THE MODULE INDEX AND INVERTIBLE IDEALS

BY

DAVID W. BALLEW(1)

Abstract. A. Fröhlich used the module index to classify the projective modules of an order in a finite dimensional commutative separable algebra over the quotient field of a Dedekind domain. This paper extends Fröhlich's results and classifies the invertible ideals of an order in a noncommutative separable algebra. Several properties of invertible ideals are considered, and examples are given.

1. Introduction. Let \( A \) be a Dedekind domain with quotient field \( K \); let \( \Sigma \) be a finite dimensional separable \( K \)-algebra. A subring \( \Lambda \) of \( \Sigma \) is said to be an \( A \)-order if \( \Lambda \) contains \( A \), \( \Lambda \) is \( A \)-torsion free and \( \Lambda \) contains a \( K \)-basis of \( \Sigma \). There is always a maximal \( A \)-order \( \Gamma \) containing \( \Lambda \). A finite dimensional torsion free \( A \)-module is called an \( A \)-lattice.

Let \( X \) be a finite dimensional torsion \( A \)-module. The order ideal, \( \text{ord}_A X \), is the product of the \( A \)-annihilators of the composition factors of an \( A \)-composition series of \( X \). Let \( M \) and \( N \) be two \( A \)-lattices which span the same vector space over \( K \); i.e., \( K \otimes_A M \cong K \otimes_A N \). Define the module index of \( M \) and \( N \), \( [M:N] \), by \( [M:N] = (\text{ord}_A M/L) \cdot (\text{ord}_A N/L)^{-1} \) where \( L \) is any \( A \)-lattice such that \( K \otimes_A L = K \otimes_A M \) and \( L \subseteq M \cap N \).

In [5], A. Fröhlich gives criteria in terms of the module index to decide the projectivity of \( \Lambda \)-modules when \( \Sigma \) is commutative. More precisely, he shows that if \( M \) is a \( \Lambda \)-module such that \( K \otimes_A M \cong \Sigma^\oplus \), then \( M \) is \( \Lambda \)-projective if and only if \( [\Gamma M: M] = [\Gamma : \Lambda] \), where \( \Gamma \) is a maximal order over \( \Lambda \) and \( \Gamma M \) denotes the smallest \( \Gamma \) module containing \( M \). The object of this paper is to give analogues to Fröhlich's theorem when \( \Sigma \) is noncommutative. We will give examples, however, to show that in the fairly simple case of a finite dimensional matrix algebra over the quotient field of a discrete rank one valuation ring, no direct extensions of either the necessity or the sufficiency of Fröhlich's theorem can be given.

It is well known that for integral domains, an ideal is invertible if and only if it is projective. However, the statement that projectivity implies invertibility is not necessarily true for noncommutative rings. Nevertheless, since invertibility is a

---

(1) Portions of this paper were included in the author's Ph.D. dissertation at the University of Illinois supervised by Robert Fossum.
property of ideals that passes from the local case to the global and vice versa, we consider it in terms of Fröhlich type index conditions. We find that if the single index condition of Fröhlich’s theorem is replaced by several index conditions in terms of idempotents and a multiplication condition on a two-sided full ideal, then we have necessary and sufficient conditions to insure the invertibility of the ideal. Further, it is not hard to see that for the case of full two-sided ideals and commutative $K$-algebras, these results imply Fröhlich’s theorem.

2. Preliminaries. We first list a few basic properties of the module index. The proofs of these results follow directly from the definitions. Let $M$, $N$, $R$ be $A$-lattices spanning the same $K$-space.

2. If $M \supseteq N$, then $[M:N] \subseteq A$.
4. If $M = M_1 \oplus M_2$, $N = N_1 \oplus N_2$ with $M_1$ (resp. $M_2$) spanning the same space as $N_1$ (resp. $N_2$), then $[M_1 \oplus M_2:N_1 \oplus N_2] = [M_1:N_1][M_2:N_2]$.
5. Let $f: v \to v'$ be a $K$-isomorphism of finite dimensional vector spaces $v$, $v'$. Let $M$, $N$ be two $A$-lattices spanning $v$. Then $[M:N] = [f(M):f(N)]$.

We will say that an ideal $I$ of $\Lambda$ is full if $I \otimes_A K \cong \Lambda \otimes_A K$.

