A CLASS OF NONLINEAR EVOLUTION EQUATIONS
IN A BANACH SPACE

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We treat the nonlinear evolution equation

(*) \[ f'(t) = A(t, f(t))f(t) \]

where the unknown function \( f \) is from a real number interval into a Banach space \( X \). For suitable real numbers \( t \) and vectors \( x \) in \( X \), \( A(t, x) \) is the infinitesimal generator of a holomorphic semigroup of linear contraction operators in \( X \), and certain regularity requirements are placed on the function \( (t, x) \to A(t, x) \).

After proving a local existence, uniqueness, and stability theorem for (*), we consider the case \( A(t, x) = H(x) \) and obtain conditions under which there is a strongly continuous semigroup of nonlinear nonexpansive transformations whose infinitesimal generator is an extension of the transformation \( Qx = H(x)x \).

We state our main results in §1 and prove them in §2. In §3, we prove some theorems about linear semigroups in a function space which yield examples of our main results and are of some interest in themselves.

1. The main results. Let \( X \) be a complex Banach space. If \( 0 < \phi \leq \pi/2 \), then let \( S_\phi = \{ z \in C : z = 0 \text{ or } |\arg z| \leq \phi \} \), where \( C \) denotes the complex plane. Following [6], we denote by \( \text{CH}(\phi) \) the collection of all semigroups \( \{ T(z) : z \in S_\phi \} \) of linear contraction operators in \( X \) which are holomorphic on \( \text{int}(S_\phi) \) and strongly continuous on \( S_\phi \). We denote by \( \text{GH}(\phi) \) the collection of all infinitesimal generators of semigroups in \( \text{CH}(\phi) \).

Let \( [a, b] \) be a closed real number interval, \( 0 < \phi \leq \pi/2 \), and \( S \) a closed set in \( X \). Let \( A \) be a function from \( [a, b] \times S \to \text{GH}(\phi) \) such that the following conditions are satisfied:

\( (C_1) \) The operators \( A(t, x) \) all have the same domain \( D_0 \).
\( (C_2) \) There is a locally bounded nonnegative function \( K \) on \( [a, b] \times S \) such that
\[ \| (I - A(t, y))(I - A(s, x))^{-1} - I \| \leq K(s, x)(|s - t| + \|x - y\|) \]
for \( a \leq s, t \leq b \) and \( x, y \in S \), where \( I \) denotes the identity transformation on \( X \).
\( (C_3) \) \( \{ \exp[zA(t, x)] : z \in S_\phi \} \subset S \) for \( z \geq 0, a \leq t \leq b, \) and \( x \in S \), where
\[ \{ \exp[zA(t, x)] : z \in S_\phi \} \]
is the class \( \text{CH}(\phi) \) semigroup generated by \( A(t, x) \).

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In connection with the condition \((C_2)\), we mention that the invertibility of 
\(I - A(s, x)\) follows from the fact that \(A(s, x) \in \text{GH}(\phi)\), see \([6]\), or \([2]\), which serves also as a general reference for semigroups of operators. The fact that 
\([I - A(t, y)] \cdot [I - A(s, x)]^{-1}\) is bounded follows from \((C_1)\) and \([3\), Lemma 2, p. 212]\).

**Theorem 1.** Suppose \(x_0 \in D_0 \cap S\) and \(a \leq t_0 < b\). Then there is a number \(c\) in 
\((t_0, b]\) such that there is a unique continuously differentiable function \(f\) from 
\([t_0, c]\) into \(D_0 \cap S\) satisfying \(f(t_0) = x_0\) and

\[
f'(t) = A(t, f(t))f(t) \quad \text{for } t_0 \leq t \leq c.
\]

Also, if \(\varepsilon > 0\), then there exists \(\delta > 0\) such that if \(x_1 \in D_0 \cap S, t_0 < c_1 \leq c, \|x_0 - x_1\| < \delta,\) and \(g\) is a continuously differentiable function from 
\([t_0, c_1]\) into \(D_0 \cap S\) such that 
\(g(t_0) = x_1\) and 
\[g'(t) = A(t, g(t))g(t) \text{ for } t_0 \leq t \leq c_1, \text{ then } \|g(t) - f(t)\| < \varepsilon \text{ for } t_0 \leq t \leq c_1.\]

**Definition 1.1.** A semi-inner product on \(X\) means a function \([\cdot, \cdot]\) from \(X \times X\) 
to \(C\) such that for each \(y \in Y\), \([\cdot, y]\) is a bounded linear functional of norm \(\|y\|\), 
and \([y, y] = \|y\|^2\) (see \([5]\)).

