THE COHOMOLOGY OF THE COMPLEX
PROJECTIVE STIEFEL MANIFOLD

BY

CARLOS ALFREDO RUIZ(1)

0. Let $U_n$ be the unitary group of order $n$. We have inclusions

$$\cdots \supset U_n \supset U_{n-1} \supset \cdots U_1 \approx S^1.$$ 

We denote by $W_{n,n-m}$ the complex Stiefel manifold $W_{n,n-m} = U_n/U_m$ and by $Y_{n,n-m}$ the complex projective Stiefel manifold which we defined as follows: $S^1$, regarded as the set of complex numbers of module 1, acts by multiplication on $U_n$. This action being compatible with the above inclusions defines an action of $S^1$ on $W_{n,n-m}$ and we define $Y_{n,n-m}$ as the set of orbits.

In particular we have

$$W_{n,1} = U_n/U_{n-1} \approx S^{2n-1},$$

$$W_{n,n} = U_n,$$

$$Y_{n,n} = PU_n,$$ the projective unitary group.

In this paper we compute $(H^* Y_{n,n-m})$. Baum and Browder [1] have obtained our result in the special case $n=p^r, m=0$.

In order to state our main result we need some notation: Let $\omega$ be the generator of $H^*(CP^\infty) = \mathbb{Z}[\omega]$ and $z_i$ the generators of $H^*(W_{n,i}) = \bigwedge (z_{m+1}, \ldots, z_n)$. Let $b_i = \text{G.C.D.}(\langle C_{n,m+1}, \ldots, C_{n,i} \rangle)$. Finally in §1, we will show there is a fibration

$$W_{n,n-m} \xrightarrow{i} Y_{n,n-m} \xrightarrow{\pi} CP^\infty$$

with $Y_{n,n-m}$ of the same homotopy type as $Y_{n,n-m}$. Then our main theorem is

**Theorem A.**

$$H^*(Y_{n,n-m}) = \mathbb{Z}[y]/I \otimes \bigwedge (v_m+2, \ldots, v_n), \text{ where } \pi^* \omega = y;$$

$i^*v_i = (b_{i-1}/b_i)z_i$ and $I$ is the ideal generated by $b_iy^i, i = m+1, \ldots, n$.

In §1 we compute $H^*(Y_{n,k}; \mathbb{Z}_p)$ and $H^*(Y_{n,k}; \mathbb{Q})$ following the Gitler and Handel proof for real case [3]. In §2 we determine Ker $\pi^*$ and Im $i^*$. In §3 we show that this information is enough to determine all relevant Bockstein homomorphisms.

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and compute the Bockstein spectral sequence for every prime $p$. This determines completely $H^*(Y_{n,k})$.

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1. First we construct a space $Y_{n,k}$ of the same homotopy type as $Y'_{n,k}$.

Observe that we have a principal bundle $S^1 \to W_{n,n-m} \to Y'_{n,n-m}$.

Let $\xi$ be the associate line bundle.

**Proposition 1.** $\xi = \xi \oplus \xi \oplus \cdots \oplus \xi$ has $n-m$ $C$-linearly independent sections and it is the universal bundle for bundles $n$ having $n-m$ $C$-linearly independent sections, where $\xi$ is a line bundle.

**Proof.** The proof is similar to the real case given in [3].

The inclusion $U_m \subset U_n$ gives rise to a fibration of classifying spaces

\[(A) \quad W_{n,n-m} \longrightarrow BU_m \longrightarrow BU_n \]

and the transgression satisfies

\[(1) \quad \tau \bar{z}_i = \sigma_i \quad \text{where } \sigma_i \text{ is the universal Chern class.} \]

Now let $\gamma$ be the canonical line bundle over $CP^\infty$ and let $f_n$ be the classifying map of $n\gamma$ and let

\[(B) \quad W_{n,n-m} \longrightarrow Y_{n,n-m} \longrightarrow CP^\infty \]

be the fibration induced from (A) by $f_n$.

It is easy to see that the bundle induced by $f_n\gamma$ is a universal bundle for the $n$-plane bundles satisfying the conditions of Proposition 1. Thus we have

**Proposition 2.** $Y_{n,n-m}$ and $Y'_{n,n-m}$ have the same homotopy type.

From (1) and naturality of Chern classes and transgression

\[(2) \quad \tau \bar{z}_i = C_{n,i} \omega_i \]

If $x \in H^*(E)$ we denote by $\bar{x}$ (resp. $\tilde{x}$) its image in $H^*(E; \mathbb{Z}_p)$ (resp. $H^*(E; \mathbb{Q})$).

