THE COHOMOLOGY OF THE COMPLEX PROJECTIVE STIEFEL MANIFOLD

BY

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0. Let $U_n$ be the unitary group of order $n$. We have inclusions

$$\cdots \supset U_n \supset U_{n-1} \supset \cdots U_1 \simeq S^1.$$ 

We denote by $W_{n,n-m}$ the complex Stiefel manifold $W_{n,n-m} = U_n/U_m$ and by $Y_{n,n-m}$ the complex projective Stiefel manifold which we defined as follows: $S^1$, regarded as the set of complex numbers of module 1, acts by multiplication on $U_n$. This action being compatible with the above inclusions defines an action of $S^1$ on $W_{n,n-m}$ and we define $Y_{n,n-m}$ as the set of orbits.

In particular we have

$$W_{n,1} = U_n/U_{n-1} \simeq S^{2n-1},$$
$$W_{n,n} = U_n,$$
$$Y_{n,n} = PU_n,$$ the projective unitary group.

In this paper we compute $(H^* Y_{n,m})$. Baum and Browder [1] have obtained our result in the special case $n = p^r, m = 0$.

In order to state our main result we need some notation: Let $\omega$ be the generator of $H^*(CP^\infty) = \mathbb{Z}[\omega]$ and $z_i$ the generators of $H^*(W_{n,n}) = \Lambda (z_{m+1}, \ldots, z_n)$. Let $b_i = \text{G.C.D.}(c_{n,m+i}, \ldots, c_{n,i})$. Finally in §1, we will show there is a fibration

$$W_{n,n-m} \xrightarrow{i} Y_{n,n-m} \xrightarrow{\pi} CP^\infty$$

with $Y_{n,n-m}$ of the same homotopy type as $Y_{n,n-m}$. Then our main theorem is

**Theorem A.**

$$H^*(Y_{n,n-m}) = \mathbb{Z}[y]/I \otimes \Lambda (v_{m+2}, \ldots, v_n),$$
where $\pi^* \omega = y$;

$i^* v_i = (b_{i-1}/b_i) z_i$ and $I$ is the ideal generated by $b_i y^i, i = m+1, \ldots, n.$

In §1 we compute $H^*(Y_{n,k}; \mathbb{Z}_p)$ and $H^*(Y_{n,k}; \mathbb{Q})$ following the Gitler and Handel proof for real case [3]. In §2 we determine $\ker \pi^*$ and $\text{Im } i^*$. In §3 we show that this information is enough to determine all relevant Bockstein homomorphisms.

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541
and compute the Bockstein spectral sequence for every prime $p$. This determines completely $H^*(Y_{n,k})$.

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1. First we construct a space $Y_{n,k}$ of the same homotopy type as $Y'_{n,k}$.

Observe that we have a principal bundle $S^1 \to W_{n,n-m} \to Y'_{n,n-m}$.

Let $\xi$ be the associate line bundle.

**Proposition 1.** $n\xi = \xi \oplus \xi \oplus \cdots \oplus \xi$ has $n-m$ $C$-linearly independent sections and it is the universal bundle for bundles $n\xi$ having $n-m$ $C$-linearly independent sections, where $\xi$ is a line bundle.

**Proof.** The proof is similar to the real case given in [3].

The inclusion $U_m \subset U_n$ gives rise to a fibration of classifying spaces

\[ W_{n,n-m} \longrightarrow BU_m \longrightarrow BU_n \]

and the transgression satisfies

\[ \tau z_i = \sigma_i \] where $\sigma_i$ is the universal Chern class.

Now let $\gamma$ be the canonical line bundle over $CP^\infty$ and let $f_n$ be the classifying map of $\gamma$ and let

\[ W_{n,n-m} \longrightarrow Y_{n,n-m} \longrightarrow CP^\infty \]

be the fibration induced from (A) by $f_n$.

It is easy to see that the bundle induced by $f_n$ is a universal bundle for the $n$-plane bundles satisfying the conditions of Proposition 1. Thus we have

**Proposition 2.** $Y_{n,n-m}$ and $Y'_{n,n-m}$ have the same homotopy type.

From (1) and naturality of Chern classes and transgression

\[ \tau z_i = C_{n,i}z_i. \]

If $x \in H^*(E)$ we denote by $\bar{x}$ (resp. $\bar{x}$) its image in $H^*(E; Z_p)$ (resp. $H^*(E; Q)$).

Let $N(p)$ be the smallest $i$ such that $m+1 \leq i < n$ and $C_{n,i}$ is not zero, mod $p$.

The following theorem is similar to [3, Theorem 1.6].

