha a - \beta = 0$.

2.3. Lemma. *Let S be simple with identity and with characteristic different from 2. Suppose $a, b \in S$ are such that $a^2 = b^2 = [a, b] = 0$. If, in addition, $[[[x, b], a], b] = 0$ for all $x \in [S, S]$, then $ab = ba = 0$.*

Proof. Since $[[[x, b], a], b] = 0$ for all $x \in [S, S]$, letting $x = [y, a]$ where y is arbitrary in S, we have $[[[[y, a], b], a], b] = 0$ for all $y \in S$. Expanding this equation and using $a^2 = b^2 = [a, b] = 0$, we obtain $4abyab = 0$ for all $y \in S$. Since S is simple, $ab = 0$.

2.4. Lemma. *Let S be a simple ring such that $[S, S]^- = S$. Suppose further that $a[S, S]b = 0$ for some $a, b \in S$. Then either $a = 0$ or $b = 0$.*

Proof. Let $x, y \in S$. Since $xy - yx \in [S, S]$, we have $a(xy - yx)b = 0$ or $axyb = ayxb$. Now let $L = \{x \in S \mid xb = 0\}$. L is a left ideal of S and $a[S, S] \subseteq L$. LS is a two-sided ideal of S, and so either $LS = 0$ or $LS = S$. If $LS = 0$, then $L = 0$, so $a[S, S] = 0$. Since $[S, S]^- = S$, this gives $aS = 0$ and hence $a = 0$. Hence we may assume that $LS = S$. Let $x \in S$, then $x = \sum_{i=1}^n l_i y_i$ where $l_i \in L$ and $y_i \in S$. Then

$$axb = a\left(\sum_{i=1}^n l_i y_i\right)b = \sum_{i=1}^n a(l_i y_i)b = \sum_{i=1}^n (ay_i l_i b) = 0.$$

Thus $aSb = 0$, so either $a = 0$ or $b = 0$.

We now state in the form of a remark a useful result which may be found in [1].

Remark 2.5. If R is a simple ring of characteristic different from 2 and is not a field, then $[R, R]^- = R$.

Henceforth R and S will be as stated in the introduction. The "off-diagonal" elements R_{ij}, $i \neq j$ of the Peirce decomposition of R are in $[R, R]$. In fact $x_{ij} = [e_i, x_{ij}]$.

REMARK 2.6. The characteristic of S is not two or three, and $([S, S])^- = S$.

Proof. Since the ideal $\{x \in R \mid 2x=0\}$ must be zero, $2R_{12} \neq 0$. Thus $2\phi(R_{12}) \neq 0$ and $2S \neq 0$. Similarly the characteristic of S is not three.

By 2.5 $[R, R]^- = R$, so $[R, R] \neq 0$. Since ϕ is a bijection $[S, S] \neq 0$. Thus by 2.5 $[S, S]^- = S$.

We now begin to examine the image of the Peirce decomposition under ϕ. If $x_{ij} \in R_{ij}$, $i \neq j$, then $x_{ij} \in [R, R]$. Thus ϕ may be applied to these elements.

2.7. LEMMA. *Let $x_{ij} \in R_{ij}$, $i \neq j$. Then $\phi(x_{ij})^2 = 0$.*

Proof. If $x_{ij}=0$, then $\phi(x_{ij})^2=0$. So we may assume that $x_{ij} \neq 0$. Since $x_{ij}^2=0$, $[[[x, x_{ij}], x_{ij}], x_{ij}]=0$ for all $x \in [R, R]$. Because ϕ is a Lie isomorphism this gives $[[[\phi(x), \phi(x_{ij})], \phi(x_{ij})], \phi(x_{ij})]=0$. But ϕ is a surjection, so

$$[[[x, \phi(x_{ij})], \phi(x_{ij})], \phi(x_{ij})] = 0$$

for all $x \in [S, S]$. By 2.2, $\phi(x_{ij}) = b+\lambda$, where $\lambda \in Z(S)$ and $b^2=0$. This is true for all i, j where $i \neq j$. Furthermore, $b \neq 0$. Otherwise $\phi(x_{ij}) \in Z$, so $[\phi(x_{ij}), \phi(x_{jk})]=0$ for $k \neq i$, $k \neq j$. Since ϕ is a Lie isomorphism, this gives $\phi([x_{ij}, x_{jk}])=0$. Hence $[x_{ij}, x_{jk}]=0$, so $x_{ij}x_{jk}=0$. But then $x_{ij}R_{jk}=0$. Hence $x_{ij}=0$, a contradiction.

For convenience in notation, let us assume that $i=1$ and $j=2$, that is, we wish to show that $\phi(x_{12})^2=0$. For this purpose let $y_{13} \in R_{13}$, $y_{12} \in R_{12}$, and $y_{32} \in R_{32}$ be arbitrary nonzero elements such that $y_{13}y_{32} \neq 0$. By the above argument we have:

(a) $\phi(y_{13}) = b+\lambda$, $\quad b^2=0$, $b \neq 0$, $\lambda \in Z(S)$,

(b) $\phi(y_{12}) = c+\mu$, $\quad c^2=0$, $c \neq 0$, $\mu \in Z(S)$,

(c) $\phi(y_{13}y_{32}) = d+\nu$, $\quad d^2=0$, $d \neq 0$, $\nu \in Z(S)$.