**Definition 1.2.** If \([\cdot, \cdot]\) is a semi-inner product on \(X\), then a transformation \(W\) 
with domain and range contained in \(X\) is said to be dissipative (with respect to 
\([\cdot, \cdot]\) if \(\Re [Wx - Wy, x - y] \leq 0\) for \(x, y \in D(W)\), the domain of \(W\).

**Remark.** Throughout this section, \([\cdot, \cdot]\) will denote a fixed semi-inner product 
on \(X\), and all results will be independent of the particular semi-inner product used.

**Theorem 2.** Suppose \(H\) is a function from \(S\) into \(\text{GH}(\phi)\) which satisfies conditions 
\((C_1), (C_2),\) and \((C_3); more precisely, the function \((t, x) \rightarrow H(x)\) satisfies these conditions. Suppose \(D_0 \cap S\) is dense in \(S\) and define \(Q\) on \(D_0 \cap S\) by 
\(Qx = H(x)x\). Suppose \(Q\) is dissipative. Then there is a unique strongly continuous semigroup 
\(\{T(t); t \geq 0\}\) of nonexpansive nonlinear transformations from \(S\) into \(S\) such that 
for each \(x\) in \(D_0 \cap S\), \(T(\cdot)x\) is a continuously differentiable function from \([0, \infty)\) into 
\(D_0 \cap S\), and \((d/dt)T(t)x = QT(t)x\) for \(t \geq 0\).

**2. Proof of the main theorems.** We will call a function \(B\) from a number interval 
\([0, R]\) into \(\text{GH}(\phi)\) regular if the following conditions are satisfied:

\((R_1)\) \(B(t)\) has domain \(D_0\) for \(0 \leq t \leq R\).

\((R_2)\) There is a positive constant \(L\) such that

\[
\|[I - B(t)][I - B(s)]^{-1} - I\| \leq L|t - s|
\]

for \(0 \leq s, t \leq R\).

\((R_3)\) \((\exp [\xi B(t)])S \subseteq S\) for \(\xi \geq 0\) and \(0 \leq t \leq R\).

We point out that a regular operator function \(B\) on \([0, R]\) also satisfies:

\((R_4)\) \(\|[I - B(t)][I - B(s)]^{-1}\| \leq 1 + LR\) for \(0 \leq s, t \leq R\), where \(L\) is as in \((R_2)\).

\[
\|[I - B(r)][I - B(s)]^{-1} - [I - B(t)][I - B(s)]^{-1}\|
\]

\[
\leq \|[I - B(r)][I - B(t)]^{-1} - I\| \cdot \|[I - B(t)][I - B(s)]^{-1}\|
\]

\[
\leq |r - t|(1 + LR)L.
\]
Lemma 2.1. Suppose $B$ is a regular operator function on $[0, R]$, and $\beta$ is a positive nonincreasing function on $[0, R]$ with Lipschitz constant $L'$. Define the operator function $A$ on $[0, R]$ by $A(t) = \beta(t)[B(t) - I]$.

Then $A$ satisfies Tanabe's conditions $1^o$ and $2^o$ of [8]. In particular, let $0 < \phi_1 < \phi$ and define

$$\Sigma = \{\lambda \in C : \lambda = 0 \text{ or } |\arg \lambda| \leq \phi_1 + \pi/2\}.$$

Also define

$$M = [\beta(R) \sin (\phi - \phi_1)]^{-1}[(1 - \sin \phi_2)/2]^{-1/2},$$

$$K = (L\beta(0) + L')(1 + LR)/\beta(R),$$

where $L$ is as in (R.2). Then $A$ satisfies the conditions:

$(T_1)$ $\rho(A(t)) \supset \Sigma$ for $0 \leq t \leq R$, and

$$\|[\lambda I - A(t)]^{-1}\| \leq M/|\lambda| + 1$$

for $\lambda \in \Sigma$ and $0 \leq t \leq R$. (If $T$ is an operator in $X$, then $\rho(T)$ denotes the resolvent set of $T$.)

$(T_2)$ $\|A(r)A(s)^{-1} - A(t)A(s)^{-1}\| \leq K|r - t|$ for $0 \leq r, s, t \leq R$.

Proof. Let $0 \leq s, t, r \leq R$. Define

$$\Delta_\phi = \{\lambda \in C : \lambda = 0 \text{ or } |\arg \lambda| \geq \phi + \pi/2\},$$

Then $\beta(t)B(t) \in \text{GH}(\phi)$, and $\Sigma \subset C \setminus \Delta_\phi$, so $\rho(\beta(t)B(t)) \supset \Sigma$ and $\|[\lambda I - \beta(t)B(t)]^{-1}\| \leq 1/d(\lambda, \Delta_\phi)$ for $\lambda \in \Sigma$, see [6]. Also $\lambda - A(t) = [\lambda + \beta(t)]I - \beta(t)B(t)$, and $\lambda + \beta(t) \in \Sigma$ if $\lambda \in \Sigma$, so $\|[\lambda I - A(t)]^{-1}\| \leq 1/d(\lambda + \beta(t), \Delta_\phi)$ for $\lambda \in \Sigma$. Property $(T_1)$ follows from this and the fact that $d(\lambda + \beta(t), \Delta_\phi) \geq (|\lambda| + 1)/M$ for $\lambda \in \Sigma$.

