

A LEMMA IN TRANSCENDENTAL NUMBER THEORY

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In this paper, we discuss a faulty lemma of Gelfond of use in the theory of algebraic independence of transcendental numbers. We then replace Gelfond's lemma by a clearly stated simpler lemma which is much easier to apply.

Gelfond's [1, p. 140] lemma is:

"LEMMA. Suppose $p, q, p > q^r, r, r_1$ are positive rational integers, $\varepsilon > 0$ and γ are fixed, and all the numbers $\alpha_1, \alpha_2, \dots, \alpha_q$, as well as the numbers $\beta_1, \beta_2, \dots, \beta_r$ are distinct and arranged in order of increasing absolute values, in other words, $|\alpha_k| \leq |\alpha_{k+1}|$ and $|\beta_k| \leq |\beta_{k+1}|$. We set $|\alpha_q| = \alpha, |\beta_r| = \beta$ and suppose that there exist constants $\gamma_0 > 0, \gamma_1 > 0, \gamma_0 + \gamma_1 < 1$, such that $\alpha < (pq)^{\gamma_1}, \beta < (pq)^{\gamma_0}$. We also suppose that there exists a constant γ_2 , such that the inequalities

$$\prod_{k=1; k \neq i}^q |\alpha_i - \alpha_k| > \exp(-\gamma_2 q \ln pq),$$

$$|\alpha_i - \alpha_k| > \exp(-\gamma_2 q \ln pq), \quad 1 \leq i \leq q, \quad 1 \leq k \leq q$$

are satisfied. Further, if the function $f(z)$ has the form

$$f(z) = \sum_{k=0}^{p-1} \sum_{s=1}^q A_{k,s} z^k e^{z\alpha_s},$$

where the numbers $A_{k,s}$ are not all zero, then at least one of the numbers

$$f^{(s)}(\beta_k), \quad 0 \leq s \leq r_1 - 1, \quad 1 \leq k \leq r,$$

$$r_1 r \geq [\lambda pq], \quad \lambda = (1 + \gamma_1 + 2\gamma_2 + \varepsilon)/(1 - \gamma_1 - \gamma_0)$$

is different from zero for sufficiently large pq ."

Šmelev [2] also makes use of this result.

We next discuss Gelfond's lemma.

First of all, the phrase "the numbers $A_{k,s}$ are not all zero," in the original is literally "the numbers $A_{k,s}$ in the aggregate are different from zero." This phrase is repeatedly mistranslated in [1] as "all different from zero."

Next one gets the impression that p and q are independent integer variables subject to $p > q^r$. This condition is also written elsewhere as $p < q^r$. The theorem as

it stands does not say whether γ is positive or negative, and the only place where “ γ ” appears in the proof is in the statement “The condition $p > q^\gamma$ ”, appearing in the formulation of the lemma serves only to simplify the proof and may be replaced by a much weaker condition.” The proof given fails at several points, and it is hard to see how to make use of either of the conditions, though both hold with different γ ’s, in the various applications. Perhaps the explanation for this peculiar condition is that in one of the uses of the lemma, the general condition with which we replace $p < q^\gamma$ reduces to a condition where $p < q^\gamma$ can be used.

One also gets the impression that either r or r_1 is running, depending on the size of pq . Indeed, r must be running, as otherwise the condition of the existence of γ_0 such that $|\beta_r| < (pq)^{\gamma_0}$ would be trivial. However, the lemma could be interpreted as having r fixed. The quantity q is running, so apparently we are dealing with a sequence of α_j ’s. When the lemma is used it turns out that we are not dealing with a sequence of α_j ’s, nor with a sequence of β_j ’s.

It appears that there are hidden assumptions which are intuitively verified when using the result.

We now give a lemma which will replace the one above. It also omits the condition of the existence of γ_2 and the associated conditions on the α_j ’s. Further, it decreases λ , and we note that Gelfond states that this can be done. The new lemma adds conditions, conditions (iii) and (iv), on the rate of growth of the parameter r_1 . In the applications, one has r_1 bounded or with a known rate of growth. In Gelfond’s “proof”, the author believes Gelfond is implicitly assuming a small rate of growth for r_1 , as he appears to be abstracting from two different situations where this occurs. Our omission of the constant γ_2 and the associated conditions does not depend on the assumptions about r_1 . This improvement of the lemma is due to a different method of estimation of the full Hermite interpolating polynomial.

