$H^p$ SPACES ON BOUNDED SYMMETRIC DOMAINS(1)

BY

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1. Introduction. Let $D$ be a bounded symmetric domain in the complex vector space $CN$ and $0 \in D$. Any bounded symmetric domain $D$, furnished with the Bergman metric $M$, is a hermitian symmetric space $(D, M)$ of noncompact type and is necessarily simply connected [3, p. 311]. Let $\Gamma$ be the group of holomorphic automorphisms of $D$; $\Gamma$ is transitive on $D$ and extends continuously to the topological boundary $\partial D$ of $D$ [7, p. 269]. The isotropy group $\Gamma_0 = \{ \gamma \in \Gamma : \gamma(0) = 0 \}$ of $\Gamma$ is a compact subgroup of $\Gamma$ and contains no normal subgroup of $\Gamma$. Thus $D$ can be identified with the coset space $\Gamma/\Gamma_0$. This realization of bounded symmetric domains enables us to study the structure of $D$, using the algebraic machinery of Lie groups. Any bounded symmetric domain may be represented as the topological product of irreducible bounded symmetric domains; the class of irreducible bounded symmetric domains consists of four types of classical Cartan domains and two exceptional ones.

A bounded symmetric domain $D$ is circular and star-shaped with respect to the origin, that is, $t z \in D$ when $z \in D$ and $t \in C$ with $|t| \leq 1$ [7]. It has Bergman-Šilov boundary $b$ which is circular and invariant under $\Gamma$ [7]. The group $\Gamma_0$ is transitive on $b$ [13, p. 922] and $b$ has a unique normalized $\Gamma_0$-invariant measure $\mu$, which is given by $d\mu = V^{-1} ds_t$, $V$ the euclidean volume of $b$ and $ds_t$ the euclidean volume element at $t \in b$.

A complex-valued function $h: D \to C$ is harmonic on $D$ if $\Delta h = 0$ for each $\Gamma$-invariant differential operator $\Delta$ of the hermitian space $(D, M)$ [6, p. 340].

For $p > 0$ the Hardy space $H^p$ is defined on $D$ by

$$H^p \equiv H^p(D) = \left\{ f : f \text{ holomorphic on } D \text{ and } \sup_{0 \leq r < 1} \left( \frac{1}{V} \int_0 |f(rt)|^p \, ds_t \right)^{1/p} = M < \infty \right\}$$

and the space $A^p$ by

$$A^p \equiv A^p(D) = \{ f : f \text{ holomorphic on } D \text{ and } |f(z)|^p \leq h(z) \text{ on } D, h \in \mathfrak{H}(D) \},$$

where

$$\mathfrak{H} \equiv \mathfrak{H}(D) = \left\{ h : h(z) = \int_0 P(z, t)\phi(t) \, ds_t \equiv (P_z, \phi)_b, z \in D, \phi \in L(b) \right\},$$

$P(z, t)$ the Poisson kernel of the domain $D$.

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In §2 we give several properties of the spaces $H^p$ and $\bar{H}^p$ and prove that these spaces are equivalent for bounded symmetric domains. Rudin pointed out this equivalence for $C^1$ in [8]. In §3 we show that $H^p$ is a Banach space for $p \geq 1$ and a complete linear Hausdorff space for $0 < p < 1$, thus generalizing the results for the unit disc [11]. §§4 and 5 consider properties of linear functionals on $H^p$. For other treatments of $H^p$ spaces ($p \geq 1$) on bounded symmetric domains see [6].

2. Properties of the spaces $H^p$ and $\bar{H}^p$.

1. Let

$$D_r = \{ rz : z \in D \}, \quad b_r = \{ rz : z \in b \}.$$ 

Since $D$ is star-shaped with respect to $0$, $D_r \subset D$ if $0 < r < 1$ and $\lim_{r \to 1} D_r = D$. Also any compact subset $K$ of $D \subset D_r$ for some $r < 1$.

Theorem 1. Let $u(z)$ be defined on $D_r$ and $v(\xi) = u(r\xi R^{-1})$. Then $u \in \mathcal{H}(D_r)$ if and only if $v \in \mathcal{H}(D_r)$.

Proof. This follows since under the transformation $z = r^{-1}\xi R$, $u(z) = (P_r z, \phi)_b$, $\phi \in L(b_r)$, goes into $v(\xi) = (P_r z, \phi)_b$, where $\psi(v) = \phi(rvR^{-1}) \in L(b_r)$ and conversely ($P_r z$ and $P_r z$ the respective Poisson kernels of $D_r$ and $D_r$).

Theorem 2. A function $u \in \mathcal{H}(D)$ is harmonic on $D$.