Let $N(p)$ be the smallest $i$ such that $m+1 \leq i < n$ and $C_{n,i}$ is not zero, mod $p$.

The following theorem is similar to [3, Theorem 1.6].

**Theorem 3.**

\[H^*(Y_{n,n-m}; \mathbb{Z}_p) = \mathbb{Z}_p[y]/[y^{N(p)}] \otimes (\bar{x}_{m+1} \cdots \bar{x}_{N(p)}), \]

\[H^*(Y_{n,n-m}; \mathbb{Q}) = \mathbb{Q}[y]/[y^{m+1}] \otimes (\bar{x}_{m+2} \cdots \bar{x}_n), \]

where $p$ is a prime and $\mathbb{Q}$ is the set of rational numbers; $i^*(\bar{x}_i) = \bar{z}_i; \quad i^*(\bar{\omega}) = \bar{\gamma};$ and $i^*(\tilde{x}_i) = \tilde{z}_i, \quad i^*(\tilde{\omega}) = \tilde{\gamma}$. 
Proof. Let $K$ be a field, $K=\mathbb{Z}_p$ or $K=\mathbb{Q}$ and let $N$ be the smallest $i$ such that $m+1 \leq i < n$ and $C_{n,i}$ is not zero in $K$.

In the spectral sequence of (B) with coefficients $K$ we have

$$E_2 = K[\omega] \otimes \wedge (z_{m+1} \cdots z_n).$$

By (2) $d_1 z_i = 0$, $r < 2i$, $d_2 z_i = \tau z_i = C_{n,i} \omega^{\tau}$, then $d_r = 0$, $r < 2N$ and then $E_2 = E_{2N}$; but

$$d_{2N} z_N = C_{n,N} \omega^N \neq 0,$$

thus, $z_N$ does not survive in $E_{2N+1}$ and the image of the ideal $[\omega^N]$ in $E_{2N+1}$ is 0. Moreover, $E_{2N+1}$ is still a tensor product:

$$E_{2N+1} \otimes E_{2N+1} = E_{2N+1}^*.$$

Now the following transgressions are zero (thus all the following differentials are zero), hence

$$E_\infty = E_{2N+1} = K[\omega]/[\omega^N] \otimes \wedge (z_{m+1} \cdots z_n).$$

The theorem now follows from Borel [2].

Corollary 4. $H^*(Y_{n,m})$ has $p$-torsion if and only if $p$ divides $C_{n,m+1}$.

2. In this section we obtain the first results about $H^*(Y_{n,k})$. The key is Proposition 5 below on $\text{Ker} \ \pi^*$. In Corollary 7 we pick some elements in $H^*(Y_{n,k})$ and using them we choose new generators for $H^*(Y_{n,k}; \mathbb{Z}_p)$ and $H^*(Y_{n,k}; \mathbb{Q})$ ((7), (7')).

Proposition 5.

$$\text{Ker} \ \pi^* = [b_{m+1} \omega^{m+1}; \ldots, b_n \omega^n].$$

Corollary 6.

$$\text{Ker} \ \tilde{\pi}^* = [b_{m+1} \omega^{m+2}; \ldots, b_n \omega^{n+1}].$$

Let $c_i = b_i - 1/b_i$, $i = m+2, \ldots, n$.

Corollary 7.

$$T^{2m+1}/\text{Im} \ i^* = \mathbb{Z}, \quad T^q/\text{Im} \ i^* = 0, \quad q \geq 2n,$$

$$T^{2i-1}/\text{Im} \ i^* = \mathbb{Z}_{c_i}, \quad m+2 \leq i \leq n.$$
We recall that in the spectral sequence of (B) $E_2^{0,2i-1} \approx H^{2i-1}(W_{n,n-m})$. This isomorphism carries the subgroup $E_2^{0,2i-1}$ onto $T^{2i-1}$, the subgroup of transgressive elements. Also $E_2^{0,0} \approx H^0(CP^\infty)$ and this isomorphism induces the isomorphism of quotient groups $E_2^{0,0} \approx H^0(CP^\infty)/\ker 2^i \pi^*$. Moreover, via these isomorphisms, $\tau$ corresponds to $d_2^{0,2i-1}$ and

$$\text{im } d_2^{0,2i-1} \approx \ker 2^i \pi^*/\ker 2^i \pi^* \subset H^2i(CP^\infty)/\ker 2^i \pi^*.$$  