Now $[[[x, y_{13}], y_{12}], y_{13}]=0$ for all $x \in [R, R]$, thus

$$[[[x, \phi(y_{13})], \phi(y_{12})], \phi(y_{13})] = 0 \quad \text{for all } x \in [S, S].$$

Since $\lambda, \mu \in Z(S)$, this gives $[[[x, b], c], b]=0$. Since $[y_{13}, y_{12}]=0$, $[\phi(y_{13}), \phi(y_{12})] =0$, and so $[b, c]=0$. By 2.3 $bc=cb=0$. Hence,

(1) $\phi(y_{13})c = (b+\lambda)c = \lambda c$.

Since $[y_{32}, y_{12}]=0$, $[\phi(y_{32}), \phi(y_{12})]=0$ and so $[\phi(y_{32}), c]=0$. Commuting (1) with $\phi(y_{32})$, we obtain

(2) $[\phi(y_{13})c, \phi(y_{32})] = [\lambda c, \phi(y_{32})] = 0$.

Since $[\phi(y_{13})c, \phi(y_{32})] = [\phi(y_{13}), \phi(y_{32})]c = \phi([y_{13}, y_{32}])c$, we have

(3) $\phi(y_{13}y_{32})c=0$.

But then from (c),

(4) $(d+\nu)c=0$.

An application of 2.3 shows that $dc=0$, hence

(5) $\nu c=0$.

(6) $\nu=0$.

We have shown that $\phi(y_{13}y_{32})^2=0$. Since $R_{12}=R_{13}R_{32}$, we may write

$$x_{12} = \sum_{i=1}^{n} y_{13}^{(i)}y_{32}^{(i)}.$$

Hence $\phi(x_{12})=\sum_{i=1}^{n}\phi(y_{13}^{(i)}y_{32}^{(i)})$. We have just shown that $\phi(y_{13}^{(i)}y_{32}^{(i)})^2=0$. Because

$$[\phi(y_{13}^{(i)}y_{32}^{(i)}), \phi(y_{13}^{(j)}y_{32}^{(j)})] = 0$$

and

$$[[[x, \phi(y_{13}^{(i)}y_{32}^{(i)})], \phi(y_{13}^{(j)}y_{32}^{(j)})], \phi(y_{13}^{(i)}y_{32}^{(i)})] = 0,$$

we have by 2.3 that $\phi(y_{13}^{(i)}y_{32}^{(i)})\phi(y_{13}^{(j)}y_{32}^{(j)})=0$. Thus

$$\phi(x_{12})^2 = \left(\sum_{i=1}^{n}\phi(y_{13}^{(i)}y_{32}^{(i)})\right)^2 = 0.$$

2.8. Lemma. *Let $x_{ij}\in R_{ij}$, $x_{kl}\in R_{kl}$ where $i\neq j$ and $k\neq l$. If $x_{kl}x_{ij}=x_{ij}x_{kl}=0$, then $\phi(x_{ij})\phi(x_{kl})=\phi(x_{kl})\phi(x_{ij})=0$.*

Proof. Since $[[[x, x_{ij}], x_{kl}], x_{ij}]=0$ for all $x\in[R, R]$ and $[x_{ij}, x_{kl}]=0$, we have

$$[[[x, \phi(x_{ij})], \phi(x_{kl})], \phi(x_{ij})] = 0 \quad \text{for all } x\in[S, S],$$

and $[\phi(x_{ij}), \phi(x_{kl})]=0$. Furthermore $\phi(x_{ij})^2=0$ and $\phi(x_{kl})^2=0$. Hence by 2.3 $\phi(x_{ij})\phi(x_{kl})=0$.

In order to continue the study of the Peirce decomposition under a Lie isomorphism, we must examine the relationship between $[R, R]$ and the "off-diagonal" components R_{ij}, $i\neq j$. To this end we have:

2.9. Lemma. *$[R, R]$ is additively generated by R_{ij}, $i\neq j$, and $[R_{ij}, R_{ji}]$ for $i\neq j$.*

Proof.

$$\begin{aligned}[R, R] &= \left[\bigoplus_{i,j=1}^{3} R_{ij}, \bigoplus_{i,j=1}^{3} R_{ij}\right] \\ &= \sum_{i\neq j} R_{ij}+\sum_{i\neq j}[R_{ij}, R_{ji}]+\sum_{i=1}^{3}[R_{ii}, R_{ii}].\end{aligned}$$

Thus we need only show that $[R_{ii}, R_{ii}]\subseteq[R_{ij}, R_{ji}]$ for $i\neq j$. Without loss of generality, we may assume that $i=1$ and $j=2$. Then $R_{11}=e_1Re_1=e_1Re_2Re_1$. Let $x, y\in R_{11}$. Write $x=\sum_{i,j}e_1x_ie_2y_je_1$ and $y=e_1we_1$. Then

$$[x, y] = \left[\sum_{i,j}e_1x_ie_2y_je_1, e_1we_1\right] = \sum_{i,j}[e_1x_ie_2y_je_1, e_1we_1].$$

So it suffices to show that $[e_1x_ie_2y_je_1, e_1we_1]\in[R_{12}, R_{21}]$. But

$$[e_1x_ie_2y_je_1, e_1we_1] = [e_1x_ie_2, e_2y_je_1we_1]-[e_1we_1x_ie_2, e_2y_je_1]$$

which is in $[R_{12}, R_{21}]$.

Since ϕ is a Lie isomorphism, 2.9 can be carried over to S.

2.10. LEMMA. *$[S, S]$ is additively spanned by $\phi(x_{ij})$, $i \neq j$, and $\phi[x_{ij}, x_{ji}]$ for $i \neq j$.*

Proof. The result is immediate from 2.9 and the fact that ϕ is a surjection.