LEMMA. *Let N be a positive integer variable. Let $p=p(N)$, $q=q(N)$, $r=r(N)$ be positive integer functions of N such that $p \cdot q = m$ tends to ∞ with N . To each $q=q(N)$ let there correspond a set A_q of q numbers $\alpha_{q,1}, \alpha_{q,2}, \dots, \alpha_{q,q}$, which we write also by an abuse of notation as $\alpha_1, \dots, \alpha_q$. To each $r=r(N)$ let there correspond a set B_r of r numbers $\beta_{r,1}, \dots, \beta_{r,r}$ which we also write as β_1, \dots, β_r . We assume also that there exist positive constants γ_0 and γ_1 such that $\gamma_0 + \gamma_1 < 1$, and for p , q , and r (determined by N),*

$$(i) \quad |\beta_{r,j}| < (p \cdot q)^{\gamma_0},$$

$$(ii) \quad |\alpha_{q,j}| < (p \cdot q)^{\gamma_1}.$$

Let ε be given, $\varepsilon > 0$. Define

$$\lambda = (1 + \gamma_1 + \varepsilon)/(1 - \gamma_1 - \gamma_0), \quad r_1 = \text{the least integer} \geq [\lambda pq]/r,$$

and assume finally, that

$$(iii) \quad r_1! \leq m^{\gamma_1 m}, \text{ for } N \text{ sufficiently large, and}$$

$$(iv) \quad rr_1 = O(m).$$

Then there is an N_0 , such that for any particular N greater than N_0 , if the function $f(z)$ has the form

$$f(z) = \sum_{k=0}^{p-1} \sum_{s=1}^q A_{k,s} z^k e^{z\alpha_s},$$

where not all $A_{k,s}$ are zero, and if $r_2 \geq r_1$ then at least one of the numbers

$$f^{(s)}(\beta_k), \quad 0 \leq s \leq r_2 - 1, \quad 1 \leq k \leq r$$

is different from zero.

Proof. Without loss of generality, we can take $r_2 = r_1$. We will arrive in the proof at various minima for N_0 , and it will be understood that we take the largest of these. We set $\varepsilon = 3\delta$. Without loss of generality, we can take $|A_{k,s}| \leq 1$, where at least one $A_{k,s}$ has absolute value 1. For we can divide all the coefficients of $f(z)$ by the coefficient with largest absolute value. Arguing by contradiction, assume $N > N_0$, and λ and r_1 given as above and that

$$f^{(s)}(\beta_k) = 0, \quad 0 \leq s \leq r_1 - 1, \quad 1 \leq k \leq r.$$

Let Γ be the circle $|z| = 1$ and Γ_1 the circle $|\xi| = m^{1-\gamma_1}$. We take N so large that $m \geq 2$. Then Γ is inside Γ_1 . Now $\gamma_0 + \gamma_1 < 1$, so $\gamma_0 < 1 - \gamma_1$. Thus, the points β_j lie in the interior of Γ_1 , as $|\beta_j| < (p \cdot q)^{\gamma_0} = m^{\gamma_0} < m^{1-\gamma_1}$. Since $f^{(s)}(\beta_k) = 0$, $0 \leq s \leq r_1 - 1$, $1 \leq k \leq r$, we have that

$$(1) \quad f(\xi) = [(\xi - \beta_1)(\xi - \beta_2) \cdots (\xi - \beta_r)]^{r_1} g(\xi),$$

where $g(\xi)$ is analytic everywhere.

Now, for z on Γ , or indeed anywhere in the interior of Γ_1 ,

$$\frac{1}{2\pi i} \int_{\Gamma_1} \frac{g(\xi)}{\xi - z} d\xi = g(z)$$

by the Cauchy integral formula, so

$$f(z) = [(z - \beta_1)(z - \beta_2) \cdots (z - \beta_r)]^{r_1} \frac{1}{2\pi i} \int_{\Gamma_1} \frac{1}{[(\xi - \beta_1) \cdots (\xi - \beta_r)]^{r_1}} \frac{f(\xi)}{\xi - z} d\xi,$$

as we can divide out $[(\xi - \beta_1) \cdots (\xi - \beta_r)]^{r_1}$ in (1) for ξ on Γ_1 . Hence, again by the Cauchy integral formula

$$(2) \quad f^{(s)}(0) = \frac{s!}{(2\pi i)^2} \int_{\Gamma} \frac{1}{z^{s+1}} \left\{ \int_{\Gamma_1} \left[\frac{(z - \beta_1) \cdots (z - \beta_r)}{(\xi - \beta_1) \cdots (\xi - \beta_r)} \right]^{r_1} \frac{f(\xi)}{\xi - z} d\xi \right\} dz.$$