Finally, $\tau$ induces an isomorphism

$$T^{i-1}/\text{im } i^* \approx \ker^i \pi^*/\ker^i \pi^*.$$ 

Consider the diagram:

$$
\begin{array}{ccc}
H^*(W_{n,n-m}) & \xleftarrow{i^*} & H^*(Y_{n,n-m}) \\
\downarrow{\theta} & & \downarrow{\theta} \\
H^*(W_{n,n-m}; A) & \xleftarrow{i^*_A} & H^*(Y_{n,n-m}; A) \\
\end{array}
$$

(C)

$$
\begin{array}{ccc}
H^*(W_{n,n-m}) & \xleftarrow{\pi^*} & H^*(Y_{n,n-m}) \\
\downarrow{\theta} & & \downarrow{\theta} \\
H^*(W_{n,n-m}; A) & \xleftarrow{\pi^*_A} & H^*(Y_{n,n-m}; A) \\
\end{array}
$$

If $A = \mathbb{Z}_p$, then Theorem 2 and $i^* \circ \pi^* = 0$ yield

(5) \hspace{1cm} \text{Ker } i^*_A = [\bar{y}], \\
(6) \hspace{1cm} \text{Ker } \pi^*_A = [\omega^{N(p)}].$

If $A = \mathbb{Q}$, we have

(5') \hspace{1cm} \text{Ker } i^*_A = [\bar{y}], \\
(6') \hspace{1cm} \text{Ker } \pi^*_A = [\omega^{m+1}].$

**Proof of Proposition 5.** The spectral sequence of (B) is trivial through $E_{2m+2}$. Thus $E_2^{2m+2,0} \approx H^{2m+2}(CP^\infty)$ and $\ker^q \pi^* = 0, q \leq 2m + 2$. From (2) and (3) we obtain

$$\text{Ker }^q \pi^* = 0, \quad q < 2m + 2 \quad \text{and} \quad \text{Ker }^{2m+2} \pi^* = [C_{n,m+1}\omega^{m+1}]^{2m+2}.$$ 

Applying (2) repeatedly we have

$$[b_{m+1}\omega^{m+1}, \ldots, b_n\omega^n] \subset \text{Ker } \pi^* \subset H^*(CP^\infty).$$

For the other inclusion, put $b_i = ap^r$, where $p$ does not divide $a$. We use diagram (C) with $A = \mathbb{Z}_p$, if $p^r\omega^s$ belongs to Ker $\pi^*$, $s < r$ and $c$ divides $a$, then $p^r\omega^s\theta(\omega)^r$ belongs to Ker $\pi^*_A$, but it is not 0 because $c$ is not a divisor of 0 in $\mathbb{Z}_p$. On the other hand in the spectral sequence of (B) with coefficients $\mathbb{Z}_p$, $p^r \omega^k = C_{n,k}\omega^k = 0, \quad k < i,$

because $p^r$ divides $C_{n,k}$ for those $k$. Thus, the spectral sequence is trivial through $2i$ and then Ker$^{2i} \pi^*_A = 0$. 

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Proof of Corollary 6. Follows from (3) and Proposition 5.

Proof of Corollary 7. First part follows from (4), second part follows trivially from first part.

The elements $v_i$ are not unique, we will choose a fixed set of such elements arbitrarily.

In diagram (C) with $Z_p = A$, we have:

(i) if $p$ does not divide $c_i$, $\theta v_i = c_i x_i + u_i$, $m + 2 \leq i \leq n$ where $u_i \in \text{Ker } i^*_p$;

(ii) if $p$ does divide $c_i$, then $\theta v_i \in \text{Ker } i^*_p$.

Let $I$ be $I = \{i; p \text{ does not divide } c_i, m + 2 \leq i \leq n\}$. Let $J = \{j; j \notin I, m + 1 \leq j \leq n\}$. Then $I = \{i; b_i \text{ is divided by the same power of } p \text{ as } b_{i-1}\}$.

The important situation occurs when $i$ belongs to $J$. For example, $m + 1$ and $N(p)$ are the smallest and the greatest elements of $J$.