We are trying to show that ϕ can be extended to either an isomorphism or the negative of an anti-isomorphism. Lemma 2.7 hints that ϕ is well-behaved. The next lemma, which is the key to the main theorem, determines ϕ on certain of the off-diagonal components.

2.11. LEMMA. *Let (i, j, k) be any permutation of $(1, 2, 3)$. Suppose $x_{ij} \in R_{ij}$ and $x_{jk} \in R_{jk}$. Then either:*
(1) $\phi(x_{ij}x_{jk}) = \phi(x_{ij})\phi(x_{jk})$, *or*
(2) $\phi(x_{ij}x_{jk}) = -\phi(x_{jk})\phi(x_{ij})$.

Proof. Without loss of generality we may assume $i=1$, $j=2$, and $k=3$. The method of proof will be to show that

$$\phi(x_{12})\phi(x_{23})[S, S]\phi(x_{23})\phi(x_{12}) = 0.$$

That this suffices, can be seen as follows:

By 2.4 either $\phi(x_{12})\phi(x_{23}) = 0$ or $\phi(x_{23})\phi(x_{12}) = 0$. Since ϕ is a Lie isomorphism,

$$\phi(x_{12}x_{23}) = \phi([x_{12}, x_{23}]) = [\phi(x_{12}), \phi(x_{23})] = \phi(x_{12})\phi(x_{23}) - \phi(x_{23})\phi(x_{12}).$$

This gives the result.

Since $[S, S]$ is additively spanned by elements of the form $\phi(x_{ij})$, $i \neq j$ and $[\phi(x_{ij}), \phi(x_{ji})]$, $i \neq j$, it suffices to consider these elements only.

$$\begin{aligned} \phi(x_{12})\phi(x_{23})\phi(y_{12})\phi(x_{23})\phi(x_{12}) &= \phi(x_{12})\phi(x_{23})\phi(y_{12})\phi(x_{23})\phi(x_{12}) \\ &\quad -\phi(x_{12})\phi(y_{12})\phi(x_{23})\phi(x_{23})\phi(x_{12}) \qquad \text{(by 2.7)} \\ (1) \qquad &= \phi(x_{12})\phi(x_{23}y_{12} - y_{12}x_{23})\phi(x_{23})\phi(x_{12}) \\ &\qquad \text{(since } \phi \text{ is a Lie isomorphism)} \\ &= -\phi(x_{12})\phi(y_{12}x_{23})\phi(x_{23})\phi(x_{12}) \\ &= 0 \quad \text{(since by 2.8, } \phi(x_{12})\phi(y_{12}x_{23}) = 0). \end{aligned}$$

(2) $\phi(x_{12})\phi(x_{23})\phi(y_{ij})\phi(x_{23})\phi(x_{12}) = 0$, for $(i, j) = (1, 3), (2, 1), (2, 3)$ (by 2.8).

$$\begin{aligned} \phi(x_{12})\phi(x_{23})\phi(y_{31})\phi(x_{23})\phi(x_{12}) &= \phi(x_{12})\phi(x_{23})\phi(y_{31})\phi(x_{23})\phi(x_{12}) \\ &\quad -\phi(x_{12})\phi(y_{31})\phi(x_{23})\phi(x_{23})\phi(x_{12}) \\ (3) \qquad &= \phi(x_{12})\phi(x_{23}y_{31} - y_{31}x_{23})\phi(x_{23})\phi(x_{12}) \\ &\qquad \text{(since } \phi \text{ is a Lie isomorphism)} \\ &= \phi(x_{12})\phi(x_{23}y_{31})\phi(x_{23})\phi(x_{12}) \\ &= 0 \quad \text{(since by 2.8, } \phi(x_{23}y_{31})\phi(x_{23}) = 0). \end{aligned}$$

$$
\begin{aligned}
\phi(x_{12})\phi(x_{23})\phi(y_{32})\phi(x_{23})\phi(x_{12}) &= \phi(x_{12})\phi(x_{23})\phi(y_{32})\phi(x_{23})\phi(x_{12}) \\
&\quad -\phi(x_{23})\phi(x_{12})\phi(y_{32})\phi(x_{23})\phi(x_{12}) \\
&= \phi(x_{12}x_{23}-x_{23}x_{12})\phi(y_{32})\phi(x_{23})\phi(x_{12}) \\
&\qquad\qquad \text{(since } \phi \text{ is a Lie isomorphism)} \\
&= \phi(x_{12}x_{23})\phi(y_{32})\phi(x_{23})\phi(x_{12}) \\
&= \phi(x_{12}x_{23})\phi(y_{32})\phi(x_{23})\phi(x_{12}) \\
&\quad -\phi(y_{32})\phi(x_{12}x_{23})\phi(x_{23})\phi(x_{12}) \\
&= \phi(x_{12}x_{23}y_{32}-y_{32}x_{12}x_{23})\phi(x_{23})\phi(x_{12}) \\
&= \phi(x_{12}x_{23}y_{32})\phi(x_{23})\phi(x_{12}) \\
&= \phi(x_{12}x_{23}y_{32})\phi(x_{23})\phi(x_{12}) \\
&\quad -\phi(x_{23})\phi(x_{12}x_{23}y_{32})\phi(x_{12}) \\
&= \phi(x_{12}x_{23}y_{32}x_{23})\phi(x_{12}) \\
&= 0 \quad \text{(by 2.8).}
\end{aligned} \tag{4}
$$

(5) $$\phi(x_{12})\phi(x_{23})([\phi(y_{ij}), \phi(y_{ji})])\phi(x_{23})\phi(x_{12}) = 0 \quad \text{for } i \neq j \qquad \text{by 2.8.}$$

This concludes the proof.