Now,

$$\begin{aligned} |f^{(s)}(0)| &\leq \frac{s!}{(2\pi)^2} 2\pi \max_{|z|=1} \left| \int_{\Gamma_1} \left[\frac{(z - \beta_1) \cdots (z - \beta_r)}{(\xi - \beta_1) \cdots (\xi - \beta_r)} \right]^{r_1} \frac{f(\xi)}{\xi - z} d\xi \right| \\ &\leq \frac{s!}{2\pi} \max_{|z|=1} \left\{ 2\pi m^{1-\gamma_1} \max_{|\xi|=m^{1-\gamma_1}} \left\{ \left| \left[\frac{(z - \beta_1) \cdots (z - \beta_r)}{(\xi - \beta_1) \cdots (\xi - \beta_r)} \right]^{r_1} \frac{f(\xi)}{\xi - z} \right| \right\} \right\}. \end{aligned}$$

Also

$$\begin{aligned} |\xi - z| &\geq |\xi| - |z| = m^{1-\gamma_1} - 1, \\ |\xi - \beta_j| &\geq |\xi| - |\beta_j| \geq m^{1-\gamma_1} - m^{\gamma_0}, \quad j = 1, \dots, r, \\ |z - \beta_j| &\leq 1 + |\beta_j| \leq 1 + m^{\gamma_0}, \quad j = 1, \dots, r. \end{aligned}$$

Thus,

$$(3) \quad |f^{(s)}(0)| \leq s! m^{1-\gamma_1} \left[\frac{1+m^{\gamma_0}}{m^{1-\gamma_1}-m^{\gamma_0}} \right]^{rr_1} \frac{1}{m^{1-\gamma_1}-1} \max_{|\xi|=m^{1-\gamma_1}} |f(\xi)|.$$

Next, we have

$$|e^{\xi \alpha_s}| = \exp(\operatorname{Re}(\xi \alpha_s)) \leq \exp(|\xi \alpha_s|) = \exp(|\alpha_s| |\xi|) \leq \exp(m^{\gamma_1} m^{1-\gamma_1}) = e^m,$$

so

$$\begin{aligned} \max_{|\xi|=m^{1-\gamma_1}} |f(\xi)| &\leq \max_{|\xi|=m^{1-\gamma_1}} \sum_{k=0}^{p-1} \sum_{s=1}^q |\xi|^k e^m \\ (4) \quad &\leq p \cdot q (m^{1-\gamma_1})^{p-1} e^m \\ &\leq m \cdot e^m \cdot m^{m(1-\gamma_1)}. \end{aligned}$$

Thus, from (3) and (4) we obtain

$$\begin{aligned} |f^{(s)}(0)| &\leq s! \frac{m^{1-\gamma_1}}{m^{1-\gamma_1}-1} m^{rr_1(\gamma_0+\gamma_1-1)} \left[\frac{1+m^{-\gamma_0}}{1-m^{\gamma_0+\gamma_1-1}} \right]^{rr_1} m e^m m^{m(1-\gamma_1)} \\ (5) \quad &\leq r_1! m^{-\gamma_1 m} m^m m^{-\lambda(1-\gamma_0-\gamma_1)m} \cdot A \\ &\leq m^m m^{-\lambda(1-\gamma_0-\gamma_1)m} \cdot A, \end{aligned}$$

using (iii), where

$$(6) \quad A = m e^m \left[\frac{1+m^{-\gamma_0}}{1-m^{\gamma_0+\gamma_1-1}} \right]^{rr_1} m^{(1-\gamma_0-\gamma_1)(\lambda m - rr_1)}.$$

