THE $C^k$-CLASSIFICATION OF CERTAIN OPERATORS
IN $L_p$. II

BY

SHMUEL KANTOROVITZ(1)

Introduction. We study the one-parameter family of operators

$$T_a = M + aJ$$

acting in $L_p(0, 1)$, $1 < p < \infty$, where $a \in \mathbb{C}$ (the complex field), $M : f(x) \to xf(x)$ and $J : f(x) \to \int_0^x f(t) \, dt$.

Our purpose is to bring the main results of [6] to the best possible form. This will be achieved by replacing Theorem 6, Proposition 13 and Proposition 15 of [6] by the following theorems.

THEOREM 1. Let $n$ be a nonnegative integer. Then $T_a$ is of class $C^n$ if and only if $|\text{Re} \, a| \leq n$.

THEOREM 2. $T_a$ is similar to $T_\beta$ if and only if $\text{Re} \, a = \text{Re} \, \beta$.

THEOREM 3. $T_a$ is spectral if and only if $\text{Re} \, a = 0$.

Theorem 2 was conjectured in [6].

The above results, along with some others, will follow from an interesting formula relating the holomorphic groups of operators $U_a(z) = \exp(zt_a)$ and $V_a(x) = (I + zJ)^x$ ($a, z \in \mathbb{C}$).

1. Preliminaries. Let $\{J_\gamma, \gamma \in \mathbb{R}\}$ be the boundary group of the Riemann-Liouville holomorphic semigroup acting in $L_p(0, 1)$, $1 < p < \infty$ (cf. [4]). It is known that

$$\|J_\gamma\| \leq \exp(\pi |\gamma|/2) \quad (\gamma \in \mathbb{R})$$

and

$$T_{\beta + i\gamma} = J^{-i\gamma} T_\beta J^{i\gamma} \quad (\beta, \gamma \in \mathbb{R})$$

(cf. [4] and [6, Lemma 2]).

For $n = 0, 1, 2, \ldots$, let $C^n[0, 1]$ denote the Banach algebra of all complex functions of class $C^n$ on $[0, 1]$ with the norm

$$\|\phi\|_n = \sum_{j=0}^n \sup_{0 < t < 1} |\phi^{(j)}(t)|/j!.$$
Let \( T \) be a bounded operator acting on a Banach space \( X \), with spectrum in \([0, 1]\). We say that \( T \) is of class \( C^n \) if there exists a continuous representation \( \tau \) of \( C^n[0, 1] \) on \( X \) which sends the functions \( \phi(t) \equiv 1 \) and \( \phi(t) \equiv t \) to the identity operator \( I \) and to \( T \), respectively. The representation \( \tau \) is unique (when it exists), and is called the \( C^n \)-operational calculus for \( T \) (cf. [5]). For example, it follows from [6, Lemma 3] that the operator \( T_n = M + nJ \) acting in \( L_{\rho}(0, 1), 1 \leq \rho < \infty \), is of class \( C^n \), and its \( C^n \)-operational calculus is given by

\[
\tau_n(\phi) = \sum_{j=0}^{\infty} \binom{n}{j} M(\phi^{(j)})J^j, \quad \phi \in C^n[0, 1],
\]

where \( M(\psi) \) denotes the operator of multiplication by the function \( \psi \).

Since \( J \) is quasi-nilpotent, the operator \( (I+zJ)^{\alpha} (\alpha, z \in C) \) is well defined by means of the analytic operational calculus:

\[
(I+zJ)^{\alpha} = \frac{1}{2\pi i} \int_{\Gamma} \lambda^{\alpha}[(\lambda - 1)I - zJ]^{-1} d\lambda,
\]

where, to fix the ideas, \( \Gamma \) is the circle \(|\lambda - 1| = 1/2\). For each fixed \( z \), \( \{(I+zJ)^{\alpha}; \alpha \in C\} \) is a holomorphic group of operators. We shall need a simple estimate on its norm. We have

\[
[(\lambda - 1)I - zJ]^{-1} = (\lambda - 1)^{-1} \sum_{n=0}^{\infty} [z/(\lambda - 1)]^n J^n, \quad \lambda \neq 1.
\]

Since \( \|J^n\| \leq 1/n! \), we see that \( \|[(\lambda - 1)I - zJ]^{-1}\| \leq 2 \exp(2|z|) (\lambda \in \Gamma) \).