This lemma points the way of the main theorem. It says that ϕ is either an associative isomorphism or the negative of an anti-isomorphism on certain parts of R. It is most convenient to break the proof of the main theorem into two cases depending on the outcome of this lemma. The two cases will occupy §§3 and 4 respectively.

3. **The isomorphism case.** In this section we will continue the proof of the theorem under the following assumption:

3.1. Assumption. There is $r_{12} \in R_{12}$, $r_{23} \in R_{23}$ such that $r_{12}r_{23} \neq 0$ and

$$\phi(r_{12}r_{23}) = \phi(r_{12})\phi(r_{23}).$$

The first task is to show that 3.1 determines the behavior of ϕ on all products $y_{12}y_{23}$.

3.2. Lemma. *If* $y_{12} \in R_{12}$, $y_{23} \in R_{23}$, *then* $\phi(y_{12}y_{23}) = \phi(y_{12})\phi(y_{23})$.

Proof. We may assume $y_{12}y_{23} \neq 0$. Otherwise, $\phi(y_{12}y_{23}) = 0$. But then

$$0 = \phi[y_{12}, y_{23}] = \phi(y_{12})\phi(y_{23}) - \phi(y_{23})\phi(y_{12}).$$

Thus $\phi(y_{12})\phi(y_{23}) = \phi(y_{23})\phi(y_{12})$. But by 2.11 one of these terms is zero. Hence $0 = \phi(y_{12}y_{23}) = \phi(y_{12})\phi(y_{23})$.

Now suppose the lemma is false. Then $\phi(y_{12}y_{23}) = -\phi(y_{23})\phi(y_{12})$ by 2.11.

We claim $r_{12}y_{23} = 0$. For this, consider

(1) $$\phi(r_{12}(y_{23}+r_{23})) = \phi(r_{12}y_{23}) + \phi(r_{12}r_{23}).$$

By 2.11, $\phi(r_{12}y_{23})=\phi(r_{12})\phi(y_{23})$ or $\phi(r_{12}y_{23})=-\phi(y_{23})\phi(r_{12})$. Suppose

$$\phi(r_{12}y_{23}) = -\phi(y_{23})\phi(r_{12}). \tag{2}$$

Then from (1) we have

$$\phi(r_{12}(y_{23}+r_{23})) = -\phi(y_{23})\phi(r_{12})+\phi(r_{12})\phi(r_{23}). \tag{3}$$

On the other hand, by 2.11 either

$$\phi(r_{12}(y_{23}+r_{23})) = \phi(r_{12})\phi(y_{23}+r_{23}) \tag{4}$$

or

$$\phi(r_{12}(y_{23}+r_{23})) = -\phi(y_{23}+r_{23})\phi(r_{12}). \tag{5}$$

If (4) is true, then using (3) we obtain

$$-\phi(y_{23})\phi(r_{12}) = \phi(r_{12})\phi(y_{23}). \tag{6}$$

It is immediate from (6) that $r_{12}y_{23}=0$, and the claim is true. If (5) is true, then again from (3) we obtain

$$\phi(r_{12})\phi(r_{23}) = -\phi(r_{23})\phi(r_{12}). \tag{7}$$

It follows from (7) that $r_{12}r_{23}=0$, a contradiction. Thus if (2) is true, the claim has been proven. If (2) is false, then we have

$$\phi(r_{12}y_{23}) = \phi(r_{12})\phi(y_{23}). \tag{8}$$

Reasoning as before, we find that either $y_{12}y_{23}=0$ or $r_{12}y_{23}=0$. Hence $r_{12}y_{23}=0$.

We claim also that $y_{12}r_{23}=0$. The proof is analogous to the above.

In order to complete the proof of this lemma we consider $\phi((r_{12}+y_{12})(r_{23}+y_{23}))$. By additivity and the claims, we have

$$\begin{aligned}\phi((r_{12}+y_{12})(r_{23}+y_{23})) &= \phi(r_{12}r_{23})+\phi(y_{12}y_{23})\\ &= \phi(r_{12})\phi(r_{23})-\phi(y_{23})\phi(y_{12})\end{aligned} \tag{9}$$

(by 3.1 and the denial of the lemma).

By 2.11 we have either

$$\phi((r_{12}+y_{12})(r_{23}+y_{23})) = \phi(r_{12}+y_{12})\phi(r_{23}+y_{23}) \tag{10}$$

or

$$\phi((r_{12}+y_{12})(r_{23}+y_{23})) = -\phi(r_{23}+y_{23})\phi(r_{12}+y_{12}). \tag{11}$$

Combining (10) with (9), we obtain

$$\phi(y_{12})\phi(y_{23}) = -\phi(y_{23})\phi(y_{12}). \tag{12}$$

This gives $y_{12}y_{23}=0$, a contradiction. Similarly (11) and (9) give

$$\phi(r_{12})\phi(r_{23}) = -\phi(r_{23})\phi(r_{12}). \tag{13}$$

Hence $r_{12}r_{23}=0$, another contradiction. This completes the proof of the lemma.

3.3. Lemma. *ϕ is a homomorphism from $R_{12} \oplus R_{23} \oplus R_{13}$ into S.*

Proof. Since ϕ is additive, it suffices to check the various products $y_{ij}y_{lm}$, $i\neq j$, $l\neq m$.

(1) $\phi(y_{12}y_{23})=\phi(y_{12})\phi(y_{23})$ (by 3.2).

