ON THE UNIVALENCE OF A CERTAIN INTEGRAL

BY

MAMORU NUNOKAWA

1. Introduction. Let $S$ be the class of functions $f(z)$ regular, univalent in $|z| < 1$ and normalized by $f(0) = 0$, $f'(0) = 1$. On the other hand, let $S^*$ and $K$ be the sub-class of $S$ starlike and convex functions respectively.

It is well known that a function $f(z) \in S$ belongs to $S^*$ if and only if

$$\text{Re} \left( \frac{zf''(z)}{f'(z)} \right) > 0 \quad \text{in } |z| < 1$$

and a function $f(z) \in S$ belongs to $K$ if and only if

$$1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) > 0 \quad \text{in } |z| < 1.$$ 

In the recent papers [2], [3], [9], [11], for the univalence of the functions

$$g(z) = \int_0^z \left( \frac{f(t)}{t} \right)^\alpha \, dt \quad \text{and} \quad g(z) = \int_0^z (f'(t))^\alpha \, dt$$

was studied.

For instance, the following theorems are obtained in [2], [9], [11].

**Theorem A.** If $f(z)$ belongs to $S$ and is close-to-convex, then

$$g(z) = \int_0^z (f'(t))^\alpha \, dt$$

belongs to $S$ for $0 \leq \alpha \leq 1$.

**Theorem B.** Suppose $f(z) \in S$ is close-to-convex. Then

$$g(z) = \int_0^z \left( \frac{f(t)}{t} \right)^\alpha \, dt$$

belongs to $S$ for $0 \leq \alpha \leq 1$.

**Theorem C.** Let $f(z) \in S$ and

$$g(z) = \int_0^z \left( \frac{f(t)}{t} \right)^\alpha \, dt.$$ 

Then $g(z) \in S$ for $0 \leq \alpha \leq ((1025)^{1/2} - 25)/100$.

In this paper we improve Theorem C and others.
2. The main theorems.

Lemma 1. Let \( f(z) = z + a_2z^2 + \cdots \) be regular in \( |z| < 1 \). If \( f(z) \) satisfies

\[
1 + \Re \frac{zf''(z)}{f'(z)} > -\frac{1}{2} \quad \text{in} \quad |z| < 1,
\]

then \( f(z) \) is univalent in \( |z| < 1 \).

We owe this lemma to Ozaki [10], [13].

Theorem 1. Let

\[
f(z) = z + \sum_{n=2}^{\infty} a_nz^n \in S^*
\]

and

\[
g(z) = \int_0^z \left( \frac{f(t)}{t} \right)^{a} \, dt.
\]

Then \( g(z) \in S \) for \( 0 \leq a \leq 1.5 \) but for \( a_0 < a \), there exists a function \( f(z) \in S^* \) such that \( g(z) \in S \) where \( a_0 \) is the smallest positive root of the equation

\[\alpha(2\alpha+1)(\alpha+1)-24 = 0.\]

Proof. It follows that

\[1 + zg''(z)/g'(z) = 1 + a(zf'(z)/f(z)-1).\]

Letting \( 0 < a \leq 1.5 \) we have

\[
1 + \Re \frac{zg''(z)}{g'(z)} = 1 - \alpha + \alpha \Re \frac{zf'(z)}{f(z)} > 1 - \alpha \geq -\frac{1}{2}.
\]

Therefore we have that \( g(z) \in S \) for \( 0 \leq a \leq 1.5 \). On the other hand, if we let \( f(z)=z/(1-z)^2 \in S^* \) and \( g(z) \in S \), then we must have from [4, p. 2] and [5, p. 134]

\[
g'(z) = \frac{1}{(1-z)^2} = 1 + 2az + \frac{2(2\alpha+1)}{2!} z^2 + \frac{2(2\alpha+1)(2\alpha+2)}{3!} z^3 + \cdots
\]

and therefore

\[
(1) \quad |2\alpha| \leq 2^2, \quad \left| \frac{2(2\alpha+1)}{2!} \right| \leq 3^2
\]

and

\[
\left| \frac{2(2\alpha+1)(2\alpha+2)}{3!} \right| \leq 4^2.
\]

Letting \( a_0 \) be a positive real number, we must have the following inequality from (1):

\[
0 < \alpha \leq a_0 < \frac{(73)^{1/2}-1}{4} < 2
\]
where $\alpha_0$ is the smallest positive root of the equation

$$\alpha(2\alpha+1)(\alpha+1)-24 = 0.$$ 

This completes our proof.

**Theorem 2.** Let

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K$$

and

$$g(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\alpha \, dt.$$ 

Then $g(z) \in S$ for $0 \leq \alpha \leq 3$ but for $\alpha_1 < \alpha$, there exists a function $f(z) \in K$ such that $g(z) \notin S$ where $\alpha_1$ is the smallest positive root of the equation $\alpha(\alpha+1)(\alpha+2)-96 = 0$.