To obtain our first main result (7) below, we need only show $A \leq m^{\delta m}$ for N sufficiently large. Now, $\lambda m < [\lambda m] + 1 \leq rr_1$, since $r_1 r$ is the least multiple of $r \geq [\lambda m]$, unless $[\lambda m] = r_1 r$, in which case $\lambda m - rr_1 < 1$. In either case,

$$m^{(1-\gamma_0-\gamma_1)(\lambda m - rr_1)} \leq m^{1-\gamma_0-\gamma_1}.$$

Let $G(m)$ be a positive function of m . We have $G(m) \leq m^{\varepsilon_1 m}$ for m sufficiently large for any given $\varepsilon_1 > 0$ if and only if $\varepsilon_1 \geq (\log G(m))/(m \log m)$ for m sufficiently large. Thus, if $\lim_{m \rightarrow \infty} (\log G(m))/(m \log m) = 0$, then we can conclude that we can assign ε_1 as we please, and then take m beyond some fixed lower bound and have $G(m) \leq m^{\varepsilon_1 m}$.

Thus, if we take $G(m)$ successively as m , e^m , $m^{1-\gamma_0-\gamma_1}$, $4^{K_1 m}$, $K_1 > 0$, we obtain $m < m^{\delta m/4}$, $e^m < m^{\delta m/4}$, $m^{1-\gamma_0-\gamma_1} < m^{\delta m/4}$, $4^{K_1 m} < m^{\delta m/4}$, for m sufficiently large. Finally, if m is sufficiently large, $1 + m^{-\gamma_0} < 2$ and $1/(1 - m^{\gamma_0+\gamma_1-1}) < 2$, so, using $rr_1 \leq K_1 m$ from (iv), we obtain

$$\left[\frac{1+m^{-\gamma_0}}{1-m^{\gamma_0+\gamma_1-1}} \right]^{rr_1} \leq 4^{rr_1} \leq 4^{K_1 m} < m^{\delta m/4},$$

for m sufficiently large.

Thus, for m sufficiently large, $A \leq m^{\delta m}$, and we have

$$(7) \quad |f^{(s)}(0)| \leq m^{(1+\delta-\lambda(1-\gamma_0-\gamma_1))m}.$$

We note that (iv) could be replaced by

$$(iv') \quad \lim_{m \rightarrow \infty} \left(rr_1 \log \left[\frac{1+m^{-\gamma_0}}{1-m^{\gamma_0+\gamma_1-1}} \right] \right) / (m \log m) = 0.$$

We now apply the theory of full Hermite interpolation. Let $\alpha_1, \dots, \alpha_q$ be distinct complex numbers and let k_1, \dots, k_q be positive integers and let $k_1 + \dots + k_q = k + 1$. Let $y_j^{(\nu)}$ be constants, $j = 1, \dots, q$, $\nu = 0, 1, \dots, k_j - 1$. J. L. Walsh [3, pp. 49–50] gives a method of proof for the existence and uniqueness of a polynomial $P_k(z)$, of degree $\leq k$, called the full Hermite interpolation polynomial, satisfying

$$(8) \quad P_k^{(\nu)}(\alpha_j) = y_j^{(\nu)}, \quad j = 1, \dots, q; \nu = 0, 1, \dots, k_j - 1.$$

I. S. Berezin and N. P. Zhidkov [4, pp. 145–147] construct such a polynomial, by first constructing polynomials $P_{i,j}(z)$ satisfying

$$(9) \quad P_{i,j}^{(\nu)}(\alpha_s) = \delta_{i,s} \delta_{j,\nu}, \quad s = 1, \dots, q; \nu = 0, 1, \dots, k_i - 1,$$

where $\delta_{i,s}$ and $\delta_{j,\nu}$ are Kronecker deltas, so that

$$(10) \quad P_k(z) = \sum_{i=1}^q \sum_{j=0}^{k_i-1} y_i^{(j)} P_{i,j}(z).$$

Writing

$$(11) \quad w(z) = (z - \alpha_1)^{k_1} (z - \alpha_2)^{k_2} \cdots (z - \alpha_q)^{k_q},$$

they find

$$P_{i,j}(z) = \frac{w(z)}{(z - \alpha_i)^{k_i-j}} \frac{1}{j!} \sum_{n=0}^{k_i-j-1} \frac{1}{n!} (z - \alpha_i)^n \left[\frac{d^n}{d\xi^n} \left(\frac{(\xi - \alpha_i)^{k_i}}{w(\xi)} \right) \right] \Big|_{\xi = \alpha_i}.$$