Proof. By definition the Poisson kernel is

$$P(z, t) = |S(z, i)|^2/S(z, \bar{z}),$$

where $S$ is the Szegő (or Cauchy) kernel of $D$. By [5, p. 88]

$$S(z, i) = \sum_{k=0}^{\infty} \sum_{v=1}^{m_v} \phi_v^{(k)}(z)\phi_v^{(k)}(t), \quad ((z, t) \in D \times b),$$

where $\{\phi_v^{(k)}\}$ is a complete orthonormal system of homogeneous polynomials on $D$, orthonormalized with respect to $b$, and $\phi_0 = \phi_0^{(0)} = V^{-1}$. Since the convergence of series (2) is uniform on compact subsets of $D \times \bar{D}$ [5, p. 89], $S(z, i)$ is holomorphic in $(z, i)$ on $D \times D$ and continuous on $D \times \bar{D}$. In particular, $S(z, \bar{z})$ is holomorphic in $(z, \bar{z})$ for $z \in D$ and from (2)

$$S(z, \bar{z}) \geq |\phi_0(z)|^2 = V^{-1}.$$ 

By the Weierstrass theorem [4, p. 6] $D_r S(z, i)$ is holomorphic on $D \times D$ and continuous on $D \times \bar{D}$ and $D^2_{\bar{z}} S(z, \bar{z})$ is holomorphic in $(z, \bar{z})$ for $z \in D$. Thus the derivatives $D^2_{\bar{z}} P(z, t)$ are bounded on $K \times b$, $K$ a compact subset of $D$, and by the Lebesgue dominated convergence theorem $\Delta u = (\Delta P_{z}, \phi) = 0$ ($\Delta = \Delta_z$ at $z$), since $P(z, t)$ is harmonic in $z$ for $z \in D, t \in b$ [6, Theorem 3.5].

2. Equivalence of $H^p$ and $\bar{H}^p$.

Lemma 1. Let $u$ be a plurisubharmonic (psh) function on $D$ and set $u_r(z) = u(rz)$ for $0 < r < 1$ and $z \in D$. Then

$$u_r(z) \leq (P_{z}, u_r).$$
Proof. Since $u$ is psh on $D$, $u_r$ is psh on $ar{D}$ for $r < 1$. Thus

$$u_r(0) \leq \frac{1}{2\pi} \int_{0}^{2\pi} u_r(te^{i\theta}) \, d\theta$$

for $t \in b$ [10, p. 73]. Let $y \in \Gamma$ and $\gamma(z) = 0$. Set $U_r(w) = u_r(\gamma^{-1}(w))$. Since psh functions are invariant under biholomorphic mappings [10, p. 81], $U_r$ is psh on $\bar{D}$ so that $U_r(0)$ satisfies (5). Integrate over $b$ and use Fubini's theorem

$$U_r(0) \geq \frac{1}{2\pi} \int_{0}^{2\pi} U_r(te^{i\theta}) \, d\theta = \frac{1}{2\pi} \int_{0}^{2\pi} \int_{b} U_r(te^{i\theta}) \, ds_t = \int_{b} U_r(t') \, ds_t,$$

since $b$ is circular and $ds_t = ds_t$ under $t' = te^{i\theta}$. Thus

$$u_r(z) \leq \frac{1}{V} \int_{b} u_r(\gamma^{-1}(t')) \, ds_t.$$

Set $t = \gamma^{-1}(t')$. Then $b = \gamma^{-1}(b)$ and by the invariance of the measure $P(z, t) \, ds_t$ under $\gamma$

$$\frac{1}{V} \, ds_t = P(0, t') \, ds_t = P(z, t) \, ds_t$$

[6, p. 339]. Thus (4) follows.

Lemma 2. The function

$$M(r) = \frac{1}{V} \int_{b} |f(rt)|^p \, ds_t$$

is a monotone nondecreasing function of $r$ on $[0, 1]$.

Proof. Since $f$ is holomorphic on $D$, $f_r$ is holomorphic on $\bar{D}$ and $|f_r|^p$ is psh on $\bar{D}$ for $p > 0$ [10, p. 74]. As in (6)

$$I = \frac{1}{2\pi} \int_{b} ds_t \int_{0}^{2\pi} |f(rt)e^{i\theta}|^p \, d\theta = \int_{b} |f(rt)|^p \, ds_t = VM(r).$$

By the definition of psh, $|f(\lambda t)|^p$ is subharmonic with respect to $\lambda$ in every component of the open set $O_t = \{\lambda : \lambda t \in D\}$. From [10, p. 62] the function

$$m(rt) = \frac{1}{2\pi} \int_{0}^{2\pi} |f(rt)e^{i\theta}|^p \, d\theta,$$

which is the mean value of a subharmonic function, is a nondecreasing function of $r$ and convex with respect to $\log r$ for all $t$. By (9) and (10) for $r < r'$

$$M(r) = \frac{1}{V} \int_{b} m(rt) \, ds_t \leq \frac{1}{V} \int_{b} m(r't) \, ds_t = M(r').$$

Also $M(r)$ is convex with respect to $\log r$ in $(0, 1)$.