We now let $A$ denote a complete discrete rank one valuation ring with quotient field $K$. Let $\pi$ be a generator of the maximal ideal of $A$, and let $\Sigma$ be a finite dimensional separable $\Lambda$-algebra.

Let $A$ denote an $\Lambda$-order in $\Sigma$, and let $\overline{\Lambda} = \Lambda/\pi\Lambda$. Then $\overline{\Lambda}$ is an algebra over the field $\Lambda/(\pi)$. Let $\overline{\Omega} = \overline{\pi} + \cdots + \overline{\pi}$ be a decomposition of $\overline{\Omega}$ into primitive orthogonal idempotents. Then since $\overline{\Lambda}$ is a complete discrete valuation ring, there are primitive orthogonal idempotents $e_1, \ldots, e_r$ in $\Lambda$ which map to $\overline{\pi}, \overline{\pi}, \ldots, \overline{\pi}$, by the natural map $\Lambda \to \overline{\Lambda}$ and such that $1 = e_1 + e_2 + \cdots + e_r$.

We say that primitive orthogonal idempotents $e_i$ and $e_j$ in $\Lambda$ are equivalent in $\Lambda$ if $(e_i, e_j)(e_j, e_i) = e_i, e_j$. We will write $e_i \sim e_j$ to denote this equivalence.

Before we show that this is an equivalence relation, we will prove the following two lemmas.

**Lemma 1.** Let $e_i$ and $e_j$ be primitive orthogonal idempotents in $\Lambda$. Then $e_i \sim e_j$ if and only if there are elements $x_{ij}$ in $e_i\Lambda e_j$ and $x_{ji}$ in $e_j\Lambda e_i$ such that $x_{ij}x_{ji} = e_i$ and $x_{ji}x_{ij} = e_j$.

**Proof.** Assume that $e_i$ is equivalent to $e_j$ in $\Lambda$, and let $x_{ij}$ in $e_i\Lambda e_j$ and $x_{ji}$ in $e_j\Lambda e_i$ be such that $x_{ij}x_{ji} = e_i$. Then, $x_{ji}x_{ij}x_{ji} = x_{ji}e_jx_{ji} = x_{ji}x_{ij}$, so $x_{ji}x_{ij}$ is idempotent in $e_j\Lambda e_j$. Since $e_j$ is a primitive idempotent, $x_{ji}x_{ij}$ is either zero or $e_j$. Because $x_{ij}x_{ji}x_{ij} = x_{ij} \neq 0$, $x_{ji}x_{ij} = e_j$.

The converse is clear. Q.E.D.

**Lemma 2.** Let $e_i$ and $e_j$ be primitive orthogonal idempotents in $\Lambda$. Then $e_i \sim e_j$ if and only if $e_i\Lambda \cong e_j\Lambda$ (as right $\Lambda$-modules) if and only if $\Lambda e_i \cong \Lambda e_j$ (as left $\Lambda$-modules).
Proof. Let $\sigma : e_i \Lambda \rightarrow e_j \Lambda$, $\tau : e_j \Lambda \rightarrow e_i \Lambda$ be inverse $\Lambda$-isomorphisms. Then $\sigma(e_i) = e_ix$ for $x$ in $\Lambda$, so $\sigma(e_i^2) = \sigma(e_i)e_i = e_ixe_i$. Thus, we can take $x$ in $e_i \Lambda e_i$. Likewise, we have $\tau(e_j) = e_jy$ for $y$ in $e_j \Lambda e_j$. Now $e_i = \tau(\sigma(e_i)) = \tau(e_i)x = e_iyx$, $e_j = \sigma(\tau(e_j)) = eyx$. But $e_iy = y$, $e_ix = x$, so $yx = e_i$, $xy = e_j$. Therefore, by Lemma 1, $e_i \sim e_j$.

If $e_i \sim e_j$, then by Lemma 1, there are elements $x$ in $e_i \Lambda e_i$, $y$ in $e_j \Lambda e_j$ such that $yx = e_i$, $xy = e_j$. Let $\sigma(e_i) = e_ix$, define a $\Lambda$-map from $e_i \Lambda$ to $e_j \Lambda$. Let $e_j \Lambda$ be in $e_j \Lambda$. Then, $\sigma(e_i^2) = e_iyx = e_j$, so $\sigma$ is onto. If $\sigma(e_i^2) = 0$, then $xy = 0$ in $\Lambda$, so $yx = 0$. Hence the proof for left modules are similar. Q.E.D.