**Theorem 3.**

\[ H^*(Y_{n,n-m}; Z_p) = Z_p[y]/[y^{N(p)}] \otimes (\bar{x}_{m+1} \cdots \bar{x}_{N(p)} \cdots \bar{x}_n), \]

\[ H^*(Y_{n,n-m}; Q) = Q[y]/[y^{m+1}] \otimes (\bar{x}_{m+2} \cdots \bar{x}_n), \]

where $p$ is a prime and $Q$ is the set of rational numbers; $i*(\bar{x}_i) = \bar{z}_i$; $i*(\bar{\omega}) = \bar{y}$; and $i*(\bar{x}_i) = \bar{z}_i$, $i*(\bar{\omega}) = \bar{y}$. 

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Proof. Let $K$ be a field, $K = \mathbb{Z}_p$ or $K = \mathbb{Q}$ and let $N$ be the smallest $i$ such that $m+1 \leq i < n$ and $C_{i,i}$ is not zero in $K$.

In the spectral sequence of (B) with coefficients $K$ we have

$$E_2 = K[\omega] \otimes \wedge (z_{m+1} \cdots z_n).$$

By (2) $d_1z_t = 0$, $r < 2i$, $d_2z_t = \tau z_t = C_{n,i} \omega^t$, then $d_t = 0$, $r < 2N$ and then $E_2 = E_{2N}$; but

$$d_{2N}z_N = C_{n,N} \omega^N \neq 0,$$

thus, $z_N$ does not survive in $E_{2N+1}$ and the image of the ideal $[\omega^N]$ in $E_{2N+1}$ is 0. Moreover, $E_{2N+1}$ is still a tensor product:

$$E_{2N+1} \otimes E_{2N+1} = E_{2N+1}.$$

Now the following transgressions are zero (thus all the following differentials are zero), hence

$$E_\infty = E_{2N+1} = K[\omega]/[\omega^N] \otimes \wedge (z_{m+1} \cdots z_n).$$

The theorem now follows from Borel [2].

**Corollary 4.** $H^*(Y, n, m)$ has $p$-torsion if and only if $p$ divides $C_{n,m+1}$.

2. In this section we obtain the first results about $H^*(Y, n, k)$. The key is Proposition 5 below on $\ker \pi^*$. In Corollary 7 we pick some elements in $H^*(Y, n, k)$ and using them we choose new generators for $H^*(Y, n, k; \mathbb{Z}_p)$ and $H^*(Y, n, k; \mathbb{Q})$ ((7), (7')).

**Proposition 5.**

$$\ker \pi^* = [b_{m+1} \omega^{m+1}, \ldots, b_n \omega^n].$$

**Corollary 6.**

$$\ker \bar{\pi}^* = [b_{m+1} \omega^{m+2}, \ldots, b_n \omega^{n+1}].$$

Let $c_i = b_{i-1}/b_i$, $i = m+2, \ldots, n$.

**Corollary 7.**

$$T^{2m+1}/\text{Im} \ i^* = \mathbb{Z}, \quad T^q/\text{Im} \ i^* = 0, \quad q \geq 2n,$$

$$T^{2i-1}/\text{Im} \ i^* = \mathbb{Z}_{c_i}, \quad m+2 \leq i \leq n.$$

Thus, there exist elements $v_i$ in $H^*(Y, n, m)$ such that $i^*v_i = c_i z_t$, $m+2 \leq i \leq n$.

Before we give the proof we recall some facts about transgression in the spectral sequence of a fibration.

We use the transgression in (B) as defined in the following diagram, $\tau = j^* \circ (\bar{\pi}^*)^{-1} \circ \delta$

$$H^*(Y, n, m) \xrightarrow{i^*} H^*(W, n, m) \xrightarrow{\delta} H^*(Y, n, m; W, n, m) \xrightarrow{\pi^*} H^*(Y, n, m; *) \xrightarrow{\pi^*} H^*(Y, n, m; \mathbb{Q})$$

$$H^*(\mathbb{C}P^\infty, *) \xrightarrow{j^*} H^*(\mathbb{C}P^\infty).$$
We recall that in the spectral sequence of (B) \( E_2^{0,2i-1} \approx H^{2i-1}(W_{n,n-m}) \). This isomorphism carries the subgroup \( E_2^{0,2i-1} \) onto \( T^{2i-1} \), the subgroup of transgressive elements. Also \( E_2^{2i,0} \approx H^{2i}(CP^\infty) \) and this isomorphism induces the isomorphism of quotient groups \( E_2^{2i,0} \approx H^{2i}(CP^\infty)/\ker 2^{2i} \pi^* \). Moreover, via these isomorphisms, \( \tau \) corresponds to \( d_2^{0,2i-1} \) and

\[
\text{Im } d_2^{0,2i-1} \approx \ker 2^{2i} \pi^*/\ker 2^{2i} \pi^* \subset H^{2i}(CP^\infty)/\ker 2^{2i} \pi^*.
\]