Theorem 3. For $p > 0, f \in H^p$ if and only if $f \in H^p$. 

Proof. Since $|f|^p$ is psh on $D$ for $p > 0$, (4) holds for $|f|^p$, $0 < r < 1$.

Let $f \in H^p$. Since $ds_t$ is a finite Borel measure and $b$ is circularly invariant, by a result of Bochner [1, Theorems 2, 3] there exists a function $\psi$, measurable on $b$, such that

$$
\lim_{r \to 1} \int_b |f(rt) - \psi(t)|^p \, ds_t = 0.
$$

Since $P_z(t)$ is uniformly continuous and nonnegative on the compact set $b$, (11)

$$
\lim_{r \to 1} \int_b |f(rt) - \psi(t)|^p P(z, t) \, ds_t = 0.
$$

From (11) follows by Minkowski's inequality for $p \geq 1$ and the inequality

$$
(a + b)^p \leq a^p + b^p,
$$

$a, b \geq 0$ and $0 < p < 1$ [11] that

$$
\lim_{r \to 1} (P_z, |f_r|^p) = (P_z, |\psi|^p).
$$

In particular since $P(0, t) = V^{-1}$ for $t \in b$, $\psi \in L^p(b)$. Let $r \to 1$. From the continuity of $f$ on $D$ and (4) and (13) follows

$$
0 \leq |f(z)|^p = \lim_{r \to 1} |f_r(z)|^p \leq \lim_{r \to 1} (P_z, |f_r|^p) = (P_z, |\psi|^p) = u^*(z).
$$

Since $u^* \in \mathcal{M}(D), f \in \mathcal{R}^p$.

Remark. (14) is proved in [12] with reference to [1] for a method of proof of inequality (4) for $|f_r|^p$. However no details are given in [1].

Conversely let $f \in \mathcal{R}^p$. Then $|f(z)|^p \leq h(z) = (P_z, \phi)$, $\phi \in L(b)$ and hence $|f_r(t)|^p \leq h_r(t)$ on $\bar{D}$ for $r < 1$. Integrate over $b$ and use Fubini's theorem. Then

$$
\frac{1}{V} \int_b |f_r(t)|^p \, ds_t \leq \frac{1}{V} \int_b h_r(t) \, ds_t = \frac{1}{V} \int_b \int_b P(r, v) \phi(v) \, ds_v \, ds_t
$$

$$
= \frac{1}{V} \int_b \phi(v) \, ds_v = h(0) < \infty,
$$

since $P(r, v) = P(r, t)$ for $v, t \in b$ [5, Theorem 4.5.2] and $\int_b P(rv, t) \, ds_t = 1$. Thus $f \in H^p$.

Another necessary and sufficient condition that $f \in H^p$ is given by

Theorem 4. Let $z_0 \in D$ and $f$ be holomorphic on $D$. Let $r, 0 < r < 1$, be such that $z_0 \in D_r$. Then $f \in H^p(D)$ if and only if there exists a constant $B(z_0)$, independent of $r$, such that

$$
(P_{z_0}, |f_r|^p) \leq B(z_0).
$$

Proof. The necessity of (15) follows from the uniform continuity of $P_{z_0}(t)$ on $b$ and Lemma 2, namely

$$
(P_{z_0}, |f_r|^p) \leq \max_{t \in b} P_{z_0}(t) VM(r) \equiv b_1(z_0) VM(r) \leq b_1(z_0) MV \equiv B(z_0).
$$
Conversely, let \( b_2(z_0) = \min_{t \in b} P_{z_0}(t); \ b_2(z_0) > 0 \) since \( P_{z_0}(t) > 0 \) on \( b \). Proof. For any \( t \in b \) there exists a holomorphic automorphism \( \gamma_t \) of \( D \) which takes \( z_0 \to 0 \) and \( b \to b \). Let \( t \to t' \). By (7) \( P_{z_0}(t) > 0 \). Since \( t \in b \) was arbitrary \( P_{z_0}(t) > 0 \) on \( b \). Then by (14)

\[
\int_b |f_t|^p \, ds_t \leq b_2^{-1}(z_0)(P_{z_0}, |f_t|^p) \leq b_2^{-1}(z_0)B(z_0)
\]

so that \( f \in H^p \).