(2) $\phi(y_{23}y_{12})=0$. But

$$\begin{aligned}\phi(y_{23})\phi(y_{12}) &= [\phi(y_{23}), \phi(y_{12})]+\phi(y_{12})\phi(y_{23})\\ &= \phi([y_{23}, y_{12}])+\phi(y_{12})\phi(y_{23})\\ &= \phi(-y_{12}y_{23})+\phi(y_{12}y_{23}) \qquad \text{(by 3.2)}\\ &= 0.\end{aligned}$$

(3) Each of the other possibilities are trivial since $y_{ij}y_{lm}=0$ and $\phi(y_{ij})\phi(y_{lm})=0$ by 2.8.

3.4. Lemma. *Let (i, j, k) be a permutation of $(1, 2, 3)$. Then ϕ is a homomorphism of $R_{ij} \oplus R_{jk} \oplus R_{ik}$ into S.*

Proof. We will prove the lemma by showing, if the lemma is true for (i, j, k), then it is true for (i, k, j) and for (j, i, k). Noting that these two transpositions generate S_3, we see then that the lemma is true either for all permutations or for none. By 3.3, we will then be done.

It clearly suffices to let $i=1, j=2, k=3$. Thus ϕ is a homomorphism on $R_{12} \oplus R_{23} \oplus R_{13}$. Suppose there are $x_{13} \in R_{13}$, $x_{32} \in R_{32}$ such that $\phi(x_{13}x_{32})\neq\phi(x_{13})\phi(x_{32})$. By 2.11, $\phi(x_{13}x_{32})=-\phi(x_{32})\phi(x_{13})$. Let $x_{23} \in R_{23}$. Then

$$\phi((x_{13}x_{32})x_{23}) = \phi(x_{13}x_{32})\phi(x_{23}) = -\phi(x_{32})\phi(x_{13})\phi(x_{23}) = 0 \qquad \text{by 2.8.}$$

Since ϕ is an injection, $x_{13}x_{32}x_{23}=0$. Hence $x_{13}x_{32}R_{23}=0$, so $x_{13}x_{32}=0$. Thus $\phi(x_{13}x_{32})=0$. On the other hand, $-\phi(x_{32})\phi(x_{13})=\phi(x_{13}x_{32})=\phi([x_{13}, x_{32}])=[\phi(x_{13}), \phi(x_{32})]$. So $\phi(x_{13})\phi(x_{32})=0$. But then $\phi(x_{13}x_{32})=\phi(x_{13})\phi(x_{32})$, a contradiction.

Now

$$\begin{aligned}\phi(x_{32})\phi(x_{13}) &= [\phi(x_{32}), \phi(x_{13})]+\phi(x_{13})\phi(x_{32})\\ &= \phi[x_{32}, x_{13}]+\phi(x_{13})\phi(x_{32})\\ &= -\phi(x_{13}x_{32})+\phi(x_{13}x_{32}) = 0 = \phi(0) = \phi(x_{32}x_{13}).\end{aligned}$$

ϕ is multiplicative on $x_{32}x_{12}$ and $x_{12}x_{32}$ by 2.8. The argument for the triple (2, 1, 3) is similar to the above.

3.5. Lemma. *Let x_{ij}, $y_{ij} \in R_{ij}$ and $x_{ji} \in R_{ji}$, $i\neq j$. Then $\phi(x_{ij}x_{ji}y_{ij})=\phi(x_{ij})\phi(x_{ji})\phi(y_{ij})$.*

Proof. In order to simplify notation, let $i=1$, and $j=2$. We will show that

$$[\phi(x_{12}x_{21}y_{12})-\phi(x_{12})\phi(x_{21})\phi(y_{12})][S, S] = 0.$$

The result will then follow from 2.4. By 2.10 it suffices to show

$$[\phi(x_{12}x_{21}y_{12})-\phi(x_{12})\phi(x_{21})\phi(y_{12})]\phi(y_{ij}) = 0 \quad \text{for } i \neq j.$$

(1) $\phi(x_{12}x_{21}y_{12})\phi(z_{12})-\phi(x_{12})\phi(x_{21})\phi(y_{12})\phi(z_{12}) = 0$ (by 2.8)

(2) $$\phi(x_{12}x_{21}y_{12})\phi(y_{13})-\phi(x_{12})\phi(x_{21})\phi(y_{12})\phi(y_{13}) = 0 \qquad \text{(by 2.8)}$$

(3)
$$\begin{aligned}\phi(x_{12}x_{21}y_{12})\phi(y_{23})-\phi(x_{12})\phi(x_{21})\phi(y_{12})\phi(y_{23})&\\ &= \phi(x_{12}x_{21}y_{12}y_{23})-\phi(x_{12})\phi(x_{21})\phi(y_{12}y_{23}) \qquad \text{(by 3.4)}\\ &= \phi(x_{12}x_{21}y_{12}y_{23})-\phi(x_{12})\phi(x_{21}y_{12}y_{23}) \qquad \text{(by 3.4)}\\ &= \phi(x_{12}x_{21}y_{12}y_{23})-\phi(x_{12}x_{21}y_{12}y_{23}) = 0 \qquad \text{(by 3.4).}\end{aligned}$$

(4) Letting $y_{21}=\sum_{i=1}^{n} y_{23}^{(i)}y_{31}^{(i)}$, we then have $\phi(y_{21})=\sum_{i=1}^{n} \phi(y_{23}^{(i)})\phi(y_{31}^{(i)})$ by 3.4. Thus

$$\phi(x_{12}x_{21}y_{12})\phi(y_{21})-\phi(x_{12})\phi(x_{21})\phi(y_{12})\phi(y_{21}) = 0 \qquad \text{(by part (3)).}$$