Now it follows from Lemma 2 that "$\sim$" is an equivalence relation.

Lemma 3. If $e_i$, $e'_i$, $e_i$, and $e'_i$ are idempotents in $\Lambda$ such that $e_i \sim e'_i$ and $e_i \sim e'_i$, then $e_i \Lambda e_i \sim e'_i \Lambda e'_j$ (as $\Lambda$-modules).

Proof. First, assume that $e_i$ is equivalent to $e'_i$. Then there is an $x$ in $e_i \Lambda e'_i$ and $y$ in $e'_i \Lambda e_i$ such that $x_i = e_i$ and $y_i = e'_i$. Define $\eta : e_i \Lambda e_i \rightarrow e'_i \Lambda e_i$ by $\eta(e_i^2) = yix$.

The proof for the case where $e_i$ is equivalent to $e'_i$ is symmetric. Q.E.D.

An $\Lambda$-order $\Lambda$ in $\Sigma$ is said to be reduced if its identity has a decomposition into primitive orthogonal idempotents $1 = e_1 + \cdots + e_n$, such that no distinct $e_i$ and $e_j$ are equivalent [6, Appendix].

For the $\Lambda$-order $\Lambda$ in $\Sigma$, let $f_1, \ldots, f_k$ denote representatives of the distinct equivalence classes of equivalent idempotents. Let $f = f_1 + \cdots + f_k$. Let $\tilde{\Lambda} = f \Lambda f$; then $\tilde{\Lambda}$ is reduced in $f\Sigma f$.

Lemma 4. The map $I \rightarrow \tilde{I} = fIf = I \cap \tilde{\Lambda}$ of two-sided $\Lambda$-modules to two-sided $\tilde{\Lambda}$-modules is one-to-one and preserves products if they are defined. Further, the map preserves sums and intersections.

Proof. Let $I$ and $J$ denote two-sided $\Lambda$-modules. The proofs that $(fIf) \cap (fJf) = f(I \cap J)f$ and that $fIf + fJf = f(I + J)f$ are ordinary set inclusion arguments and are omitted.

We note that $\Lambda f \Lambda = \Lambda$. For, if $e_i$ is one of the idempotents of $\Lambda$, there is an $f_i$ such that $e_i$ is equivalent to $f_i$; i.e., $e_i \Lambda e_i = (e_i \Lambda f_i)(f_i \Lambda e_i)$. Hence, $e_i = e_i x f_i y e_i$ for elements $x$ and $y$ in $\Lambda$. Thus, in particular, $e_i$ is in $\Lambda f \Lambda$, so $1 = \sum f_i e_i$ is in $\Lambda f \Lambda$. Therefore, $\Lambda f \Lambda = \Lambda$.

Since $fIf = fIf fIf = fI f\Lambda f fIf = (fIf)(fJf)$, it is clear that products are preserved when defined.

Finally, if $fIf = fJf$, then $\Lambda f \Lambda f \Lambda = \Lambda f \Lambda J \Lambda f \Lambda$. So, $\Lambda f J \Lambda = I = J = \Lambda f \Lambda$. Hence, the map $I \rightarrow \tilde{I}$ is one-to-one. Q.E.D.

We will say that a two-sided (fractional) ideal $I$ of $\Lambda$ is invertible if there is a two-sided $\Lambda$-module $J$ such that $IJ = JI = \Lambda$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
3. Index conditions in reduced orders. In this section, let $A$ be a complete discrete rank one valuation ring, let $\Lambda$ be a reduced $A$-order in $\Sigma$, and let $\Gamma$ be a maximal $A$-order containing $\Lambda$. Let $f_1, \ldots, f_k$ denote the nonequivalent primitive orthogonal idempotents of $\Lambda$, and assume they are primitive in $\Gamma$.

**Theorem 5.** Let $I$ be an invertible two-sided ideal in reduced $A$-order $A$. Then $I$ is $\Lambda$-cyclic, $\Lambda$-free, and there is a permutation $\sigma$ of the set $\{1, 2, \ldots, k\}$ such that for all $i$ and $j$, $i, j=1, 2, \ldots, k$,

(a) $[f_i I f_j : f_i \Lambda / f_j] = [f_{\sigma^{-1}(i)} I f_{\sigma^{-1}(j)} : f_{\sigma^{-1}(i)} \Lambda / f_{\sigma^{-1}(j)}]$,

(b) $[f_i I f_j : f_i \Lambda f_j] = [f_{\sigma(\sigma(0))} I f_{\sigma(\sigma(0))}]$,

(c) $(f_{\sigma^{-1}(i)} I f_i)(f_i \Lambda f_i) = f_{\sigma^{-1}(i)} I f_i$,

(d) $(f_i \Lambda f_i)(f_i I f(i)) = f_i I f(i)$.