Finally, \( \tau \) induces an isomorphism

\[
T^{2i-1}/\text{Im } i^* \approx \ker 2^{2i} \pi^*/\ker 2^{2i} \pi^*.
\]

Consider the diagram:

\[
\begin{array}{ccc}
H^*(W_{n,n-m}) & \overset{i^*}{\leftarrow} & H^*(Y_{n,n-m}) & \overset{\pi^*}{\leftarrow} & H^*(CP^\infty) \\
\downarrow \theta & & \downarrow \theta & & \downarrow \theta \\
H^*(W_{n,n-m} \cap A) & \overset{i_A^*}{\leftarrow} & H^*(Y_{n,n-m} \cap A) & \overset{\pi_A^*}{\leftarrow} & H^*(CP^\infty \cap A).
\end{array}
\]

If \( A = \mathbb{Z}_p \), then Theorem 2 and \( i^* \circ \pi^* = 0 \) yield

\[
\text{(5)} \quad \ker i_2^* \approx [\bar{y}],
\]

\[
\text{(6)} \quad \ker \pi_2^* \approx [\omega^{n(p)}].
\]

If \( A = \mathbb{Q} \), we have

\[
\text{(5') } \quad \ker i_q^* \approx [\bar{y}],
\]

\[
\text{(6') } \quad \ker \pi_q^* \approx [\omega^{n+1}].
\]

**Proof of Proposition 5.** The spectral sequence of (B) is trivial through \( E_{2m+2} \). Thus \( E_{2m+2}^{2m,0} \approx H^{2m+2}(CP^\infty) \) and \( \ker 2^{m+2} \pi^* = 0, q \leq 2m+2 \).

From (2) and (3) we obtain

\[
\ker 2^{2m+2} \pi^* = [C_{n,m+1} \omega^{m+1}]^{2m+2}.
\]

Applying (2) repeatedly we have

\[
[b_m, \omega^k, \ldots, b_n \omega^k] \subset \ker \pi^* \subset H^*(CP^\infty).
\]

For the other inclusion, put \( b_i = ap^s \), where \( p \) does not divide \( a \). We use diagram (C) with \( A = \mathbb{Z}_{pf} \), if \( p^s \omega^t \) belongs to \( \ker \pi^* \), \( s < r \) and \( c \) divides \( a \), then \( p^s c \theta(\omega)^t \) belongs to \( \ker \pi_{2r}^* \) but it is not 0 because \( c \) is not a divisor of 0 in \( \mathbb{Z}_{pf} \). On the other hand in the spectral sequence of (B) with coefficients \( \mathbb{Z}_{pf} \)

\[
\tau_{Z_k} = C_{n,k} \omega^k = 0, \quad k < i,
\]

because \( p^s \) divides \( C_{n,k} \) for those \( k \). Thus, the spectral sequence is trivial through \( 2i \) and then \( \ker 2^{2i} \pi_{2r}^* = 0 \).
Proof of Corollary 6. Follows from (3) and Proposition 5.

Proof of Corollary 7. First part follows from (4), second part follows trivially from first part.

The elements $v_i$ are not unique, we will choose a fixed set of such elements arbitrarily.

In diagram (C) with $Z_p = A$, we have:

(i) if $p$ does not divide $c_i$, $\theta v_i = c_i \bar{x}_i + u_i$, $m + 2 \leq i \leq n$ where $u_i \in \text{Ker } i^{\infty}_p$;
(ii) if $p$ does divide $c_i$, then $\theta v_i \in \text{Ker } i^{\infty}_p$.

Let $I$ be $I = \{i; p$ does not divide $c_i, m + 2 \leq i \leq n\}$. Let $J = \{j; j \notin I, m + 1 \leq j \leq n\}$. Then $I = \{i; b_i$ is divided by the same power of $p$ as $b_{i-1}\}$.

The important situation occurs when $i$ belongs to $J$. For example, $m + 1$ and $N(p)$ are the smallest and the greatest elements of $J$.