Write \( \alpha = \beta + i\gamma \) (\( \beta, \gamma \in R \)) and \( \lambda = re^{i\theta} \). For \( \lambda \in \Gamma \), we have \( 1/2 \leq r \leq 3/2 \) and \(|\theta| \leq \pi/6\). Consequently

\[
|\lambda^{\beta+i\gamma}| = r^\beta e^{-\gamma\theta} \leq (1 \pm 1/2)^\beta \exp(\pi|\gamma|/6), \quad \lambda \in \Gamma,
\]

where the sign is (+) if \( \beta \geq 0 \) and (−) if \( \beta < 0 \). By (3), it follows that

\[
\|(I+zJ)^{\alpha}\| \leq (1 \pm 1/2)^\beta \exp(\pi|\gamma|/6) \exp(2|z|) \quad (\alpha = \beta + i\gamma).
\]

2. The basic results.

**Theorem 4.** For all \( \alpha, z \in C \),

\[
\exp(zT_\alpha) = e^{zM}(I+zJ)^{\alpha} = (I-zJ)^{-\alpha}e^{zM}.
\]

**Proof.** Note first that all the operator functions involved are entire functions of the complex variables \( \alpha, z \). Moreover, it suffices to prove the first identity (or the second), because once this is done, we get

\[
\exp(zT_\alpha) = [\exp(-zT_\alpha)]^{-1} = [e^{-zM}(I-zJ)^{\alpha}]^{-1} = (I-zJ)^{-\alpha}e^{zM},
\]

as wanted.
For $z$ fixed, consider the operator-valued entire function

$$\Phi_2(\alpha) = e^{-zM} \exp(zT_\alpha) - (I + zJ)^a.$$  

We verify the hypothesis of [3, Theorem 3.13.7]. Since

$$\|e^{-zM} \exp(zT_\alpha)\| \leq \exp(\|z\| M) \exp(\|z\| M + \alpha J) \leq \exp(\|z\|(2 + |\alpha|)),$$

it follows from (4) that

$$\|\Phi_2(\alpha)\| \leq \exp(2|z|)\left\{ \exp(|z| |\alpha|) + (3/2)^{\theta} \exp(\pi |\gamma|/6) \right\}$$

for $\alpha = \beta + i\gamma$, $\beta \geq 0$.

Thus

$$\|\Phi_2(re^{i\theta})\| \leq Ce^{\lambda(\theta)}, \quad -\pi/2 \leq \theta \leq \pi/2,$$

where $C = 2e^{2|z|}$ and

$$\lambda(\theta) = \max\{|z|, \log(3/2) \cos \theta + (\pi/6) |\sin \theta|\}.$$  

Clearly, $\lambda(\theta)$ is bounded, even, and

$$\lambda(\pm \pi/2) \leq \pi$$

for $|z| \leq \pi$.

Moreover, if $|z| < \pi$, we have

$$\limsup_{\delta \to 0^+} \delta^{-1}\{\pi - \lambda(\pi/2 - \delta)\} = \infty.$$  

Using (2) with $\phi(t) = \phi_2(t) = e^{zt}$, we obtain

$$\exp(zT_\alpha) = \sum_{j=0}^{\infty} \binom{n}{j} z^j M(\phi_2) J^j$$

$$= e^{zM} \sum_{j=0}^{\infty} \binom{n}{j} (zJ)^j$$

i.e.,

$$\exp(zT_\alpha) = e^{zM}(I + zJ)^n, \quad n = 0, 1, 2, \ldots.$$  

Thus $\Phi_2(n) = 0, n = 0, 1, 2, \ldots$. By Theorem 3.13.7 in [3], it follows that

$$\Phi_2(\alpha) = 0, \quad \Re \alpha \geq 0, \ |z| < \pi.$$  

Since $\Phi_2(\alpha)$ is entire in both variables, we conclude that $\Phi_2(\alpha) = 0$ for all $\alpha, z \in C$.