(5) $$\phi(x_{12}x_{21}y_{12})\phi(y_{31})-\phi(x_{12})\phi(x_{21})\phi(y_{12})\phi(y_{31}) = 0-0 = 0 \qquad \text{(by 3.4).}$$

(6) $$\phi(x_{12}x_{21}y_{12})\phi(y_{32})-\phi(x_{12})\phi(x_{21})\phi(y_{12})\phi(y_{32}) = 0 \qquad \text{(by 3.4).}$$

The key to extending ϕ to all of R is given by:

3.6. LEMMA. *Let (i, j, k) be any permutation of* $(1, 2, 3)$. *Suppose that*

$$\sum_{s=1}^{n} x_{ij}^{(s)}x_{ji}^{(s)} = \sum_{t=1}^{m} x_{ik}^{(t)}x_{ki}^{(t)},$$

then

$$\sum_{s=1}^{n} \phi(x_{ij}^{(s)})\phi(x_{ji}^{(s)}) = \sum_{t=1}^{m} \phi(x_{ik}^{(t)})\phi(x_{ki}^{(t)}).$$

Proof. To simplify notation, take $(i, j, k)=(1, 2, 3)$. As before, the method of proof will be to show:

$$\left[\sum_{s=1}^{n} \phi(x_{12}^{(s)})\phi(x_{21}^{(s)})-\sum_{t=1}^{m} \phi(x_{13}^{(t)})\phi(x_{31}^{(t)})\right][S, S] = 0.$$

By 2.10 it suffices to verify this for elements of $[S, S]$ of the form $\phi(y_{ij})$, $i\neq j$.

(1)
$$\begin{aligned}\sum_{s=1}^{n} \phi(x_{12}^{(s)})\phi(x_{21}^{(s)})\phi(y_{12})-\sum_{t=1}^{m} \phi(x_{13}^{(t)})\phi(x_{31}^{(t)})\phi(y_{12})&\\ &= \sum_{s=1}^{n} \phi(x_{12}^{(s)}x_{21}^{(s)}y_{12})-\sum_{t=1}^{m} \phi(x_{13}^{(t)})\phi(x_{31}^{(t)}y_{12}) \qquad \text{(by 3.5 and 3.4)}\\ &= \sum_{s=1}^{n} \phi(x_{12}^{(s)}x_{21}^{(s)}y_{12})-\sum_{t=1}^{m} \phi(x_{13}^{(t)}x_{31}^{(t)}y_{12}) \qquad \text{(by 3.4)}\\ &= \phi\left(\sum_{s=1}^{n} x_{12}^{(s)}x_{21}^{(s)}y_{12}-\sum_{t=1}^{m} x_{13}^{(t)}x_{31}^{(t)}y_{12}\right) \qquad \text{(since } \phi \text{ is additive)}\\ &= \phi(0) = 0.\end{aligned}$$

$$\sum_{s=1}^{n} \phi(x_{12}^{(s)})\phi(x_{21}^{(s)})\phi(y_{13})-\sum_{t=1}^{m} \phi(x_{13}^{(t)})\phi(x_{31}^{(t)})\phi(y_{13})$$

(2)

$$= \sum_{s=1}^{n} \phi(x_{12}^{(s)})\phi(x_{21}^{(s)}x_{13})-\sum_{t=1}^{m} \phi(x_{13}^{(t)}x_{31}^{(t)}y_{13}) \qquad \text{(by 3.5 and 3.4)}$$

$$= \sum_{s=1}^{n} \phi(x_{12}^{(s)}x_{21}^{(s)}x_{13})-\sum_{t=1}^{m} \phi(x_{13}^{(t)}x_{31}^{(t)}y_{13}) \qquad \text{(by 3.4)}$$

$$= \phi(0) = 0 \qquad \text{(since } \phi \text{ is additive).}$$

(3) $$\sum_{s=1}^{n} \phi(x_{12}^{(s)})\phi(x_{21}^{(s)})\phi(y_{21})-\sum_{t=1}^{m} \phi(x_{13}^{(t)})\phi(x_{31}^{(t)})\phi(y_{21}) = 0-0 = 0 \qquad \text{(by 2.8).}$$

(4) $$\sum_{s=1}^{n} \phi(x_{12}^{(s)})\phi(x_{21}^{(s)})\phi(y_{23})-\sum_{t=1}^{m} \phi(x_{13}^{(t)})\phi(x_{31}^{(t)})\phi(y_{23}) = 0-0 = 0 \qquad \text{(by 3.4).}$$

(5) The computations for y_{31} and y_{32} are the same as (4).

3.7. COROLLARY. *Suppose $i\neq j$ and $\sum_{s=1}^{n} x_{ij}^{(s)}x_{ji}^{(s)}=0$. Then $\sum_{s=1}^{n} \phi(x_{ij}^{(s)})\phi(x_{ji}^{(s)})=0$.*

Proof. Choose $x_{ik}^{(1)}=x_{ki}^{(1)}=0$, $k\neq i$, $k\neq j$, and $m=1$ in 3.6. The result is then immediate.

Corollary 3.7 gives the necessary information to allow us to extend ϕ to all of R. This is done as follows:

3.8. DEFINITION. Let ψ be the mapping of R into S defined by:

(1) For $x\in R_{ij}$, $i\neq j$, $\psi(x)=\phi(x)$.