Let $A_{\xi} = A(\xi), B_{\xi} = B(\xi)$, and $\beta_{\xi} = \beta(\xi)$ for $0 \leq \xi \leq R$. Then

$$A_{\xi}A_{\xi}^{-1} - A_{\xi}A_{\xi}^{-1} = \beta_{\xi}^{-1}[\beta_{\xi}]^{-1}[I - B_{\xi}]^{-1}[(I - B_{\xi})^{-1} - (I - B_{\xi})^{-1}],$$

so that $(T_2)$ follows from $(R.4)$ and $(R.5)$.

Lemma 2.2. Let $F$ be a function from $[0, R]$ into $[a, b]$ with Lipschitz constant $L_1$, and $\psi$ a function from $[0, R]$ into $S$ with Lipschitz constant $L_2$. Define the operator function $B$ on $[0, R]$ by

$$B(t) = A(F(t), \psi(t)).$$

Then $B$ is regular, where we can take the constant $L$ of $(R.2)$ as

$$L = (L_1 + L_2) \sup_{0 \leq t \leq R} K(F(t), \psi(t)),$$

see $(C.2)$.

Thus if $\beta$ is positive, nonincreasing, and Lipschitz continuous on $[0, R]$, and we define $A(t) = \beta(t)[B(t) - I]$ for $0 \leq t \leq R$, then $A$ satisfies $(T_1)$ and $(T_2)$.

If in addition, $\beta(t) - \beta(s) \leq -(t - s)\beta(0)L$ for $0 \leq s \leq t \leq R$, then $A$ also satisfies

$(T_3)$ $\|A(t)A(s)^{-1}\| \leq 1$ for $0 \leq s \leq t \leq R$.

Proof. Only the last statement needs proof, and it follows from the fact that $\beta(t)\beta(s)^{-1} \leq e^{[\beta(t) - \beta(s)]/\beta(s)}$, and $\|[I - B(t)](I - B(s))^{-1}\| \leq e^{L(t - s)}$ for $0 \leq s \leq t \leq R$. 

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Lemma 2.3. Let the operator function $A : [0, R] \to GH(\phi)$ be as in Lemma 2.1, and let $x_0 \in D_0$. Then there is a unique continuously differentiable function $f$ from $[0, R]$ into $D_0$ such that $f(0) = x_0$ and $f'(t) = A(t)f(t)$ for $0 \leq t \leq R$.

Proof. Tanabe establishes much more than this in [8].

Lemma 2.4. Let $A$, $x_0$, and $f$ be as in Lemma 2.3. If $\Delta = \{t_0, \ldots, t_n\}$ is a partition of $[0, R]$, then let $f_\Delta$ be defined on $[0, R]$ by $f_\Delta(0) = x_0$, and

$$f_\Delta(t) = T_k(t-t_{k-1})f_\Delta(t_{k-1}) \quad \text{for} \ t_{k-1} \leq t \leq t_k,$$

where $T_k(\xi) = \exp[\xi A(t_k)]$. Then $f_\Delta$ converges uniformly to $f$ on $[0, R]$ as the norm of $\Delta$ approaches zero.

Proof. Define $A_\Delta$ on $[0, R]$ by $A_\Delta(0) = A(t_1)$ and $A_\Delta(t) = A(t_k)$ for $t_{k-1} < t \leq t_k$. Then $f_\Delta(t) = A_\Delta(t)f_\Delta(t)$ for $t \in [0, R] \setminus \Delta$.

Let $h_\Delta(t) = f(t) - A_\Delta(t)f_\Delta(t)$ for $0 \leq t \leq R$. Then

$$h_\Delta'(t) = \left[A(t) - A_\Delta(t)\right]f(t) + A_\Delta(t)h_\Delta(t) = \left[1 - A_\Delta(t)A(t)^{-1}\right]f'(t) + A_\Delta(t)h_\Delta(t).$$

By [4, Lemma 1.3, p. 510], $\|h_\Delta(t)\|_A(d/dt)\|h_\Delta(t)\| = \text{Re}[h_\Delta'(t), h_\Delta(t)]$ a.e. on $[0, R]$. Thus $\|h_\Delta(t)\| \leq K|\Delta| A$ a.e. on $[0, R]$, where $K$ is as in (T2),

$$\Lambda = \sup_{0 \leq t \leq R} \|A(t)\|,$$

and $|\Delta|$ denotes the norm of $\Delta$. We have used the fact that $A_\Delta(t)$ is dissipative, see [5].