**Q.E.D.**

**Remarks.** 1. Theorem 4 is also valid for $p = 1$, since we used only Lemma 3 of [6], which is true in this case as well.

2. Let $n$ be a nonnegative integer. The second identity in Theorem 4 shows that

$$\exp(zT_{-n}) = (I - zJ)^n e^{zM}.$$  

This formula follows also from Lemma 5 in [6], and could be used instead of (5) to prove Theorem 4, thus relying on Lemma 5 in [6] rather than on Lemma 3.
there. As a matter of fact, the proof of Theorem 4 can be used to show that the two lemmas are consequences of each other (cf. [5, proof of Lemma 2.11]).

3. It follows from Theorem 4 that the holomorphic groups $U_\alpha(z) = \exp(zT_\alpha)$ satisfy the "cocycle" identity:

$$U_{\alpha + \beta}(z) = U_\alpha(z)e^{-zM}U_\beta(z) \quad (\alpha, \beta, z \in \mathbb{C}).$$

By Theorem 4 and the spectral mapping theorem, the spectrum of the operator $e^{-zM}\exp(zT_\alpha) = (I+zJ)^\alpha$ consists of the single point $\lambda = 1$. Therefore, the analytic operational calculus may be used to define powers $[e^{-zM}\exp(zT_\alpha)]^\beta$ for $\beta \in \mathbb{C}$, and by Theorem VII.3.12 in [2], one has

$$[e^{-zM}\exp(zT_\alpha)]^\beta = (I+zJ)^{\alpha\beta} = e^{-zM}\exp(zT_{\alpha\beta}).$$

A similar relation follows from the second identity in Theorem 4:

**Corollary 5.** For all $\alpha, \beta, z \in \mathbb{C}$,

$$\exp(zT_{\alpha\beta}) = e^{zM}[e^{-zM}\exp(zT_\alpha)]^\beta = [\exp(zT_\alpha)e^{-zM}]^\beta e^{zM}.$$ 

We consider now the one-parameter groups of operators

$$G_\alpha(t) = \exp(itT_\alpha) \quad (t \in \mathbb{R}).$$

**Theorem 6.** For each $\beta \in \mathbb{R}$, there exists a constant $C_\beta > 0$ such that

$$Ce^{-\pi|\gamma|} \leq (1 + |r|)^{-|\beta|}\|G_{\beta+i\gamma}(t)\| \leq e^{a|\gamma|}e^{\pi|\gamma|}$$

for all $\gamma, t \in \mathbb{R}$.

**Proof.** By (0) and (1), it suffices to prove the theorem for $\gamma = 0$.

Fix $t \in \mathbb{R}$, and consider the operator-valued entire function

$$\psi_\beta(\alpha) = \exp(\pi\alpha^2)G_\alpha(t) \quad (\alpha \in \mathbb{C}).$$

By (1)

$$\|\psi_\beta(\alpha)\| \leq \exp(\pi(\beta^2 + \frac{1}{4}))\|G_\alpha(t)\| \quad (\alpha = \beta + i\gamma).$$

In particular, $\psi_\beta(\beta + i\gamma)$ is bounded in the strip $n-1 \leq \beta \leq n$, for any integer $n$. By (5), (6) and (8),

$$\|\psi_\beta(\beta + i\gamma)\| \leq \exp(\pi(n^2 + \frac{1}{4}))\|(I + itJ)^{|n|}\|
\leq \exp(\pi(n^2 + \frac{1}{4})(1 + |t|)^{|n|})^{|\beta|}.$$ 

Write $\beta$ as the convex combination $b\beta + c(n-1) = b\beta + c(n-1)$. Then, by the "three lines theorem" [2, VI.10.3] and the preceding inequalities,

$$\|\psi_\beta(\beta + i\gamma)\| \leq \exp \pi\left[b(n^2 + \frac{1}{2}) + c((n-1)^2 + \frac{1}{2})](1 + |t|)^{|\beta|}
= \exp \pi\left[n^2 - 2cn + c + \frac{1}{2}](1 + |t|)^{|\beta|}
= \exp \pi\left[(n-c)^2 + c(1-c) + \frac{1}{2}][1 + |t|]^{|\beta|}
\leq \exp \pi[b^2 + \frac{1}{2}](1 + |t|)^{|\beta|}.$$
Thus
\[ \|G_\beta(t)\| = \exp(-\pi \beta^2)\|\psi_\beta(\beta)\| \leq \exp(\pi/2)(1 + |t|)^{1/\beta}. \]