(2) For $x\in R_{ii}$, let $x=\sum_{t=1}^{n} x_{ij}^{(t)}x_{ji}^{(t)}=\sum_{s=1}^{m} x_{ik}^{(s)}x_{ki}^{(s)}$ where (i,j,k) is a permutation of $(1,2,3)$. Then

$$\psi(x) = \sum_{t=1}^{n} \phi(x_{ij}^{(t)})\phi(x_{ji}^{(t)}) = \sum_{s=1}^{m} \phi(x_{ik}^{(s)})\phi(x_{ki}^{(s)}) \qquad \text{by 3.6.}$$

(3) For $x\in R$, let $x=\sum_{i,j=1}^{3} x_{ij}$ where $x_{ij}\in R_{ij}$. Then $\psi(x)=\sum_{i,j=1}^{3}\psi(x_{ij})$.

3.9. REMARKS. Part (2) of the definition gives a well defined mapping by 3.7. Part (3) is legitimate since the Peirce decomposition is a direct sum. The mapping ψ is an additive mapping of R into S since ϕ is additive on R_{ij}, $i\neq j$, and ψ is by its nature additive on R_{ii}. We hope to show that ψ is the desired extension of ϕ and that it is an associative isomorphism. We begin with,

3.10. LEMMA. *ψ is an extension of ϕ to R.*

Proof. We must show that $\psi|_{[R,R]}=\phi$. By 2.9 $[R, R]$ is additively generated by elements of the form x_{ij}, $i\neq j$, and by $x_{ik}x_{ki}-x_{ki}x_{ik}$ where $i\neq k$. Thus it suffices to check ψ on elements of this type.

(1) By definition, $\psi(x_{ij})=\phi(x_{ij})$ for $i\neq j$.

(2)

$$\begin{aligned}\psi(x_{ij}x_{ji}-x_{ji}x_{ij}) &= \psi(x_{ij}x_{ji})-\psi(x_{ji}x_{ij})\\ &= \phi(x_{ij})\phi(x_{ji})-\phi(x_{ji})\phi(x_{ij}) = \phi(x_{ij}x_{ji}-x_{ji}x_{ij}),\end{aligned}$$

by the definition of ψ.

3.11. Lemma. *ψ is a homomorphism of R into S.*

Proof. Since ψ is already known to be additive, it suffices to show that $\psi(x_{ij}x_{kl}) = \psi(x_{ij})\psi(x_{kl})$ $i, j, k, l = 1, 2, 3$.

(1) $i \neq j$, $k \neq l$, $j \neq k$. Then $\psi(x_{ij}x_{kl}) = \psi(0) = 0$. But then $\psi(x_{ij})\psi(x_{kl}) = \phi(x_{ij})\phi(x_{kl})$. But this product is zero by 3.4 if $i = l$, and by 2.8 if $i \neq l$.

(2) $i \neq j$, $k \neq l$, $j = k$. If $i = l$, then $\psi(x_{ij}x_{jl}) = \phi(x_{ij})\phi(x_{jl})$ by the definition of ψ. But $\phi(x_{ij})\phi(x_{jl}) = \psi(x_{ij})\psi(x_{jl})$. If $i \neq l$, then $\psi(x_{ij})\psi(x_{jl}) = \phi(x_{ij})\phi(x_{jl}) = \phi(x_{ij}x_{jl}) = \psi(x_{ij}x_{jl})$ by 3.4.

(3) $i = j$, $k \neq l$, $i \neq k$. By 3.9 we may assume $x_{ii} = x_{ik}x_{ki}$. Then

$$\psi(x_{ik}x_{ki})\psi(x_{kl}) = \phi(x_{ik})\phi(x_{ki})\phi(x_{kl}) = \phi(x_{ik})0 = 0 = \psi(0) = \psi((x_{ik}x_{ki})x_{kl}).$$

(4) $i = j$, $k \neq l$, $i = k$. We may assume $x_{ii} = x_{il}x_{li}$. Then

$$\begin{aligned}\psi(x_{il}x_{li})\psi(y_{il}) &= \phi(x_{il})\phi(x_{li})\phi(y_{il}) = \phi(x_{il})\phi(x_{li})\phi(y_{il}) \\ &= \phi(x_{il}x_{li}y_{il}) = \psi(x_{il}x_{li}y_{il}) = \psi((x_{il}x_{li})x_{kl})\end{aligned}$$

by 3.5.

(5) $i \neq j$, $k = l$. This case is handled exactly as cases (3) and (4).

(6) $i = j$, $k = l$, $i \neq k$. Then we may assume $x_{ii} = x_{ik}x_{ki}$ and $x_{kk} = y_{ki}y_{ik}$, then

$$\psi(x_{ik}x_{ki})\psi(y_{ki}y_{ik}) = \phi(x_{ik})\phi(x_{ki})\phi(y_{ki})\phi(y_{ik}) = 0 \qquad \text{(by 2.8)}.$$

But $\psi((x_{ik}x_{ki})(y_{ki}y_{ik})) = \psi(0) = 0$.

(7) $i = j$, $k = l$, $i = k$. In this case we may assume $x_{ii} = x_{ip}x_{pi}$ and $x_{kk} = y_{ip}y_{pi}$. Then

$$\begin{aligned}\psi(x_{ip}x_{pi})\psi(y_{ip}y_{pi}) &= \phi(x_{ip})\phi(x_{pi})\phi(y_{ip})\phi(y_{pi}) \\ &= \phi(x_{ip}x_{pi}y_{ip})\phi(y_{pi}) = \psi((x_{ip}x_{pi}y_{ip})y_{pi})\end{aligned}$$

by 3.5 and the definition of ψ.

3.12. Lemma. *ψ is an isomorphism of R onto S.*

Proof. All that is needed is to show that ψ is a bijection.

(1) ψ is an injection; ψ is nonzero, since ϕ is nonzero. The kernel of ψ is an ideal of the simple ring R. Hence ψ is one-to-one.