**Proof.** It is well known [6], [12] that

$$\text{Re } \left(\frac{zf'(z)}{f(z)}\right) > \frac{1}{2} \text{ in } |z| < 1.$$ 

Applying the same method as in the proof of Theorem 1 we have

$$1 + \text{Re } \frac{2g'(z)}{g(z)} = 1 - \alpha + \text{Re } \alpha \frac{zf'(z)}{f(z)} > 1-\alpha + \frac{1}{2} \alpha \geq -\frac{1}{2}$$

if $0 < \alpha \leq 3$. Therefore $g(z) \in S$ for $0 \leq \alpha \leq 3$.

Putting $f(z) = z/(1-z) \in K$ and $g(z) \in S$, then we have

$$g'(z) = \frac{1}{(1-z)^{\alpha}} = 1 + \alpha z + \frac{\alpha(\alpha+1)}{2!} z^2 + \frac{\alpha(\alpha+1)(\alpha+2)}{3!} z^3 + \ldots$$

and therefore we have also as in the proof of Theorem 1

$$(2) \quad |\alpha| \leq 2^2, \quad \left|\frac{\alpha(\alpha+1)}{2!}\right| \leq 3^2$$

and

$$\left|\frac{\alpha(\alpha+1)(\alpha+2)}{3!}\right| \leq 4^2.$$ 

Letting $\alpha$ be a positive real number, we must have from (2) the following

$$0 < \alpha \leq \alpha_1 < ((72)^{1/3}-1)/2 < 4$$

where $\alpha_1$ is the smallest positive root of the equation

$$\alpha(\alpha+1)(\alpha+2)-96 = 0.$$ 

This completes our proof and Theorem 2 is a stronger result than [9, Theorem 4].
Theorem 3. Let
\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K \]
and
\[ g(z) = \int_0^z (f'(t))^\alpha \, dt. \]

Then \( g(z) \in S \) for \( 0 \leq \alpha \leq 1.5 \) but for \( \alpha_0 < \alpha \), there exists a function \( f(z) \in K \) such that \( g(z) \notin S \) where \( \alpha_0 \) is the smallest positive root of the equation
\[ \alpha(2\alpha+1)(\alpha+1)-24 = 0. \]

Proof. We have
\[ 1 + \frac{zg''(z)}{g'(z)} = 1 + \alpha \frac{zf''(z)}{f'(z)} \]
and so
\[ 1 + \text{Re} \frac{zg''(z)}{g'(z)} = 1 - \alpha + \text{Re} \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \]
\[ > 1 - \alpha \geq -\frac{1}{2} \]
if \( 0 < \alpha \leq 1.5 \).

Therefore \( g(z) \in S \) if \( 0 \leq \alpha \leq 1.5 \) and this is a stronger result than [9, Theorem 3].
Putting \( f(z) = z/(1-z) \in K \) and \( g(z) \in S \), then we have \( g'(z) = 1/(1-z)^{3\alpha} \).
By the same reason as in the proof of Theorem 1 we can complete our proof.

Theorem 4. Let
\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \]
be regular in \( |z| < 1 \), \( \text{Re} \, f'(z) > 0 \) and
\[ g(z) = \int_0^z (f'(t))^\alpha \, dt. \]

Then \( g(z) \in S \) for \( -1 \leq \alpha \leq 1 \).

Proof. It follows that
\[ \text{Re} \, g'(z) = \text{Re} \, (f'(z))^\alpha > 0 \quad \text{in} \quad |z| < 1 \]
if \( -1 \leq \alpha \leq 1 \).

By Noshiro [8] we have \( g(z) \in S \) for \( -1 \leq \alpha \leq 1 \).

Lemma 2. Let
\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \]
be regular in \( |z| < 1 \) and
\[ |f(z)| < \frac{2}{(1-r^2)^2} \]
for all \( z, |z| = r < 1 \), where

\[
\{f, z\} = \left( \frac{f''(z)}{f'(z)} \right) - \frac{1}{2} \left( \frac{f'(z)}{f'(z)} \right)^2
\]

is the Schwarzian derivative. Then \( f(z) \) is univalent in \( |z| < 1 \).

The proof of this lemma can be found in [7].

**Lemma 3.** If \( f(z) \) is regular in \( |z| < 1, f(0) = 0 \) and satisfying \( |f(z)| < 1 \) there, then

\[
|f'(z)| < 1 \quad \text{or} \quad |f'(z)| < \frac{(1 + |z|^3)^2}{4|z|(1 - |z|^2)}
\]

according as

\[
|z| < \sqrt{2} - 1 \quad \text{or} \quad \sqrt{2} - 1 \leq |z| < 1.
\]

These bounds are sharp.

A proof of this lemma can be found in [1].