Lemma 2.5. Let $A$, $x_0$, and $f$ be as in Lemma 2.3. Then

$$\|f(t)\| \leq \|f(0)\| \exp \left[ -\int_0^t \beta \right]$$

for $0 \leq t \leq R$.

If $A$ satisfies (T3), then $\|f'(t)\| \leq \|f'(0)\| \exp \left[ -\int_0^t \beta \right]$ for $0 \leq t \leq R$.

Proof. Let $\Delta, f_\Delta$, and $T_\Delta$ be as in Lemma 2.4. Then $T_\Delta(\xi) = e^{-\xi R(t_k)}\exp[\xi B(t_k)]$, so that $\|T_\Delta(\xi)\| \leq e^{-\xi R(t_k)}$. Therefore,

$$\|f_\Delta(t)\| \leq \|f(0)\| \exp \left[ -\beta(t_k)(t-t_{k-1}) - \sum_{j=1}^{k-1} \beta(t_j)(t_j-t_{j-1}) \right]$$

for $t_{k-1} \leq t \leq t_k$, and the first conclusion follows.

Define $X_k = T_\Delta(t_k-t_{k-1})$, $A_k = A(t_k)$, and $\beta_k = \beta(t_k)$ for $k = 0, 1, \ldots, n$.

If $t_{k-1} < t < t_k$, then

$$f_\Delta'(t) = A_kf_\Delta(t)$$

$$= A_kT_k(t-t_{k-1})X_{k-1} \cdots X_1$$

$$= T_k(t-t_{k-1})A_kA_{k-1}^{-1}A_{k-1}X_{k-1} \cdots X_1$$

$$= T_k(t-t_{k-1})A_kA_{k-1}^{-1}A_{k-1}X_{k-1} \cdots X_1A_1A_0^{-1}A_0x_0,$$

and the second conclusion follows.
Lemma 2.6. Let $A$, $x_0$, and $f$ be as in Lemma 2.3, but add the condition that $x_0 \in S$. Then $(\exp \left[ \int_0^t \beta \right]) f(t) \in S$ for $0 \leq t \leq R$.

Proof. Let $\Delta$ and $f_\Delta$ be as in Lemma 2.4. From (R3) and the construction of $f_\Delta$, we get

$$\left( \exp \left[ (t-t_{k-1})\beta(t_{k-1}) + \sum_{j=1}^{k-1} \beta(t_j)(t_j-t_{j-1}) \right] \right) f_\Delta(t) \in S$$

for $t_{k-1} \leq t \leq t_k$.

2.7. Proof of Theorem 1. Choose $\delta > 0$ so that $K(t, x)$ (see condition (C2)) is bounded for $|t-t_0| \leq \delta$, $\|x-x_0\| \leq \delta$. Let $K_0$ be an upper bound for $K(t, x)$ on this set, with $K_0 > 1$, $(1/\delta)$. Let

$$y_0 = A(t_0, x_0)x_0, \quad \gamma = 2K_0(1 + 2\|x_0\| + \|y_0\|),$$

$$c = \min \left\{ b, t_0 + (1/2\gamma) \right\}, \quad R = -\gamma^{-1} \ln \left( 1 - \gamma(c-t_0) \right).$$

We will need the following two inequalities, which follow immediately from the above definitions:

$$(2.7.1) \quad R(2\|x_0\| + \|y_0\|) \leq \delta,$$

$$(2.7.2) \quad c - t_0 \leq \delta.$$

Define $F$ from $[0, R]$ onto $[t_0, c]$ by $F(t) = t_0 + \gamma^{-1}[1 - e^{-\gamma t}]$. Define $\beta$ on $[0, R]$ by $\beta(t) = e^{-\gamma t}$, and define $G$ from $[t_0, c]$ onto $[0, R]$ by $G(t) = -\gamma^{-1} \ln [1 - \gamma(t-t_0)]$. Then

$$(2.7.3) \quad F(G(t)) = t, \quad G(F(t)) = t,$$

$$(2.7.4) \quad G'(t) \beta(G(t)) = 1,$$

$$(2.7.5) \quad \int_0^t \beta = F(t) - t_0.$$

Define $\alpha$ on $[0, R]$ by $\alpha(\tau) = \exp \left( \int_0^\tau \beta \right)$.

We intend to solve (*) by first solving

$$(***) \quad g'(\tau) = \beta(\tau)[A(F(\tau), \alpha(\tau)g(\tau)) - I]g(\tau),$$

and then making the substitution $f(t) = e^{t_0}g(G(t))$. (2.7.3), (2.7.4), and (2.7.5) are the pertinent identities for showing that this yields a solution of (*).