In the special case of $k_1 = k_2 = \dots = k_q = p$, we obtain

$$(12) \quad P_{i,j}(z) = \frac{w(z)}{(z - \alpha_i)^{p-j}} \frac{1}{j!} \sum_{n=0}^{p-j-1} \frac{1}{n!} (z - \alpha_i)^n \left[\frac{d^n}{d\xi^n} \left(\frac{(\xi - \alpha_i)^p}{w(\xi)} \right) \right] \Big|_{\xi = \alpha_i},$$

where $1 \leq i \leq q$ and $0 \leq j \leq p - 1$, and $w(z) = [(z - \alpha_1) \cdots (z - \alpha_q)]^p$. Next, we write

$$P_{i,j}(z) = \sum_{k=0}^{m-1} C_k z^k,$$

so $C_k = P_{i,j}^{(k)}(0)/k!$.

Using (12), Gelfond now estimates $|C_k|$ by a complicated algebraic expansion of

$$B_{i,n} = \frac{d^n}{d\xi^n} \left(\frac{(\xi - \alpha_i)^p}{w(\xi)} \right) \Big|_{\xi = \alpha_i},$$

and arrives at his intricate conditions on the α_j 's. We proceed by an integral estimate.

Let $\alpha = m^{\gamma_1}$. We have

$$\begin{aligned} C_k &= P_{i,j}^{(k)}(0)/k! \\ &= \frac{1}{2\pi i} \int_{|z|=2\alpha} \left(\frac{w(z)}{(z-\alpha_i)^{p-j}} \frac{1}{j!} \sum_{n=0}^{p-j-1} \frac{1}{n!} (z-\alpha_i)^n B_{i,n} \right) \frac{dz}{z^{k+1}}, \end{aligned}$$

so

$$|C_k| \leq 2\alpha \max_{|z|=2\alpha} \left| \frac{w(z)}{(z-\alpha_i)^{p-j}} \frac{1}{j!} \sum_{n=0}^{p-j-1} \frac{1}{n!} (z-\alpha_i)^n B_{i,n} \right| \cdot \frac{1}{(2\alpha)^{k+1}}.$$

Now, on $|z|=2\alpha$, since $|\alpha_j| \leq \alpha$, we have

$$(13) \quad \alpha \leq |z-\alpha_j| \leq 3\alpha,$$

so

$$|C_k| \leq 2\alpha(3\alpha)^{q(p-1)}(3\alpha)^j \frac{1}{j!} \left(\sum_{n=0}^{p-j-1} \frac{1}{n!} (3\alpha)^n |B_{i,n}| \right) (2\alpha)^{-1-k}.$$

Next,

$$B_{i,n} = \frac{n!}{2\pi i} \int_{|\xi|=2\alpha} \frac{(\xi-\alpha_i)^p}{w(\xi)} \cdot \frac{1}{(\xi-\alpha_i)^{n+1}} d\xi,$$

so, using (13)

$$|B_{i,n}| \leq \frac{n!}{2\pi} 4\alpha\pi \frac{1}{\alpha^{q(p-1)}} \cdot \frac{1}{\alpha^{n+1}} = n! \frac{2}{\alpha^{q(p-1)+n}}.$$

Thus,

$$\begin{aligned} |C_k| &\leq 2\alpha(3\alpha)^{q(p-1)}(3\alpha)^j \frac{1}{j!} \left(\sum_{n=0}^{p-j-1} (3\alpha)^n \frac{2}{\alpha^{q(p-1)+n}} \right) \frac{1}{(2\alpha)^{k+1}} \\ &\leq \frac{1}{(2\alpha)^k} 3^{q(p-1)}(3\alpha)^{p-1} \left(\sum_{n=0}^{p-j-1} 2 \cdot 3^n \right) \\ &\leq 3^{qp} 3^p \alpha^m(p) \cdot 2 \cdot 3^{p-1} \\ &\leq 2 \cdot p \cdot 3^{qp} 3^{2p} \alpha^m \\ &\leq m^2 3^{3m} \alpha^m = m^2 3^{3m} m^{\gamma_1 m}, \end{aligned}$$

and taking $G(m)=m^2$ respectively 3^{3m} , we obtain $m^2 < m^{\delta m/2}$, $3^{3m} < m^{\delta m/2}$, for m sufficiently large. Thus,

$$(14) \quad |C_k| \leq \exp((\delta + \gamma_1)m \log m).$$

We next show that for k and r nonnegative integers, we have

$$\frac{d^k}{dz^k} z^r e^{\alpha z} \Big|_{z=0} = \frac{d^r}{dz^r} z^k \Big|_{z=\alpha}$$