3. The topology of \( H^p \) spaces. For \( p > 0 \) set

\[
\|f\|_p = \sup_{0 < r < 1} \left( \frac{1}{V} \int_b |f(rt)|^p \, ds_t \right)^{1/p}.
\]

Remark. From (2.14) with \( z = 0 \) and Lemma 2 follows \( \|f\|_p = u^*(0)^{1/p} \), where \( u^* \in \Sigma(D) \).

For \( p \geq 1 \) the triangle inequality follows for \( f, g \in H^p \) by Minkowski's inequality and properties of sup. For \( 0 < p < 1 \) Minkowski's inequality does not hold but (2.12) applied to \( f, g \) gives

\[
\|f + g\|_p \leq \|f\|_p + \|g\|_p \quad (0 < p < 1).
\]

Lemma 3. Let \( f \in H^p \). For any \( z \in D \) there exists a constant \( C(z) \), depending on \( z, p, \) and \( D \) but not on \( f \), such that

\[
|f(z)| \leq C(z)\|f\|_p.
\]

For any compact set \( K \) of \( D \) there exists a constant \( C = C(D, K, p) \), depending on \( D, K \) and \( p \) but not on \( f \), such that for \( z \in K \)

\[
|f(z)| \leq C\|f\|_p.
\]

Hence if \( \|f - f_n\|_p \to 0 \) as \( n \to \infty \), then \( f_n \to f \) uniformly on compact subsets of \( D \).

Proof. From (2.8) and (2.14) follows \( |f(z)|^p \leq b_1(z)M(r) \leq b_1(z)V\|f\|_p^p \) for \( z \in D \).

Letting \( r \to 1 \) gives (2) with \( C(z) = V^{1/p}b_1(z)^{1/p} \). If \( K \) is a compact subset of \( D \), then

\[
|f(z)|^p \leq \max_{z \in K} P(z, t) V\|f\|_p^p \equiv C^p(K, D, p)\|f\|_p^p
\]

and (3) follows when \( r \to 1 \).

From (2), \( \|f\|_p = 0 \), implies \( f(z) = 0 \) on \( D \) and conversely. Thus \( H^p \) is a metric space for \( p \geq 1 \) and satisfies all the axioms except the triangle inequality for \( 0 < p < 1 \). As usual a subset \( O \) of \( H^p \) is said to be open if for every \( f_0 \in O \) there exists \( \rho > 0 \) such that \( \{f : \|f - f_0\|_p < \rho\} \subset O \); for \( p \geq 1 \) this gives the usual topology induced by the metric. It is easy to prove that the Hausdorff separation axiom holds. Thus \( H^p \) is a linear Hausdorff space. From the last statement of Lemma 3 the completeness of \( H^p \) follows by well-known procedures from the completeness of \( C^1 \), the triangle inequality for \( p \geq 1 \), and inequality (1) for \( 0 < p < 1 \).
Hence:

**Theorem 5.** For $p \geq 1$ $H^p$ is a Banach space and for $0 < p < 1$ a complete linear Hausdorff space.

**Theorem 6.** $H^p$ is equivalent to a closed subspace of $L^p(b)$ (that is, there exists an algebraic isomorphism $\sigma$ of $H^p$ onto a closed subspace of $L^p(b)$, the isomorphism being norm-preserving).

**Proof.** From the monotonicity of $M(r)$ and (2.14) with $z = 0$ follows

$$
\|f\|_p = \lim_{r \to 1} M(r)^{1/p} = \left( \frac{1}{V} \int_b |\psi(t)|^p \, ds_t \right)^{1/p} = \|\psi\|_p.
$$

Define a mapping $\sigma$ by $\sigma(f) = \psi$. From $\|f\|_p = \|\psi\|_p$ follows $\sigma$ is 1-1 from $H^p$ onto a subspace of $L^p(b)$. Also $\sigma(H^p)$ is closed in $L^p(b)$ [11, p. 802].

A final property is that $H^p$ spaces are perfectly separable as follows by the same proof as in [11].