We will change the generators of $H^*(Y_{n,n-m}; Z_p)$ to the following

$$c_i \bar{x}_i = \theta v_i, \quad i \in I,$$
$$x_i = \bar{x}_i, \quad i \in J, i \neq N(p);$$

then we obtain:

$$H^*(Y_{n,n-m}; Z_p) = Z_p[\bar{y}] /[\bar{y}^{N(p)}] \otimes \wedge (x_{m+1}, \ldots, \bar{x}_{N(p)}, \ldots, x_n)$$

where

(7) $\theta v_i = c_i \bar{x}_i, \quad i \in I; \quad i^{\infty}_p \bar{x}_i = \bar{z}_i, \quad m + 1 \leq i \leq n, \quad i \neq N(p)$ and $\pi^{\infty}_p(\bar{\omega}) = \bar{y}$.

Again in diagram (C), this time with $A = Q$, we have

$$\theta v_i = c_i \bar{x}_i + u_i, \quad u_i \in \text{Ker } i^p, \quad m + 2 \leq i \leq n.$$

We define $c_i w_i = \theta v_i, m + 2 \leq i \leq n$ and we obtain

(7') $H^*(Y_{n,n-m}; Q) = Q[\bar{y}] /[\bar{y}^{m+1}] \otimes \wedge (w_{m+2}, \ldots, w_n)$

where $\theta v_i = c_i w_i, \quad i^{\infty}_p w_i = \bar{z}_i, \quad \pi^p(\bar{\omega}) = \bar{y}$.

3. Next, we compute the Bockstein spectral sequence of the couple

$$H^*(Y_{n,n-m}) \xrightarrow{(p)^*} H^*(Y_{n,n-m})$$

$$\delta \quad \theta$$

$H^*(Y_{n,n-m}; Z_p)$

It follows from (7) that

$$E_1 = H^*(Y_{n,n-m}; Z_p) = Z_p[y]/[y^{N(p)}] \otimes \wedge (x_{m+1}, \ldots, \bar{x}_{N(p)}, \ldots, x_n)$$

and from (7') that

$$E_\infty = H^*(Y_{n,n-m})/\text{torsion} \otimes Z_p = Z_p[y]/[y^{m+1}] \otimes \wedge (w_{m+2}, \ldots, w_n).$$
Recall that the differentials are Bockstein homomorphisms \( \beta \), and an element \( x \in E_1 \), belongs to \( \text{Im} \theta \) if and only if

\[
\beta_r x = 0 \quad \text{for all } r.
\]

An element \( y \in H^*(Y_{n,n-m}) \) has torsion \( p^r \), that is \( p^r ay = 0 \) where \( p \) does not divide \( a \), if and only if \( \theta y \notin \text{Im} \beta_j \), for \( j < r \), but

\[
\theta y \in \text{Im} \beta_r.
\]

First we will give some easy results:

If \( x \in E_r \), call \( \phi(x) \) its image in \( E_\infty \), then

\[
\phi(\tilde{y}) = \tilde{y}; \quad \phi(x_i) = w_i, \quad i \in I.
\]

By (7) and (8)

\[
\beta_r(\tilde{y}) = 0, \quad \beta_r(x_i) = 0, \quad \text{all } r, i \in I.
\]

By (10), since \( w_i \neq 0, \)

\[
x_i \notin \text{Im} \beta_r, \quad \text{all } r, i \in I.
\]

We arrange \( J \) so that \( m+1 = i(0) < i(1) < \cdots < i(j) < \cdots < i(t) = N(p) \) and put \( b_i = p^{\alpha(i)} a_i \), where \( p \) does not divide \( a_i \); then \( r(j) > r(j+1) \) and \( b_i = p^{\alpha(i)} a_i, \quad i(j) \leq i < i(j+1). \)

By (9) and Proposition 5:

\[
y^i \notin \text{Im} \beta_r, \quad r < r(j), \quad y^i \in \text{Im} \beta_{r(j)}, \quad i(j) \leq i < i(j+1).
\]

Trivially

\[
E^q_i = E^q_{r(j)}, \quad q < 2i(0) - 1.
\]

Now, we will compute \( \beta_r \):

**Lemma 8.** The following formulae hold for every \( j \)

\[
\beta_r x_{i(j)} = 0, \quad r < r(j),
\]

\[
\beta_r x_{i(j)} = k_j y_i^{i(j)}, \quad k_j \in \mathbb{Z}_p, \quad k_j \neq 0,
\]

\[
E^q_i = E^q_{r(j)}, \quad q < 2i(j+1) - 1.
\]

**Proof.** By (13) there is an element \( x \) such that \( \beta_{r(j)} x = y_i^{i(j)} \) but \( x \) can only be a scalar multiple of \( x_{i(0)} \), then (15) and (16) hold for \( j = 0 \).