Next, fix \( \beta \in \mathbb{R} \) and let
\[ C_\beta = \inf_{t \in \mathbb{R}} (1 + |t|)^{-1/\beta} \|G_\beta(t)\| \geq 0. \]

We must show that \( C_\beta > 0 \). This is obvious for \( \beta = 0 \), since \( \|G_0(t)\| = 1 \). So consider \( \beta \neq 0 \), and fix an integer \( n \geq 1 \) such that \( n|\beta| > 1 \). Trivially, \( (1 + |t|)^{-1/\beta} \|G_\beta(t)\| > 0 \) for each \( t \in \mathbb{R} \). Assume \( C_\beta = 0 \). There exists then a sequence \( \{t_k\} \) in \( \mathbb{R} \) such that \( |t_k| \to \infty \) and \( (1 + |t_k|)^{-1/\beta} \|G_\beta(t_k)\| \to 0 \) as \( k \to \infty \).

Fix \( e > 0 \) and choose \( k_0 \) such that
\[ (1 + |t_k|)^{-1/\beta} \|G_\beta(t_k)\| < e^{1/\beta} \quad \text{for} \quad k \geq k_0. \]

For \( k \geq k_0 \) fixed, consider the entire functions
\[ F_k^\pm(\zeta) = (1 + |t_k|)^{\mp n/\beta} \psi_{t_k}(\zeta), \quad \zeta \in \mathbb{C}. \]

It follows from (8) that \( F_k^\pm(\zeta) \) is bounded in each vertical strip \( a \leq \text{Re} \zeta \leq b \) \((a, b \in \mathbb{R})\) and
\[ \|F_k^\pm(i\eta)\| \leq \exp(\pi/4) \quad (\eta \in \mathbb{R}). \]

By Corollary 5,
\[ \|G_{n\beta}(t_k)\| = \|\exp(-it_k M)G_\beta(t_k)\|^n \leq \|G_\beta(t_k)\|^n. \]

Therefore, by (9),
\[ (1 + |t_k|)^{-n/\beta} \|G_{n\beta}(t_k)\| \leq [(1 + |t_k|)^{-1/\beta} \|G_\beta(t_k)\|]^n \leq e^{n/\beta}. \]

By (1), we then have
\[ \|F_k^\pm(n\beta + i\eta)\| \leq \exp(\pi(n^2\beta^2 + 1))e^{n/\beta} \]
where the superscript of \( F_k \) is \((+)\) if \( \beta > 0 \) and \((-)\) if \( \beta < 0 \). By the "three lines theorem" applied to \( F_k^+ \) (resp. \( F_k^- \)) in the strip \( 0 \leq \xi \leq n\beta \) (resp. \( n\beta \leq \xi \leq 0 \)), we obtain
\[ \|F_k^\pm(\xi + i\eta)\| \leq \exp(\pi(n^2\beta^2 + 1))e^{1/\beta} \]
in the respective strips.

This is true in particular for \( \xi = \pm 1 \) (resp. \( -1 \)), since \( n|\beta| > 1 \). Thus
\[ (1 + |t_k|)^{-1}\|G_{\pm 1}(t_k)\| = e^{-\pi}\|F_k^\pm(\pm 1)\| \leq C e \]
where \( C \) does not depend on \( k \).

This proves that
\[ \lim_{k \to \infty} (1 + |t_k|)^{-1}\|G_{\pm 1}(t_k)\| = 0. \]

However, by (5) and (6), this limit is equal to
\[ \lim_{k \to \infty} (1 + |t_k|)^{-1}\|I \pm it_k J\| = \|J\| \neq 0 \]
(since \( |t_k| \to \infty \) as \( k \to \infty \)). This contradiction shows that \( C_\beta > 0 \), and the proof is complete.