4. **Linear functionals.** Let $\gamma$ be a functional in $H^p$. Then $\gamma \in (H^p)^*$ if and only if $\gamma$ is bounded on the unit sphere in $H^p$. Topologize $(H^p)^*$ by setting $\|\gamma\| = \sup_{f \neq 0} |\gamma(f)|$. Then $(H^p)^*$ is a Banach space [11]. The class $[L^p(0, 2\pi)]^*$ of linear functionals on $L^p(0, 2\pi)$, $0 < p < 1$, contains only the zero functional but, as in the case of the unit disc, $(H^p)^*$ contains other elements [11]. Set

$$
\gamma_{\nu,n}^p(f) = (1/n!) D^n f(z), \quad \nu = 1, \ldots, m_n, \quad D^n z = \partial^n/\partial z_1 \cdots \partial z_N^n,
$$

where $m_n$ is the number of derivatives of order $n$. Similar to Theorem 6 in [11], we obtain precise bounds for the norms $\|\gamma_{\nu,n}^p\|$. Such bounds will be used in the proof of Theorem 9 and later papers.

**Theorem 7.** $\gamma_{\nu,n}^p \in (H^p)^*$ ($n = 0, 1, 2, \ldots; \nu = 1, \ldots, m_n$) for $z \in D$ and

1. $\|\gamma_{0,n}^p\| \leq 1/(1 - z)^{2N/p}$,
2. $\|\gamma_{\nu,n}^p\| \leq \frac{r_{n,z}^{N/2} V^{1/2}}{n!(r_{n,z} - r_z)^{N/2}(1 - r_{n,z})^{2N/p}} [D^n_s S(Z_{n, \nu}, Z_{n, r})]^{1/2}$ ($n > 0$),

($Z_{n, r} = (r_{n,z} - r_z)^{-1} z$), where $r_z, 0 < r_z < 1$, depends on $z$ only and $r_{n,z}$ is the value of $r$ on $(r_z, 1)$, which minimizes the right side of (8). (If $z = 0$, replace $r_z$ by $\frac{1}{2} r_{n,z}$)

**Proof of (1).** Fix $r_1 \in (0, 1)$. By (2.4) applied to $|f_{r_1}(z)|^p$ and (2.16)

$$
|f_{r_1}(z)|^p \leq \max_{t \in [0, 1]} P(z, i)V \|f\|_p^p.
$$

From (2.1) and (2.2) and the fact that $\phi^{(k)}(z)$ is homogeneous of degree $k$

$$
P(z, t) = S(z, \bar{z})^{-1} \lim_{n \to \infty} \left| \sum_{k=0, \nu} \phi^{(k)}(z, \nu^{-1}) \phi^{(k)}(t)^{r^k} \right|^2.
$$
where \( r \) is chosen so that \( zr^{-1} \in \overline{D} \). By the maximum principle and Schwarz inequality the right side is

\[
\leq S(z, z)^{-1} \lim_{r \to 1} \max_{\lambda \in \partial D} \sum_{k=0}^{n} |\phi^{(k)}_{\lambda}(z)|^{2r^{-k}} \leq 1/V(1-r)^{2N}
\]

by (2.3) and [5, Theorem 4.5.1]. Since \( D \) is circular and star-shaped, it is clear that there is a unique \( r_z \in [0, 1) \) such that \( z \in \overline{D}_{r_z} \) and \( z \notin D_r \) for \( r < r_z \).

Thus

\[
P(z, t) \leq 1/V(1-r_z)^{2N};
\]

and from (3) and (4)

\[
|f_r(z)| \leq (1-r_z)^{-2N/p} \|f\|_p.
\]

(1) follows by letting \( r_1 \to 1 \).

**Proof of (2).** Since \( f \) is holomorphic on \( \overline{D} \), for \( r < 1 \), the Cauchy integral formula gives \( f(z) = (S_{r_2}, f)_{b_r}, (z \in D_r), S_{r_2} \) the Szegö kernel of \( D_r \). By [4, p. 7, Corollary 2] \( D^2_{r_2} \) and \( \int_{b_r} \) can be interchanged, giving

\[
|D^2_r f(z)| \leq \int_{b_r} |D^2_{r_2} S_r(z, i)| |f(t)| dt \leq \max_{t \in b_r} |D^2_{r_2} S_r(z, i)| \int_{b_r} |f(t)| dt.
\]

(5)

But

\[
S_r(z, i) = \sum_{k \in \mathbb{N}, \lambda} \phi^{(k)}_{\lambda}(r^{-1}z) \overline{\phi^{(k)}_{\lambda}(r^{-1}t)} = \sum_{k \in \mathbb{N}, \lambda} \phi^{(k)}_{\lambda}(z) \overline{\phi^{(k)}_{\lambda}(t)} r^{-k},
\]

where the convergence is uniform for \( z \in \) compact subsets of \( D_r \) and \( t \in b_r \). By Weierstrass's theorem [4, p. 6]