We will change the generators of $H^*(Y_{n,n-m}; Z_p)$ to the following

$$c_i x_i = \theta v_i, \quad i \in I,$$
$$x_i = \bar{x}_i, \quad i \in J, i \neq N(p);$$

then we obtain:

$$H^*(Y_{n,n-m}; Z_p) = Z_p[Y]/[Y^{N(p)}] \otimes \wedge (x_{m+1}, \ldots, x_{N(p)}, \ldots, x_n)$$

where

$$\theta v_i = c_i x_i, \quad i \in I; \quad i^*_p x_i = \bar{z}_i, \quad m+1 \leq i \leq n, \quad i \neq N(p) \quad \text{and} \quad \pi^*_p(\bar{\omega}) = \bar{y}.$$

Again in diagram (C), this time with $A = Q$, we have

$$\theta v_i = c_i w_i, \quad m+2 \leq i \leq n.$$

We define $c_i w_i = \theta v_i$, $m + 2 \leq i \leq n$ and we obtain

$$H^*(Y_{n,n-m}; Q) = Q[Y]/[Y^{m+1}] \otimes \wedge (w_{m+2}, \ldots, w_n)$$

where

$$\theta v_i = c_i w_i, \quad i^*_p w_i = \bar{z}_i, \quad \pi^*_p(\bar{\omega}) = \bar{y}.$$

3. Next, we compute the Bockstein spectral sequence of the couple

$$H^*(Y_{n,n-m}) \xrightarrow{(p)^*} H^*(Y_{n,n-m})$$

$$\delta \quad \theta$$

$$H^*(Y_{n,n-m}; Z_p)$$

It follows from (7) that

$$E_1 = H^*(Y_{n,n-m}; Z_p) = Z_p[Y]/[Y^n] \otimes \wedge (x_{m+1} \cdots \bar{x}_{N(p)} \cdots x_n)$$

and from (7') that

$$E_\infty = H^*(Y_{n,n-m})/\text{Torsion} \otimes Z_p = Z_p[Y]/[Y^{m+1}] \otimes (w_{m+2}, \ldots, w_n).$$
Recall that the differentials are Bockstein homomorphisms $\beta$, and an element $x \in E_1$, belongs to $\text{Im } \theta$ if and only if
\begin{equation}
\beta_r x = 0 \text{ for all } r.
\end{equation}

An element $y \in H^*(Y_{n,n-m})$ has torsion $p^r$, that is $p^*ay = 0$ where $p$ does not divide $a$, if and only if $\theta y \notin \text{Im } \beta_r$ for $j < r$, but
\begin{equation}
\theta y \in \text{Im } \beta_r.
\end{equation}

First we will give some easy results:
If $x \in E_r$, call $\phi(x)$ its image in $E_\infty$, then
\begin{equation}
\phi(y) = \tilde{y}; \quad \phi(x_i) = w_i, \quad i \in I.
\end{equation}

By (7) and (8)
\begin{equation}
\beta_r(\tilde{y}) = 0, \quad \beta_i(x_i) = 0, \quad \text{all } r, i \in I.
\end{equation}

By (10), since $w_i \neq 0$,
\begin{equation}
x_i \notin \text{Im } \beta_r, \quad \text{all } r, i \in I.
\end{equation}

We arrange $J$ so that $m+1 = i(0) < i(1) < \cdots < i(j) < \cdots < i(t) = N(p)$ and put $b_{i(j)} = p^r a_j$, where $p$ does not divide $a_j$; then $r(j) > r(j+1)$ and $b_i = p^r a_k$, $i(j) \leq i < i(j+1)$.

By (9) and Proposition 5:
\begin{equation}
y^i \notin \text{Im } \beta_r, \quad r < r(j), \quad y^i \in \text{Im } \beta_{r(j)}, \quad i(j) \leq i < i(j+1).
\end{equation}

Trivially
\begin{equation}
E_q^r = E_q^a, \quad q < 2i(0) - 1.
\end{equation}

Now, we will compute $\beta_r$:

**LEMMA 8.** The following formulae hold for every $j$
\begin{align}
\beta_r x_{i(j)} &= 0, \quad r < r(j), \quad \text{(15)} \\
\beta_r x_{i(j)} &= k_j y^{i(j)}, \quad k_j \in \mathbb{Z}_p, k_j \neq 0, \quad \text{(16)} \\
E_q^r &= E_q^{a(j)}, \quad q < 2i(j+1) - 1. \quad \text{(17)}
\end{align}

**Proof.** By (13) there is an element $x$ such that $\beta_{r(0)} x = y^{i(0)}$ but $x$ can only be a scalar multiple of $x_{i(0)}$, then (15) and (16) hold for $j = 0$.

By the same argument (15) and (16) hold for $j = h$ provided that (17) holds for $j = h - 1$.