The proof is as follows. For $r=0$, $k=0$, we have $e^{\alpha z}=1$, $z^0=1$. For $r=0$, $k>0$, we have

$$\frac{d^k}{dz^k} e^{\alpha z} \Big|_{z=0} = (\alpha)^k e^{\alpha z} \Big|_{z=0} = \alpha^k, \quad z^k \Big|_{z=\alpha} = \alpha^k,$$

so we have the result. For $k=0$, $r>0$, we have

$$\left. \frac{d^k}{dz^k} z^r e^{\alpha z} \right|_{z=0} = z^r e^{\alpha z} \Big|_{z=0} = 0, \quad \left. \frac{d^r}{dz^r} 1 \right|_{z=\alpha} = 0.$$

For $k=1$, $r>0$, we have

$$\begin{aligned} \left. \frac{d}{dz} z^r e^{\alpha z} \right|_{z=0} &= r z^{r-1} e^{\alpha z} + \alpha z^r e^{\alpha z} \Big|_{z=0} = r z^{r-1} \Big|_{z=0} = 1 \quad \text{if } r=1, \\ &= 0 \quad \text{otherwise,} \\ \frac{d^r}{dz^r} z &= 1 \quad \text{if } r=1, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Finally, let $k>1$, $r>0$, then

$$\begin{aligned} \left. \frac{d^k}{dz^k} z^r e^{\alpha z} \right|_{z=0} &= \sum_{p=0}^k \binom{k}{p} (z^r)^{(p)} \alpha^{k-p} e^{\alpha z} \Big|_{z=0} = r! \binom{k}{p} \alpha^{k-r} \quad \text{if } r \leq k, \\ &= 0 \quad \text{if } r > k; \\ \left. \frac{d^r}{dz^r} z^k \right|_{z=\alpha} &= (k)(k-1) \cdots (k-r+1) \alpha^{k-r} \quad \text{if } r \leq k, \\ &= 0 \quad \text{if } r > k, \end{aligned}$$

and the results are easily seen to be the same.

Hence, we obtain

$$\begin{aligned} (15) \quad P_{i,j}^{(r)}(\alpha_s) &= \sum_{k=0}^{m-1} C_k \left. \frac{d^r}{dz^r} z^k \right|_{z=\alpha_s} \\ &= \sum_{k=0}^{m-1} C_k \left. \frac{d^k}{dz^k} z^r e^{\alpha_s z} \right|_{z=0}. \end{aligned}$$

Thus,

$$\begin{aligned} A_{j,i} &= \sum_{r=0}^{p-1} \sum_{s=1}^q A_{r,s} P_{i,j}^{(r)}(\alpha_s) \\ &= \sum_{r=0}^{p-1} \sum_{s=1}^q A_{r,s} \sum_{k=0}^{m-1} C_k \left. \frac{d^k}{dz^k} z^r e^{\alpha_s z} \right|_{z=0} \\ &= \sum_{k=0}^{m-1} C_k f^{(k)}(0). \end{aligned}$$

Choose $A_{j,i}$ so $|A_{j,i}|=1$. Then from (7) and (14)

$$\begin{aligned} 1 &\leq \sum_{k=0}^{n-1} |C_k| |f^{(k)}(0)| \leq m \exp [(\gamma_1 + \delta + 1 + \delta - \lambda(1 - \gamma_0 - \gamma_1))m \log m] \\ &= m \exp [(\gamma_1 + 1 + 2\delta - (1 + \gamma_1 + \varepsilon))m \log m] \\ &= m \exp (-\delta m \log m). \end{aligned}$$

This is a contradiction for m sufficiently large, thus proving the lemma.

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