\[
D^2_{r_2} S_r(z, i) = \sum_{k \in \mathbb{N}, \lambda} D^2_{r_2} \phi^{(k)}_{\lambda}(r^{-1}z) \overline{\phi^{(k)}_{\lambda}(r^{-1}t)} r^{-k} = \sum_{k \in \mathbb{N}, \lambda} D^2_{r_2} \phi^{(k)}_{\lambda}(r^{-1}z) \overline{\phi^{(k)}_{\lambda}(r^{-1}t)} \left(\frac{r}{r_2}\right)^k,
\]

if \( z \neq 0 \), \( \notin \overline{D}_{r_z} \). By the Schwarz inequality

\[
|D^2_{r_2} S_r(z, i)|^2 \leq \sum_{k \in \mathbb{N}, \lambda} |D^2_{r_2} \phi^{(k)}_{\lambda}(r^{-1}z)|^2 \left(\frac{r}{r_2}\right)^k \sum_{k \in \mathbb{N}, \lambda} |\phi^{(k)}_{\lambda}(r^{-1}t)|^2 \left(\frac{r}{r_2}\right)^k
\]

(6)

\[
\leq \frac{r^N \mathcal{S}_n((rr_z)^{-1/2}z)}{V(r-r_z)^N},
\]

where \( \mathcal{S}_n(Z) = \sum_{k \in \mathbb{N}, \lambda} |D^2 \phi^{(k)}_{\lambda}(Z)|^2. \) If \( z = 0 \), \( r_z \) can be replaced by \( \frac{1}{2}r \), say. Now if \( t \in b_r \), then \( t \notin \overline{D}_{r_2} \) for \( r < r \). Thus in (4) we can take \( r = r \) for all \( t \). Hence by (3) with \( r_z = 1 \) and (4)

\[
|f(t)| \leq \|f\|_p / (1-r)^{2N/p}.
\]

Using (6) and (7) in (5) gives

\[
|D^2_r f(z)| \leq \frac{V^{1/2} r^{-N/2} \mathcal{S}_n((rr_z)^{-1/2}z)}{(r-r_z)^{N/2}(1-r)^{2N/p}} \|f\|_p \quad (Z = (rr_z)^{-1/2}z).
\]
On \((r, 1)\) the function
\[
Y(r) = r^{N/2}(r-r_2)^{-N/2}(1-r)^{-2N/p}\mathcal{S}_{n}^{1/2}(Z)
\]
is positive and continuous and \(\to \infty\) as \(r \to 1^+\) and \(\to r_2^+\), where \(0 < \mathcal{S}_{n}^{1/2}(z_2^{-1}) \leq \infty\).
Hence \(Y(r)\) assumes its minimum value on \((r, 1)\) on the compact set \(I_{\sigma} = \{r_2 + \sigma, 1 - \sigma\}\) if \(\sigma > 0\) is sufficiently small, and this minimum value is positive. Suppose \(Y\) assumes its minimum value at \(r = r_2\). Since \(Z_{n,r} = (r_2 z, r_2 z)^{-1/2} \in D\), \(\mathcal{S}_{n}^{1/2}(Z_{n,r}) = D_{n,2}^{S_{n}}(Z_{n,r}, Z_{n,r})\) and \((2)\) follows.

**Corollary 1** [11]. There exists a countable collection of linear functionals \(\{\eta_n\}\) on \(H^p\) such that if \(f \in H^p\) and \(f \neq 0\), then there is an \(n\) with \(\eta_n(f) \neq 0\).

**Proof.** Let \(z_0 \in D\) and set \(\eta_n = \gamma_{n, z_0}\). Since \(f \neq 0\) on \(D\), by the identity theorem \(f(z) \neq 0\) in any polydisc neighborhood \(N\) of \(z_0 \subset D\). Hence by the power series expansion of \(f\) in \(N\) some derivative \(D_{n,2}^{S_{n}}(z_0) \neq 0\). Thus \(\gamma_{n, z_0}(f) \neq 0\).

**Theorem 8.** Let \(F = \{f \in H^p : \gamma(f)\text{ is bounded on } F\text{ for fixed } \gamma \in (H^p)^*\}. Then there exists \(B > 0\), independent of \(f\), such that

\[
\begin{align*}
(a) \ |\gamma(f)| & \leq B\|\gamma\|, \\
(b) \ |f(z)| & \leq B(1-r_2)^{-2N/p}, \\
(c) \ |\gamma_{n,z}^{(p)}(f)| & \leq B^V_{N/2} \frac{1}{n! (r_2 z - r_2 z)^{N/2} (1 - r_2 z)^{2N/p}} \{D_{n,2}^{S_n}(Z_{n,r}, Z_{n,r})\}^{1/2} \quad (n > 0),
\end{align*}
\]
\((Z_{n,r} = (r_2 z, r_2 z)^{-1/2})\) for all \(f \in F\). (If \(z = 0\), set \(r_0 = \frac{1}{2} r_{n,0}\).)