We define inductively the sequence $\{g_n\}$ of functions on $[0, R]$ as follows:

$$g_0(\tau) = x_0, \quad g_{n+1}(0) = x_0, \quad g_{n+1}(\tau) = A_n(\tau)g_n(\tau),$$

where

$$A_n(\tau) = \beta(\tau)[A(F(\tau), \psi_n(\tau)) - I], \quad \psi_n(\tau) = \alpha(\tau)g_n(\tau).$$

We see that this inductive definition is possible by Lemmas 2.2, 2.3, and 2.6.
We will need the fact that each of the operator functions $A_n$ has property $(T_3)$; in fact this is the reason for our change of variable. Define $B_n(\tau) = A(F(\tau), \psi_n(\tau))$ for $0 \leq \tau \leq R$, and $n = 0, 1, 2, 3, \ldots$. Then each $B_n$ is regular by Lemma 2.2. For each $n$, let $L^{(n)}$ denote the least constant $L$ that will work in condition $(R_2)$ for $B_n$. Notice that

\[
(\beta(\tau) - \beta(\sigma))/\beta(0) = -\gamma B(R)(\tau - \sigma) \leq -\gamma/2 (\tau - \sigma)
\]

for $0 \leq \sigma \leq \tau \leq R$. Thus by Lemma 2.2, $A_n$ satisfies $(T_3)$ if $L^{(n)} \leq \gamma/2$. In order to show this we will need

(2.7.6) \quad |F'(\tau)| = |e^{-\tau r}| \leq 1,

(2.7.7) \quad |\alpha'(\tau)| = \left| \exp \left[ -\gamma \tau + \int_0^\tau \beta \right] \right| \leq 1,

(2.7.8) \quad |F(\tau) - t_0| \leq \delta.

Thus, we have $L_0 \leq K_0(1 + \|x_0\|) = \gamma/2$ since $\|\psi_0(\tau) - x_0\| < R \|x_0\| \leq \delta$, $\|\psi_0(\tau)\| \leq \|x_0\|$, so that $A_0$ has property $(T_3)$.

Suppose $A_n$ has property $(T_3)$. Then

\[
\psi_{n+1}(\tau) = a(\tau)\psi_n(\tau) + \alpha'(\tau)g_n(\tau),
\]

\[
\|\psi_{n+1}(\tau)\| \leq \|g_n(0)\| + \|x_0\| \leq 2\|x_0\| + \|y_0\|
\]

by Lemma 2.5, and (2.7.7). Therefore,

(2.7.9) \quad \|\psi_{n+1}(\tau) - x_0\| \leq \delta

by (2.7.1). Thus $L^{(n+1)} \leq K_0(1 + 2\|x_0\| + \|y_0\|) = \gamma/2$ by (2.7.6), (2.7.8), and Lemma 2.2. Thus $A_{n+1}$ also has property $(T_3)$.

Thus, we have

(2.7.10) \quad \|g_n(\tau)\| \leq (\|y_0\| + \|x_0\|)/\alpha(\tau)

for $0 < \tau \leq R$, and $n = 0, 1, 2, 3, \ldots$ by Lemma 2.5.

For each $n = 1, 2, 3, \ldots$, define $h_n$ on $[0, R]$ by $h_n(\tau) = g_{n+1}(\tau) - g_n(\tau)$. Then

\[
h_n(\tau) = A_n(\tau)g_{n+1}(\tau) - A_{n-1}(\tau)g_n(\tau)
\]

\[
= [A_n(\tau) - A_{n-1}(\tau)]g_{n+1}(\tau) + A_{n-1}(\tau)h_n(\tau)
\]

\[
= [I - A_{n-1}(\tau)A_n(\tau)^{-1}]g_{n+1}(\tau) + A_{n-1}(\tau)h_n(\tau).
\]

By [4, Lemma 1.3, p. 510], we have

\[
\|h_n(\tau)(d/d\tau)\| h_n(\tau)\| = \text{Re} \ [h_n(\tau), h_n(\tau)]
\]

a.e. on $[0, R]$, so that $(d/d\tau)\|h_n(\tau)\| \leq K_0(\|x_0\| + \|y_0\|)\|h_{n-1}(\tau)\|$ a.e. on $[0, R]$. We have used the fact that $A_{n-1}(\tau)$ is dissipative (see [5]), property $(C_2)$, (2.7.8), (2.7.9), and (2.7.10).