**Theorem 5.** Let

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S
\]

and

\[
g(z) = \int_{0}^{z} \left( \frac{f(t)}{t} \right)^{\alpha} dt.
\]

Then \( g(z) \in S \) for \( 0 \leq \alpha \leq \alpha_2 \) where \( \alpha_2 \) is the smallest positive root of the simultaneous equations (5) and

\[
\frac{(18425)^{1/2} - 75}{800} < \alpha_2 < \frac{(24841)^{1/2} - 125}{384}.
\]

**Proof.** Let \( \alpha \) be a positive real number and

\[
F(z) = g^{\alpha}(z)/g'(z).
\]

Then \( F(z) \) is regular in \( |z| < 1 \) and we have also from [9, p. 396]

\[
|F(z)| < 8\alpha \quad \text{in} \quad |z| < \frac{1}{2}.
\]

Let

\[
G(z) = \{F(z/2) - F(0)/10\alpha \} \quad \text{in} \quad |z| < 1.
\]

Applying Lemma 3 and the same method as in the proof of [9] we have

\[
|G'(z)| = \frac{1}{20\alpha} \left| F' \left( \frac{z}{2} \right) \right| \leq \frac{(1 + \rho^3)^2}{4\rho(1 - \rho^2)} \quad \text{in} \quad \sqrt{2} - 1 \leq |z| \leq \rho < 1.
\]

From the maximum principle we have

\[
|F'(z)| \leq \frac{5\alpha(1 + \rho^3)^2}{\rho(1 - \rho^2)} \quad \text{in} \quad |z| \leq \frac{\rho}{2}.
\]
Hence we get

$$\begin{align*}
|\{g, z\}| & \leq \left| \frac{G'(z)}{G(z)} \right| + \frac{1}{2} \left| \frac{G'(z)}{G(z)} \right|^2 \\
& = |F'(z)| + \frac{1}{2} |F(z)|^2 \leq \left\{ 32\alpha^2 + \frac{5\alpha(1 + \rho^2)^2}{\rho(1 - \rho^2)} \right\} \left( 1 - r^2 \right)^2 \quad \text{in } |z| = r \leq \frac{\rho}{2}.
\end{align*}$$

In $\rho/2 \leq |z| = r < 1$ we have from [9, p. 397]

$$\begin{align*}
|F'(z)| & \leq \frac{2\alpha}{r(1 - r)(1 - \sqrt{r})} \\
& = \frac{2\alpha(1 + r)^2(1 + \sqrt{r})}{r(1 - r^2)^2} \\
& < \frac{2\alpha}{(1 - r^2)^2} \left\{ 2 \left( 1 + \frac{\rho}{2} \right)^2 \left( 1 + \left( \frac{\rho}{2} \right)^{1/3} \right) \rho^{-1} \right\}
\end{align*}$$

and

$$\begin{align*}
|F(z)| & \leq \frac{2\alpha}{r(1 - r)} \\
& = \frac{2\alpha(1 + r)}{r(1 - r^2)} \\
& < \frac{2\alpha}{(1 - r^2)} \left( 1 + \frac{2}{\rho} \right).
\end{align*}$$

Therefore we have

$$\begin{align*}
|\{g, z\}| & \leq \frac{2\alpha}{(1 - r^2)^2} \left\{ 2(1 + \rho/2)^2(1 + (\rho/2)^{1/2}) \rho^{-1} \right\} + \frac{4\alpha^2}{(1 - r^2)^2} \left( 1 + 2/\rho \right)^2 \\
& = \left\{ 4\alpha^2(1 + 2/\rho)^2 + 4\alpha(1 + \rho/2)^2(1 + (\rho/2)^{1/2}) \rho^{-1} \right\}/(1 - r^2)^2 \\
& \quad \text{in } \rho/2 \leq |z| < 1.
\end{align*}$$

Putting

$$32\alpha^2 + \frac{5\alpha(1 + \rho^2)^2}{\rho(1 - \rho^2)} = 4\alpha^2(1 + 2/\rho)^2 + 4\alpha(1 + \rho/2)^2(1 + (\rho/2)^{1/2}) \rho^{-1}$$

and

$$\sqrt{2} - 1 \leq \rho < 1.$$

Let $\alpha_2$ be the smallest positive root of the simultaneous equations (5). Then we have

$$\begin{align*}
|\{g, z\}| & \leq \frac{2}{(1 - r^2)^2} \quad \text{in } |z| = r < 1
\end{align*}$$

if $0 < \alpha < \alpha_2$.

From Lemma 2 we have $g(z) \in S$ for $0 \leq \alpha \leq \alpha_2$. (For the case $\alpha = 0$, Theorem 5 is trivial.)
It can be verified that
\[
\frac{(18425)^{1/2} - 75}{800} < \alpha_2 < \frac{(24841)^{1/2} - 125}{384}.
\]

**Remark.** If we put \( \rho = \frac{1}{2} \) in (3) and (4) we have
\[
|\{g, z\}| \leq \frac{2}{(1-r^2)^2} \quad \text{in} \ |z| = r < \frac{1}{4}
\]
if
\[
0 \leq \alpha \leq \frac{(24841)^{1/2} - 125}{384}
\]
and
\[
|\{g, z\}| \leq \frac{2}{(1-r^2)^2} \quad \text{in} \ 1 \leq |z| = r < 1
\]
if
\[
0 \leq \alpha \leq \frac{(18425)^{1/2} - 75}{800}
\]

Therefore we have \( g(z) \in S \) for at least
\[
0 \leq \alpha \leq \frac{(18425)^{1/2} - 75}{800}
\]

This is an improvement of [9, Theorem 1].

The author would like to acknowledge helpful suggestions made by Professor W. M. Causey.

**References**


Gunma University, 
Maebashi, Japan