**Proof.** The proof of \((a)\) uses only functional analysis and is the same as the proof of Theorem 7 in [11]. Inequalities \((b)\) and \((c)\) follow by setting \(\gamma = \gamma_{n, z}^{(p)}\) and using \((1)\) and \((2)\).

5. **Weak convergence.** A sequence \(\{f_n\} \subset H^p\) converges weakly to \(f \in H^p\), \(f_n \rightharpoonup f\), if \(\lim_n \gamma(f_n) = \gamma(f)\) for every \(\gamma \in (H^p)^*\). By Corollary 1 the limit is unique.

The following lemma is more general than necessary but has some independent interest. Let \(\Delta_R\) be a polydisc of radius \(R\) and center 0.

**Lemma 4** (Vitali’s Convergence Theorem for \(C^N\)). Let \(\{f_n\}\) be a sequence of holomorphic functions on the closed polydisc \(\overline{\Delta_R}\), which are bounded independently of \(z\) and \(n\) on \(\overline{\Delta_R}\). Also \(f_n \to a\) limit as \(n \to \infty\) on a set \(\{z'\}\) with limit point 0 and such that for each \(j, 1 \leq j \leq N\), \(\{z'_j\}\) is an infinite set. Then \(\{f_n\}\) tends uniformly to \(a\) limit on compact subsets of \(\Delta_R\).

**Proof.** It is sufficient to consider the case \(N = 2\). Then \(f_n\), holomorphic on \(\overline{\Delta_R}\), has a power series representation
\[
f_n(z) = \sum_{j,k=0}^\infty a_{jk}^n z_j^j z_k^k \quad (z \in \overline{\Delta_R}),
\]
where the convergence is absolute and uniform on compact subsets of \( \bar{\Delta}_R \). Following the proof in [9] for \( N=1 \), we show that \( \lim_n a_{00}^{(n)} \) exists for each \( j, k \) and equals \( a_{jk} \), say. By Schwarz's Lemma [2] and the uniform boundedness it follows as in [9] that \( \{a_{00}^{(n)}\} \) is a Cauchy sequence and hence convergent. The function

\[
g_{10}^{(n)}(z_1) = \sum_{j=1}^{\infty} a_{00}^{(n)} z_1^{-j-1} = \frac{f_n(z_1, 0) - a_{00}^{(j)}}{z_1}
\]

\((z_1 \neq 0)\) satisfies the same hypotheses as \( f_n(z) \).

**Proof.** The functions \( f_n(z_1, z_2) \) are holomorphic and uniformly bounded independently of \( n \) and \( z_1 \) on the closed disc \( |z_1| \leq R \). Also 0 is a limit point of the set \( (z_1')_2 \) and \( \lim_n f_n(z_1', z_2') \) exists. Thus by Vitali's convergence theorem for \( N=1 \), \( \lim_n f_n(z_1, z_2) \) exists uniformly on compact subsets of \( |z_1| < R \). Since also \( \lim_{z_2 \to 0} f_n(z_1, z_2) = f_n(z_1, 0) \), the hypotheses of the Moore-Osgood theorem are satisfied so that \( \lim_n \lim_{z_2 \to 0} f_n(z_1', z_2') = \lim_n f_n(z_1, 0) \) exists. Thus \( \lim g_{10}^{(n)}(z') \) exists. Also \( g_{10}^{(n)} \) is holomorphic on \( \bar{\Delta}_R \) with a removable singularity at \( z_1 = 0 \). Now \( |g_{10}^{(n)}(z_1)| \leq 2MR^{-1} \) on \( |z_1| = R \) so that by the maximum principle \( |g_{10}^{(n)}(z_1)| \leq 2MR^{-1} \) on \( |z_1| < R \). Hence similarly as for \( \{a_{00}^{(n)}\} \), \( \lim_n a_{10}^{(n)} \) exists. By an analogous argument \( \lim_n a_{01}^{(n)} \) and \( \lim_n a_{00}^{(n)} \) exist for all \( j \geq 1 \). Next set

\[
g_{11}^{(n)}(z) = \sum_{j,k=1}^{\infty} a_{jk}^{(n)} z_1^{-1} z_2^{-1} = \frac{f_n(z_1, z_2) - f_n(z_1, 0) - f_n(0, z_2) + f_n(0, 0)}{z_1 z_2}
\]