**Proof.** Since $I$ is invertible, there is a two-sided $A$-module $J$ such that $IJ=JI=\Lambda$. Thus

$$f_i \Lambda f_i = \sum_{s=1}^{k} (f_i I f_s)(f_s I f_i).$$

In particular,

$$f_i \Lambda f_i = \sum_{s=1}^{k} (f_i I f_s)(f_s I f_i).$$

Since $f_i$ is a primitive idempotent, $f_i \Lambda f_i$ has a single maximal ideal $M_i$, and the factor algebra $f_i \Lambda f_i / M_i$ is a division ring. Further, since $f_i \Lambda f_i$ is an $A$-algebra, every ideal is regular, so the Jacobson radical of $f_i \Lambda f_i$, being the intersection of all maximal right (or left) regular ideals, must be $M_i$ [7]. Consider

$$f_i \Lambda f_i / M_i = \sum_{s=1}^{k} (f_i I f_s)(f_s I f_i) / M_i.$$ 

Then, since not all of the products $(f_i I f_s)(f_s I f_i)$ are in $M_i$ and since $f_i \Lambda f_i / M_i$ has no nontrivial ideals, there is an $s=\sigma(i)$, $\sigma$ a function of $\{1, 2, \ldots, k\}$ into itself, such that

$$f_i \Lambda f_i = (f_i I f_{\sigma(i)})(f_{\sigma(i)} I f_i) + M_i.$$ 

Then by [9, Theorem 4.1, p. 12],

$$(f_i I f_{\sigma(i)})(f_{\sigma(i)} I f_i) = f_i \Lambda f_i.$$ 

Therefore, there is an $x_i$ in $f_i I f_{\sigma(i)}$ and a $y_i$ in $f_{\sigma(i)} I f_i$ such that $x_i y_i = f_i$. Since $y_i x_i y_i x_i = x_i y_i = f_i$, then $y_i x_i$ is in $f_{\sigma(i)} I f_{\sigma(i)}$, and since $f_{\sigma(i)}$ is a primitive idempotent, then $y_i x_i$ is either zero or $f_{\sigma(i)}$. But $x_i y_i x_i = x_i \neq 0$, so $y_i x_i = f_{\sigma(i)}$. Such $x_i$ and $y_i$ can be found for all $i$, $i=1, 2, \ldots, k$.

We claim that $\sigma$ is a permutation. Consider

$$x_1 \Lambda + \cdots + x_k \Lambda \subseteq I.$$
The left-hand side of (3) contains all of the $x_i$, so

(4) \[(x_1\Lambda + \cdots + x_k\Lambda)I = \Lambda,\]  

since the left-hand side of this expression contains all of the $x_iy_i$; therefore, it contains 1. Hence, multiplying equation (4) on the right by $I$, we have

$$I = (x_1\Lambda + \cdots + x_k\Lambda)JI,$$

or

(5) \[I = x_1\Lambda + \cdots + x_k\Lambda.\]

Similarly,

(6) \[J = \Lambda y_1 + \cdots + \Lambda y_k.\]

On noting that $y_ix_i = 0$ if $i \neq j$, we have that

$$\Lambda = JJ = \sum_{j=1}^{k} \Lambda y_jx_j\Lambda = \sum_{j=1}^{k} \Lambda f_{\sigma(j)}\Lambda.$$

Now, if $f_\alpha$ is a member of the set \{\(f_1, \ldots, f_k\)\}, consider the equation

$$f_\alpha I f_\alpha = \sum_{\sigma} (f_\alpha f_{\sigma(\beta)})(f_{\sigma(\beta)} f_\alpha).$$

As before, there must be a $\beta$ in \{1, 2, \ldots, $k$\} such that

(7) \[f_\alpha f_\sigma = (f_\alpha f_{\sigma(\beta)})(f_{\sigma(\beta)} f_\alpha).\]

But equation (7) implies that $f_\alpha \sim f_{\sigma(\beta)}$ which implies $\alpha = \sigma(\beta)$.