By the same argument (15) and (16) hold for \( j = h \) provided that (17) holds for \( j = h - 1 \).

In turn, (15) for every \( j \leq h \) and (11) together imply (17) for \( j = h \) because Bockstein homomorphisms are derivations.

**Corollary 9.** For every \( j \)

\[
\beta_{r(j)}(x_{i(j)} y^s) = k_j y_i^{i(j) + s} \neq 0, \quad 0 \leq s < i(j+1) - i(j),
\]

\[
\beta_{r(j)}(x_{i(j)} y_i^{i(j) - i(j)}) = 0.
\]
Proof. (18) follows from (16) and (17). (19) follows from (16).

We call \( u_{i(t+1)} \) the image of \( x_{i(t)}y^{j(t+1) - j(t)} \) in \( E_{r(t+1)} \).

It remains to prove that \( \beta_r = 0 \) unless \( r = r(j) \) for some \( j \). This is part of the following lemma.

**Lemma 10.** \( \beta_r = 0 \) unless \( r = r(j) \) and \( E_r = E_r(0) \).

We use induction. Assign \( \bar{y} \) to \( \bar{y} \) and \( w_i \) to \( x_i \) for \( i \in I \), \( i < i(1) \).

By (15), ..., (19) and \( \dim E_\alpha \leq E_{r(0)} \), this correspondence determines an isomorphism from \( E_\alpha \) onto \( E_{r(0)} \), up to degree \( 2i(1) - 2 \).

Moreover, \( \beta_t = 0 \) up to degree \( 2i(1) - 2 \) unless \( r = r(0) \).

Suppose we have elements \( \bar{u}_{i(j)} \), \( j = 1, \ldots, h \), such that:

1. \( \gcd \bar{u}_{i(j)} = 2i(j) - 1 \).
2. \( \bar{u}_{i(j)}x_i = -x_i\bar{u}_{i(j)} ; \bar{u}_{i(j)}u_{i(j+1)} = -\bar{u}_{i(j)'}\bar{u}_{i(j)} \), \( j' < j \); \( (\bar{u}_{i(j)})^2 = 0 \).
3. If we assign \( \bar{y} \) to \( \bar{y} \); \( w_i \) to \( x_i \) for \( i \in I \), \( i < i(j+1) \) and \( w_{i(t)} \) to \( \bar{u}_{i(t)} \) we determine an isomorphism from \( E_\alpha \) onto \( E_{r(0)} \) up to degree \( 2i(h+1) - 2 \).

Suppose besides that \( \beta_t = 0 \) up to degree \( 2i(h+1) - 2 \) unless \( r = r(j) \), \( j = 0, \ldots, t-1 \).

From these assumptions and (15), ..., (19) we have \( \dim E_{r(0)} = \dim E_{r(0)}^2 \), \( q \leq 2i(h+1) - 1 \) and all differentials are determined on all elements of degree \( \leq 2i(h+1) - 1 \) except \( u_i(h+1) \) belonging to \( E_{r(h+1)} \) and its images in \( E_r \), \( r > r(h+1) \).

Thus, for every \( r \), \( \beta_t u_{i(h+1)} \) must lie in the subspace of \( E_r \), spanned by \( (\beta_t a) \), where \( a \) ranges over products. That means \( \beta_t u_i(h+1) = 0 \) for \( r < r(h+1) \); and there is an element \( u' \) in \( E_{r(h+1)} \), such that \( \beta_r u_{i(h+1)} = 0 \) and \( u' \) does not belong to the subalgebra generated by elements with degree \( < 2i(h+1) - 1 \). It is easy to see that \( u' \) satisfies (ii). Again, \( \beta_t u' = 0 \) for \( r < r(h-2) \) and we repeat the argument until we reach \( E_{r(0)} \), then we obtain an element \( \bar{u}_{i(h+1)} \) in \( E_{r(0)} \) to which we may assign \( w_{i(h+1)} \).

Now we assign \( w_i \) to \( x_i \) for \( i \in I \), \( i < i(h+2) \) and obtain an isomorphism up to degree \( 2i(h+2) - 1 \). Then we have finished the proof of (i), (ii), (iii) with \( j = h+1 \). From (11) we see \( \beta_t = 0 \) up to degree \( 2i(h+2) - 1 \), unless \( r = r(j) \) for \( j = 0, \ldots, t-1 \). This completes the proof.

We have identified \( H^*(Y_{n,n-\alpha}; \mathbb{Q}) \) with \( E_{r(0)} \) as algebras, for every prime \( p \). Then we have completed the proof of Theorem A.

**References**


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