(2) ψ is a surjection: Since ψ is an extension of ϕ, $[S, S] \subseteq$ image of ψ. Since the image of a homomorphism is a subring, this gives $[S, S]^- \subseteq \text{im}\ \psi$. But $[S, S]^- = S$. Hence ψ is onto.

4. **The general case.**

Suppose that assumption 3.1 is false. Then in particular, we have by 2.11:

4.1. Assumption. There is $r_{12} \in R_{12}$, $r_{23} \in R_{23}$, such that $r_{12}r_{23} \neq 0$ and

$$\phi(r_{12}r_{23}) = -\phi(r_{23})\phi(r_{12}).$$

By using the opposite ring S^* of S, and the results of §3, we will prove the main result.

Let S^* denote the opposite ring of S. Let η denote the canonical anti-isomorphism of S onto S^*. Let $\nu = -\eta$. ν is the negative of an anti-isomorphism of S onto S^*, and hence ν is a Lie isomorphism of S onto S^*. It is clear that $\nu([S, S]) = [S^*, S^*]$. Let κ denote the restriction of ν to $[S, S]$. κ is a Lie isomorphism of $[S, S]$ onto $[S^*, S^*]$. Thus $\kappa \circ \phi$ is a Lie isomorphism of $[R, R]$ onto $[S^*, S^*]$. Furthermore, $\kappa \circ \phi(r_{12}r_{23}) = \kappa(-\phi(r_{23})\phi(r_{12})) = \kappa \circ \phi(r_{12})\kappa \circ \phi(r_{23})$. Thus $\kappa \circ \phi$ is a Lie isomorphism of $[R, R]$ onto $[S, S]$ satisfying 3.1. By 3.13, $\kappa \circ \phi$ can be extended to an associative isomorphism ξ of R onto S. Let $\psi = \nu^{-1} \circ \xi$. ψ is clearly the negative of an anti-isomorphism R onto S. We must show that ψ extends ϕ. Let $[r_1, r_2] \in [R, R]$.

$$\psi([r_1, r_2]) = \nu^{-1} \circ \xi([r_1, r_2]) = \nu^{-1} \circ \kappa \circ \phi[r_1, r_2] = \kappa^{-1} \circ \kappa \circ \phi[r_1, r_2] = \phi[r_1, r_2].$$

We have now completed the proof of:

4.2. Main Theorem. *Let R be a simple ring containing three nonzero orthogonal idempotents $\{e_i\}_{i=1}^3$ such that $1 = \sum_{i=1}^3 e_i$. Let S be a simple ring with 1. If ϕ is a Lie isomorphism of $[R, R]$ onto $[S, S]$, and the characteristic of R is not two or three, then ϕ can be extended to an additive bijection ψ of R onto S such that ψ is either an associative isomorphism or the negative of an anti-isomorphism of R onto S.*

We give now two corollaries of the main result. The first is the classical result, which is due to Landherr [4].

4.3. Corollary. *Let F_n be the $n \times n$ matrices over F, a field of characteristic 0. Let G_m be the $m \times m$ matrices over G, a field of characteristic 0. Suppose either $n \geqq 3$ or $m \geqq 3$. If ϕ is a Lie isomorphism of $[F_n, F_n]$ onto $[G_m, G_m]$ then ϕ may be extended to a mapping ψ of R onto S such that ψ is either an associative isomorphism or the negative of an anti-isomorphism of F_n onto G_m.*

Proof. The corollary follows immediately from 4.2, by using $e_i = e_{ii}$, the standard matrix units.

4.4. Corollary (Martindale [5]). *Let R, S be simple rings of characteristic not 2 or 3. Suppose R contains three orthogonal idempotents $\{e_i\}_{i=1}^3$ such that $\sum_{i=1}^3 e_i = 1$. Let ϕ be a Lie isomorphism of R onto S. Then $\phi = \sigma + \tau$, where σ is either an isomorphism of R onto S or the negative of an anti-isomorphism of R onto S, and τ is an additive mapping of R into $Z(S)$, the center of S, such that τ maps $[R, R]$ to zero.*

Proof. Let η be the restriction of ϕ to $[R, R]$. It is clear that $\eta[R, R] = [S, S]$, and so η is a Lie isomorphism of $[R, R]$ onto $[S, S]$. Let σ be the extension of η to R guaranteed by 4.2. Let $\tau = \phi - \sigma$. τ is an additive mapping of R into S, since both ϕ and σ are additive. Furthermore, $\tau[x, y] = (\phi - \sigma)([x, y]) = \phi[x, y] - \sigma[x, y] = \eta[x, y] - \eta[x, y] = 0$. It remains to show that τ maps R into the center of S. That is we wish to show that $[\tau(x), S] = 0$ for all $x \in R$. Since $[S, S]^- = S$, it suffices to show that $[\tau(x), [S, S]] = 0$ for all $x \in R$. But η is a surjection of $[R, R]$

onto $[S, S]$, hence it suffices to show $[\tau(x), \eta(y)]=0$ for all $x \in R$ and for all $y \in [R, R]$. But

$$\begin{aligned}[\tau(x), \eta(y)] &= [\phi(x)-\sigma(x), \eta(y)] = [\phi(x), \eta(y)]-[\sigma(x), \eta(y)] \\ &= [\phi(x), \phi(y)]-[\sigma(x), \sigma(y)] = \phi[x, y]-\sigma[x, y] \\ &= (\phi-\sigma)[x, y] = 0,\end{aligned}$$

since both ϕ and σ are Lie isomorphisms.

By definition, $\phi=\sigma+\tau$, and the proof is complete.

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