Therefore, $\{g_n\}$ converges uniformly to a function $g$ on $[0, R]$. Also

\[
\|g(\tau) - g(\sigma)\| \leq (\|x_0\| + \|y_0\|)|\tau - \sigma|
\]
for $0 \leq \sigma, \tau \leq R$. Let $\psi(\tau) = \alpha(\tau)g(\tau), 0 \leq \tau \leq R$. Then
\[
\|\psi(\tau) - \psi(\tau)\| \leq (2\|x_0\| + \|y_0\|)|\tau - \sigma|,
\]
and $\|\psi(\tau) - x_0\| \leq \delta$ for $0 \leq \tau \leq R$. Define the operator function $A$ on $[0, R]$ by
\[
A(\tau) = \beta(\tau)[A(F(\tau), \psi(\tau)) - f].
\]
Then $A$ has properties (T1), (T2), and (T3) (we will not need (T3)). Define $u$ on $[0, R]$ by $u(0) = x_0, u'(\tau) = A(\tau)u(\tau)$.

We wish to show that $u = g$. Let $u_n = u - gn$. An argument similar to the one used to show that $\{h_n\}$ converges to 0 will show that $\{u_n\}$ converges to 0. Therefore $g$ satisfies (**), and the function $f$ defined on $[t_0, c]$ by $f(t) = e^{t-t_0}g(G(t))$ satisfies (*).

Note also that $f(t) = \psi(G(t)), \|f(t) - x_0\| \leq \delta$ for $y_0 \leq t \leq c$.

Suppose $x \in D_0 \cap S, t_0 < c_1 \leq c$, and $v$ is a continuously differentiable function from $[t_0, c_1]$ into $D_0 \cap S$ such that $v(t_0) = x_1$ and $v'(t) = A(t, v(t))v(t)$ for $t_0 \leq t \leq c_1$. Define $w$ on $[t_0, c_1]$ by $w(t) = v(t) - v(t)$. Then
\[
\begin{align*}
w'(t) &= [A(t, f(t)) - A(t, v(t))]f(t) + A(t, v(t))w(t) \\
&= ((1 - A(t, v(t)))^{-1} - I)(f(t) - f'(t)) + A(t, v(t))w(t),
\end{align*}
\]
a.e. on $[t_0, c_1]$. The stability claim, and hence the uniqueness claim, follow from this differential inequality.

2.8. Proof of Theorem 2. First we mention that if we prove that for each $x \in D_0 \cap S$, there exists a continuously differentiable function $f$ from $[0, \infty)$ into $D_0 \cap S$ such that $f(0) = x$ and $f'(t) = Qf(t)$ for $t \geq 0$, then the rest of the theorem follows in routine manner. We define $T_0(t)x = f(t)$ for $x \in D_0 \cap S$ and $t \geq 0$. The fact that $T_0$ is nonexpansive on $D_0 \cap S$ follows from the fact that $Q$ is dissipative. Thus each $T_0(t)$ has a unique extension to a nonexpansive transformation $T(t)$ from $S$ into $S$. $\{T(t); t \geq 0\}$ is the desired semigroup.

Now we return to the first question. Let $x_0 \in D_0 \cap S$. Then by Theorem 2, there is a number $c > 0$ such that there is a unique continuously differentiable function $f$ from $[0, \infty)$ into $D_0 \cap S$ such that $f(0) = x_0$ and $f'(t) = Qf(t)$ for $0 \leq t \leq c$. Let $\zeta$ denote the supremum of the set of all such numbers $c$, and suppose that $\zeta < \infty$. Let $f$ denote the unique continuously differentiable function from $[0, \zeta)$ into $D_0 \cap S$ such that $f(0) = x_0$ and $f'(t) = Qf(t)$ for $0 \leq t < \zeta$. If $0 < h < \zeta$, then define $f_h$ on $[0, \zeta - h]$ by $f_h(t) = f(t + h) - f(t)$. Then $f_h(t) = Qf(t + h) - Qf(t)$, and $(d/dt)f_h(t) \leq 0$ a.e. on $[0, \zeta - h]$ since $Q$ is dissipative, so that $t(h)[f(t + h) - f(t)] \leq \|f(t + h) - f(t)\| \leq \|f'(0)\|$, for $0 \leq t < \zeta - h$. Therefore $\|f'(t)\| \leq \|f'(0)\|$ for $0 \leq t \leq \zeta$, and thus $x_1 = \lim_{t \to \zeta}f(t)$ exists. Therefore $f([0, \zeta))$ is relatively compact and $K(t, f(t))$ (see (C2)) is bounded on $[0, \zeta)$. Using this and the fact that $f(t)$ and $f'(t)$ are also bounded on $[0, \zeta)$, we see by examining the argument for Theorem 1 that there is a positive constant $\eta$ such that for each $t$ in $[0, \zeta)$, there is a unique continuously differentiable function $g$ from $[t, t + \eta]$ into $D_0 \cap S$ such that $g(t) = f(t)$ and $g'(s) = Qg(s)$ for $t \leq s \leq t + \eta$. Simply take $\zeta - \eta < t < \zeta$, and use the corresponding function $g$ to extend $f$ beyond $\zeta$. 