\( g_{11}^{(n)}(z_1) \) satisfies the same hypotheses as \( f_n(z) \). \( \lim_n g_{11}^{(n)}(z') \) exists since the four limits on the right side of (1) exist. Also \( g_{11}^{(n)} \) is holomorphic on \( \bar{\Delta}_R \) if \( z_1 z_2 \neq 0 \) and is locally bounded in the neighborhood of points \( (z_1, 0), (0, z_2), (0, 0) \) since

\[
\lim_{z_2 \to 0} g_{11}^{(n)}(z) = \sum_{j,k=1}^{\infty} a_{jk}^{(n)} z_1^{-1} = z_1^{-1} \frac{\partial f_n(z_1, 0)}{\partial z_2} - \frac{\partial f_n(0, 0)}{\partial z_2}
\]

for \( z_1 \neq 0 \) and similarly for the other two limits. Thus by Riemann's theorem on removable singularities \( [2] \) \( g_{11}^{(n)} \) is holomorphic on \( \bar{\Delta}_R \). Also on \( c = \{z : |z_j| = R, j=1, 2\} \), \( |g_{11}^{(n)}(z)| \leq 4MR^{-2} \) and by the maximum principle for polydiscs \( |g_{11}^{(n)}(z)| \leq 4MR^{-2} \) on \( \bar{\Delta}_R \). Thus as above \( \lim_n a_{11} \) exists. Similarly \( \lim_n a_{1k}^{(n)} \) exists.

Finally we show that

\[
\lim_n \sum_{j,k=0}^{\infty} a_{jk}^{(n)} z_1^j z_2^k = \sum_{j,k=0}^{\infty} a_{jk} z_1^j z_2^k.
\]

By the Cauchy inequality for derivatives \( [2] \) and the uniform boundedness follows \( |a_{jk}^{(n)}| \leq MR^{-j-k} \) and hence \( |a_{jk}| \leq MR^{-j-k} \). Thus the series on the right of (2) converges absolutely and uniformly on compact subsets of \( \Delta_R \). Now given \( \epsilon > 0 \) there exists \( K = K(\sigma, \epsilon) \) such that

\[
\sum_{j=k+1}^{\infty} \left( \frac{R-\sigma}{R} \right)^j < \frac{\epsilon A_\sigma}{4RM} \quad (\sigma > 0),
\]

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and \(N = N(K, \varepsilon)\) such that for \(n > N\)

\[
|a_{jk}^{(n)} - a_{jk}| < \frac{1}{2} e B_K, \quad B_K = \left( \frac{R - 1}{R^2 + 1 - 1} \right)^2 \quad (j, k = 1, \ldots, K).
\]

Then

\[
\sum_{j, k = 0}^{\infty} a_{jk}^{(n)} z_j^k - \sum_{j, k = 0}^{\infty} a_{jk} z_j^k \leq \left| \sum_{j, k = 0}^{K} (a_{jk}^{(n)} - a_{jk}) z_j^k \right| + \left| \sum_{j = K + 1}^{\infty} \sum_{k = 0}^{\infty} + \sum_{j = 0}^{K} \sum_{k = K + 1}^{\infty} \right) (a_{jk}^{(n)} - a_{jk}) z_j^k \right| \leq \left| \sum_{j, k = 0}^{K} |a_{jk}^{(n)} - a_{jk}| R^{j+k} + 2M \left( \sum_{j = K + 1}^{\infty} \sum_{k = 0}^{\infty} + \sum_{j = 0}^{K} \sum_{k = K + 1}^{\infty} \right) (R - \sigma)^j (R - \sigma)^k \right| < \varepsilon
\]

for \(n > N\). Thus (2) holds and Lemma 4 is proved.

We have

THEOREM 9. If \(f_n \to f\) in \(H^p\), then \(\lim_n f_n(z) = f(z)\) uniformly on compact subsets of \(D\).

Proof. Since \(\lim_n \gamma(f_n) = \gamma(f)\) for \(\gamma \in (H^p)^*\), \(\{\gamma(f_n)\}\) is bounded independently of \(n\). From inequality (4.9b) follows \(|f_n(z)| \leq B(1 - r)^{-2N/p}\) for \(z \in \overline{D}_r\), which bound is independent of \(n\) and \(z\). In particular \(\gamma_{0,2}(f_n) \to \gamma_{0,2}(f)\), that is, \(f_n(z) \to f(z)\) for \(z \in \overline{D}_r\). Hence by Lemma 4, \(f_n(z) \to f(z)\) uniformly on compact subsets of \(D_r\) \((r < 1)\). Hence \(f_n(z) \to f(z)\) uniformly on compact subsets of \(D\). (Lemma 4 was proved for a polydisc but the compact set \(\overline{D}\), can be covered by a finite number of closed polydiscs and the conclusion of Lemma 4 will hold for \(\overline{D}\), also.)

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