In turn, (15) for every $j \leq h$ and (11) together imply (17) for $j = h$ because Bockstein homomorphisms are derivations.

**COROLLARY 9.** For every $j$
\begin{align}
\beta_r x_{i(j)} y^s &= k_j y^{i(j)+s} \neq 0, \quad 0 \leq s < i(j+1) - i(j), \quad \text{(18)} \\
\beta_r x_{i(j)} y^{i(j)+s} &= 0. \quad \text{(19)}
\end{align}
Proof. (18) follows from (16) and (17). (19) follows from (16).

We call \( u_{i(l+1)} \) the image of \( x_{(l+1)} \) in \( E_{r(l+1)+1} \).

It remains to prove that \( \beta_r = 0 \) unless \( r = r(j) \) for some \( j \). This is part of the following lemma.

**Lemma 10.** \( \beta_r = 0 \) unless \( r = r(j) \) and \( E_r = E_r(0) \).

We use induction. Assign \( y \) to \( y \) and \( w_i \) to \( x_i \) for \( i \in I \), \( i < i(1) \).

By (15),..., (19) and \( \dim E_\alpha \leq E_{r(0)} \), this correspondence determines an isomorphism from \( E_\alpha \) onto \( E_{r(0)} \), up to degree \( 2i(1) - 2 \).

Moreover, \( \beta_r = 0 \) up to degree \( 2i(1) - 2 \) unless \( r = r(0) \).

Suppose we have elements \( \tilde{u}_{i(l)} \), \( j = 1, \ldots, h \), such that:

1. \( \text{gr} \tilde{u}_{i(l)} = 2i(l) - 1 \).
2. \( \tilde{u}_{i(l)} x_i = -x_i \tilde{u}_{i(l)} \); \( \tilde{u}_{i(l)} u_{i(l)} = -\tilde{u}_{i(l)} \tilde{u}_{i(l)}, j' < j \); \( (\tilde{u}_{i(l)})^2 = 0 \).
3. If we assign \( y \) to \( y \); \( w_i \) to \( x_i \) for \( i \in I \), \( i < i(j+1) \) and \( w_{i(l)} \) to \( \tilde{u}_{i(l)} \) we determine an isomorphism from \( E_\alpha \) onto \( E_{r(0)} \) up to degree \( 2i(h+1) - 2 \).

Suppose besides that \( \beta_r = 0 \) up to degree \( 2i(h+1) - 2 \) unless \( r = r(j), j = 0, \ldots, t-1 \).

From these assumptions and (15),..., (19) we have \( \dim E_{r(0)} = \dim E_{r(0)}^2, q \leq 2i(h+1) - 1 \) and all differentials are determined on all elements of degree \( \leq 2i(h+1) - 1 \) except \( u_k(h+1) \) belonging to \( E_{r(h+1)+1} \) and its images in \( E_r, r > r(h)+1 \).

Thus, for every \( r \), \( \beta_r u_{i(h+1)} \) must lie in the subspace of \( E_r \) spanned by \( \beta_r a \), where \( a \) ranges over products. That means \( \beta_r u_k(h+1) = 0 \) for \( r < r(h-1) \); and there is an element \( u' \) in \( E_{2i(h+1) - 1}^{r(h-1)+1} \), such that \( \beta_{r(h-1)} u' = 0 \) and \( u' \) does not belong to the subalgebra generated by elements with degree \( < 2i(h+1) - 1 \). It is easy to see that \( u' \) satisfies (ii). Again, \( \beta_r u' = 0 \) for \( r < r(h-2) \) and we repeat the argument until we reach \( E_{r(0)} \), then we obtain an element \( \tilde{u}_{i(h+1)} \) in \( E_{r(0)} \) to which we may assign \( w_i(h+1) \).

Now we assign \( w_i \) to \( x_i \) for \( i \in I \), \( i < i(h+2) \) and obtain an isomorphism up to degree \( 2i(h+2) - 1 \). Then we have finished the proof of (i), (ii), (iii) with \( j = h+1 \). From (11) we see \( \beta_r = 0 \) up to degree \( 2i(h+2) - 1 \), unless \( r = r(j) \) for \( j = 0, \ldots, t-1 \). This completes the proof.

We have identified \( H^*(Y_{n,n,n}; \mathbb{Q}) \) with \( E_{r(0)} \) as algebras, for every prime \( p \).

Then we have completed the proof of Theorem A.

**REFERENCES**


**Northwestern University, Evanston, Illinois**

**Universidad de Buenos Aires, Buenos Aires, Argentina**