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3. Semigroups in a function space. Let \( E \) be a set, \( B(E) \) the Banach space of bounded complex valued functions on \( E \) with supremum norm, and \( Y \) a closed real or complex subspace of \( B(E) \). We denote by \( \Omega \) the collection of all positive bounded functions \( p \) on \( E \) which are bounded away from zero and have the property that \( p \leq Y \).

If \( Y \) is complex, we take \( CH(\phi) \) and \( GH(\phi) \) as defined in §1, with \( X = Y \). If \( Y \) is a real Banach lattice, then \( CP \) denotes the collection of all strongly continuous semigroups of linear positive contraction operators in \( Y \), and \( GP \) denotes the collection of infinitesimal generators of such semigroups. In either case, \( G \) denotes the collection of all infinitesimal generators of strongly continuous semigroups of linear contraction operators in \( Y \).

If \( y \in Y \), then \( y(y) \) denotes a multiplicative linear functional on \( B(E) \) such that \( \langle y, y(y) \rangle = \| y \| \). We define the semi-inner product \([\cdot, \cdot]\) on \( Y \times Y \) by \( [x, y] = \langle x, y(y) \rangle \langle y, y(y) \rangle^* \), where * denotes complex conjugation. All reference to a semi-inner product in this section will be to this one just defined. One special property of \([\cdot, \cdot]\) which is useful to us is that \( [px, y] = \langle p, y(y) \rangle [x, y] \) for \( x, y \in Y \), \( p \in B(E) \), \( px \in Y \). Also, if \( Y \) is a real Banach lattice, then \([\cdot, \cdot]\) has the special properties required in [7]. That is, \([\cdot, y]\) is a positive linear functional if \( y \geq 0 \), and \([x, x^+] = \| x^+ \|^2 \) for each \( x \) in \( Y \), where \( x^+ \) denotes the positive part of \( x \).

By Definition 1.2, a linear operator \( A \) in \( Y \) is dissipative if \( Re \langle Ay, y \rangle \leq 0 \) for \( y \in D(A) \). Following [6], in case \( Y \) is complex, we say that a linear operator \( A \) in \( Y \) is \( \phi \)-sectorial if \( e^{i\theta}A \) is dissipative for \( |\theta| \leq \phi \). Following [7], in case \( Y \) is a real lattice, we say that a linear operator \( A \) in \( Y \) is dispersive if \( [Ax, x^+] \leq 0 \) for all \( x \in D(A) \).

Lemma 3.1. A linear operator \( A \) in \( Y \) is in \((G, GP, GH(\phi))\) if and only if \( D(A) \) is dense in \( Y \), the range of \( I - A \) is all of \( Y \), and \( A \) is (dissipative, dispersive, \( \phi \)-sectorial).

Proof. The proof of this lemma is contained in [5], [7], and [6], respectively. We merely state the lemma here for reference in proving the next theorem, which is a generalization of the author’s earlier theorem in [1].

Theorem 3.1. Suppose \( A \in G \), and \( A = A_1 + \cdots + A_n \), where each \( A_j \) has domain \( D(A_j) \), and each \( A_j \) has a closed extension. If each \( A_j \) is (dissipative, dispersive, \( \phi \)-sectorial), and \( p_1, \ldots, p_n \in \Omega \), then \( p_1A_1 + \cdots + p_nA_n \in (G, GP, GH(\phi)) \).

Proof. \( p_1A_1 + \cdots + p_nA_n \) is easily seen to be (dissipative, dispersive, \( \phi \)-sectorial). Thus by Lemma 3.1, we need only show that the range of \( I - (p_1A_1 + \cdots + p_nA_n) \) is all of \( Y \).

We will first prove that the range of \( I - (p_1A_1 + A_2 + \cdots + A_n) \) is all of \( Y \). By [3, Lemma 2, p. 212], the operator \( U_1 = A_1(I-A)^{-1} \) is bounded. Since \( F(p_1)Y \subset Y \) for every polynomial \( F \), then \( p_1^{(1)} \in \Omega \) for every positive integer \( m \) by the classical Weierstrass theorem. Choose \( m \) so that \( \| 1 - p_1^{(1)} \| < \| U_1 \|^{-1} \), and let \( r = p_1^{(1)}m \). Then \( I - (rA_1 + A_2 + \cdots + A_n) = I - A + (1-r)A_1 = (I + (1-r)U_1)(I-A) \).
Thus the range of \( I - (rA_1 + A_2 + \cdots + A_n) \) is all of \( Y \). Replacing \( A_1 \) by \( rA_1 \), \( r^2A_1 \), etc., we see that the range of \( I - (p_1A_1 + A_2 + \cdots + A_n) \) is all of \( Y \).