Proof of Theorem 1. If \(|\text{Re} \alpha| \leq n\), \(T_\alpha\) is of class \(C^\infty\) by Theorem 6 in [6]. Suppose then that \(T_\alpha\) is of class \(C^\infty\) for some \(\alpha = \beta + iy\) with \(|\beta| > n\). It follows that (cf. [5, Lemma 2.11]) \(\|G_\alpha(t)\| \leq C(1 + |t|^n)\), and therefore

\[
(1 + |t|)^{-|\beta|} \|G_{\beta + iy}(t)\| \leq C(1 + |t|^n)^{1 - |\beta|} \rightarrow 0
\]
as \(|t| \rightarrow \infty\), contradicting Theorem 6.

Proof of Theorem 2. By (1), \(T_\alpha\) and \(T_\lambda\) are similar if \(\text{Re} \alpha = \text{Re} \lambda\). It then remains to show that \(T_\beta\) and \(T_\lambda\) are not similar for distinct real numbers \(\beta\) and \(\lambda\). Suppose \(\beta, \lambda \in \mathbb{R}, \beta \neq \lambda, \) and \(T_\beta = Q^{-1}T_\lambda Q\) with \(Q\) nonsingular. First, assume \(|\lambda| < |\beta|\). Then, by Theorem 6,

\[
(1 + |t|)^{-|\beta|} \|G_\beta(t)\| = (1 + |t|)^{-|\beta|}\|Q^{-1}G_\lambda(t)Q\|
\]

\[
\leq e^{\pi/2} \|Q\| \|Q^{-1}(1 + |t|)^{1 - |\lambda| - |\beta|} \rightarrow 0\quad \text{as} \quad |t| \rightarrow \infty,
\]

contradicting Theorem 6.

The following argument, which was kindly communicated to me by Professor G. K. Kalisch, disposes of the case \(|\lambda| = |\beta|\). Suppose \(T_\beta P = PT_\beta\) for \(P\) nonsingular and \(\beta > 0\). By Lemma 1 in [6], it follows that the compact operator \(J^{\beta}PJ^{\beta}\) commutes with \(M\), and hence must be 0, a contradiction (cf. Lemma 2 in G. Kalisch, On isometric equivalence of certain Volterra operators, Proc. Amer. Math. Soc. 12 (1961), 93–98).

Proof of Theorem 3. By (1), \(T_\alpha\) is trivially spectral (of scalar type) for \(\text{Re} \alpha = 0\), and we already know that \(T_\alpha\) is not spectral for \(|\text{Re} \alpha| \geq 1\) [6, Proposition 15]. Suppose then that \(T_\alpha\) is spectral for some \(\alpha = \beta + iy\) with \(0 < |\beta| < 1\). By (1), it follows that \(T_\beta\) is spectral. Since \(T_\beta\) is of class \(C^1\) (Theorem 1), it is necessarily of type \(\leq 1\), i.e., \(T_\beta = S + N\) with \(S, N\) commuting, \(S\) spectral of scalar type and \(N^2 = 0\) (cf. [1]). Thus \(G_\beta(t) = e^{itS}e^{itN} = e^{itS}(I + itN)\). Since \(S\) has real spectrum (the spectrum of \(T_\beta\)), \(\|e^{itS}\| \leq M\), and therefore, by Theorem 6, we have as \(|t| \rightarrow \infty\):

\[
\|N\| \leq \lim (1 + |t|)^{-1}\|I + itN\| = \lim (1 + |t|)^{-1}\|e^{-itS}G_\beta(t)\|
\]

\[
\leq M \lim \sup (1 + |t|)^{-1}\|G_\beta(t)\| \leq Me^{\pi/2} \lim \sup (1 + |t|)^{1 - |\beta| - 1} = 0.
\]

Thus \(T_\beta = S\) and \((1 + |t|)^{-|\beta|}\|G_\beta(t)\| \leq M(1 + |t|)^{-|\beta|} \rightarrow 0\), contradicting Theorem 6.

REFERENCES


**University of Illinois at Chicago Circle, Chicago, Illinois**