Now we consider the operator \( A' = A_2 + p_1A_1 + A_3 + \cdots + A_n \) and repeat the previous argument to prove that the range of \( I - (p_1A_1 + p_2A_2 + A_3 + \cdots + A_n) \) is all of \( Y \). Repeating this process proves the theorem.

**Example.** Let \( E \) denote real Euclidean \( n \)-space, and let \( Y \) denote any of the subspaces of \( B(E) \) in which the Laplacian operator generates a strongly continuous semigroup. The semigroup will then consist of contraction operators and will be in \( CH(\phi) \) if \( Y \) is complex, in \( CP \) if \( Y \) is a real lattice. Let \( A \) denote the Laplacian operator in \( Y \), and for each \( j = 1, \ldots, n \), let \( A_j \) denote the restriction of \( (\partial^2/\partial s^2) \) to the domain of \( A \).

**Lemma 3.2.** Let \( A \) be in \( G \) with \( A = A_1 + \cdots + A_n \), where each \( A_j \) has domain \( D(A_j) \), each \( A_j \) has a closed extension, and each \( A_j \) is dissipative. Define the function \( P \) from \( \Omega^{(n)} \) into \( G \) by \( P(p) = p_1A_1 + \cdots + p_nA_n \).

Then there is a locally bounded nonnegative function \( K \) on \( \Omega^{(n)} \) such that
\[
\| [I - P(q)][I - P(p)]^{-1} - I \| \leq \left( \sum q_i - p_i \right) K(p)
\]
for \( p, q \in \Omega^{(n)} \).

**Proof.** If \( p, q \in \Omega^{(n)} \), then
\[
[I - P(q)][1 - P(p)]^{-1} - I = [P(p) - P(q)][I - P(p)]^{-1} = \sum (p_i - q_i) A_i [I - P(p)]^{-1}.
\]

There we can take \( K(p) = \max_i \| A_i [I - P(p)]^{-1} \|^{-1} \).

To see that \( K(p) \) is locally bounded, notice that
\[
A_i (I - P(r))^{-1} = A_i (I - P(p))^{-1} (I + \sum (p_i - r_i) A_i (I - P(p))^{-1})^{-1},
\]
so that \( K(r) \leq K(p)/(1 - K(p) \sum p_i - r_i) \) for
\[
K(p) \sum p_i - r_i < 1.
\]

**Theorem 3.3.** Let \( B \) be in \( GH(\phi) \) with \( B = B_1 + \cdots + B_n \), where each \( B_i \) has a closed extension, each \( B_i \) has domain \( D(B_i) = D_0 \), and each \( B_i \) is \( \phi \)-sectorial. Let \( S \) be a closed set in \( Y \), \( [a, b] \) a closed interval, and \( p \) a Lipschitz continuous function from \( [a, b] \times S \) into \( \Omega^{(n)} \). Define the operator function \( A \) from \( [a, b] \times S \) into \( GH(\phi) \) by \( A(t, x) = \sum p(t, x) A_i \). Then \( A \) satisfies conditions \( (C_1) \) and \( (C_2) \).

**Proof.** This follows from Lemma 3.2.

There are a variety of ways in which the set \( S \) in Theorem 3.2 could be chosen in order that the operator function \( A \) will satisfy \( (C_3) \), and it seems inappropriate to state any theorems about this. It is not quite so easy to choose \( H \) and \( S \) so that \( Q \) will be dissipative as in Theorem 2, but we will indicate one way in which it can be done.

Let \( Y \) be complex, let \( Y_0 \) denote the space of real functions in \( Y \), and suppose \( Y_0 \) is a lattice. Let \( A \in GH(\phi) \), \( A = A_1 + \cdots + A_n \) as in Theorem 3.2. Let \( D_{00} = D_0 \cap Y_0 \),

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A^0 = A|_{D_{00}}, A_j^0 = A_j|_{D_{00}}, and suppose that A^0, A_j^0 satisfy the portion of Theorem 3.1 dealing with positive semigroups. Let Y_{00} denote the nonpositive functions in Y_0, let S_0 = \bigcap (A_j^0)^{-1} Y_{00}, and let S denote the closure of S_0. Let p_1, \ldots, p_n be Lipschitz continuous accretive (-p_t dissipative) functions from S into \Omega. Define H from S onto \text{GH} (\phi) by H(x) = \sum p_i(x) A_i. Then the hypothesis of Theorem 2 is satisfied. This can all be done taking Y, A, A_j as in the example after Theorem 3.1 which dealt with the Laplacian operator.

REFERENCES


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