Thus, since $\sigma$ is a permutation, and $x_iy_j = 0$ for $i \neq j$, $(x_1 + \cdots + x_k)\Lambda I$ has all of the elements $x_iy_i$, and so it has 1. Hence,

(8) \[(x_1 + \cdots + x_k)\Lambda J = \Lambda.\]

Multiplying equation (8) on the right by $I$, we have

$$I = (x_1 + \cdots + x_k)\Lambda JJ = (x_1 + \cdots + x_k)\Lambda,$$

so $I$ is cyclic as a right module.

Since

$$I = \sum_{i,j=1}^{k} (x_1 + \cdots + x_k)f_if_j,$$

then

$$f_\sigma^{-1} f_\alpha I f_\beta = x_\sigma^{-1} f_\alpha \Lambda f_\beta,$$

and so

$$f_\sigma^{-1} f_\alpha I f_\beta = x_\sigma^{-1} f_\alpha \Lambda f_\beta = x_\sigma^{-1} f_\alpha \Lambda f_j = f_\sigma^{-1} f_\alpha I f_j,$$

which establishes (c).

We claim that $I$ is isomorphic to $\Lambda$ and is therefore free. Set $x = x_1 + \cdots + x_k$.

Define $\eta: \Lambda \to I$ by $\eta(\lambda) = x$. $IJJI = \Lambda$, so there is a $y$ in $J$ such that $xy = yx = 1$;
hence $x$ is a unit in $\Sigma$, and $\eta$ is one-to-one. $\eta$ is obviously onto, so it is an isomorphism.

By methods symmetric to the preceding, we have

$$I = \Lambda(x_1 + \cdots + x_k) = \Lambda x,$$

$$J = (y_1 + \cdots + y_k)\Lambda.$$

Therefore,

$$II = (x_1 + \cdots + x_k)\Gamma = x\Gamma,$$

$$\Gamma I = \Gamma(x_1 + \cdots + x_k) = \Gamma x,$$

$$\Gamma J = \Gamma(y_1 + \cdots + y_k),$$

$$J\Gamma = (y_1 + \cdots + y_k)\Gamma.$$

Since $x$ is a unit in $\Sigma$, we have

$$[\Gamma : \Lambda] = [x\Lambda : \Lambda x] = [\Gamma : I],$$

and

$$[\Gamma : \Lambda] = [\Lambda x : \Lambda] = [\Gamma : I].$$

Define maps $\theta_i : f_i\Lambda f_j \to f_{\sigma^{-1}(i)}f_i\Lambda f_j$ by $\theta_i(f_i\Lambda f_j) = x_{\sigma^{-1}(i)}f_i\Lambda f_j$. Since we have that $x_{\sigma^{-1}(i)}f_i\Lambda f_j = f_{\sigma^{-1}(i)}f_i\Lambda f_j$, $\theta_i$ is onto. If $x_{\sigma^{-1}(i)}f_i\Lambda f_j = f_{\sigma^{-1}(i)}f_i\Lambda f_j = 0$, so the $\theta_i$ are one-to-one and are isomorphisms. The $\theta_i$ extend to isomorphisms

$$\theta_i : f_i\Gamma f_j \to f_{\sigma^{-1}(i)}\Gamma f_j$$

by $\theta_i(f_i\Gamma f_j) = x_{\sigma^{-1}(i)}f_i\Gamma f_j$. Hence, for all $i$ and $j, i, j = 1, \ldots, k$,

$$[f_i\Gamma f_j : f_i\Lambda f_j] = [f_{\sigma^{-1}(i)}\Gamma f_j : f_{\sigma^{-1}(i)}f_i\Lambda f_j].$$

In order to obtain the other index conditions, we proceed symmetrically. Thus we show that

$$(f_i\Lambda f_j)(f_i\Gamma f_j) = f_i\Gamma f_j,$$

and show that the maps $\mu_i : f_i\Lambda f_j \to f_{\sigma(i)}f_i\Lambda f_j$ defined by $\mu_i(f_i\Lambda f_j) = f_{\sigma(i)}f_i\Lambda f_j$ are isomorphisms which extend to isomorphisms $\mu_i : f_i\Gamma f_j \to f_i\Gamma f_{\sigma(i)}$. From this, the index condition (b) is immediate. Q.E.D.

In Example 14 we shall see that $[\Gamma : \Lambda] = [\Gamma : I] = [\Gamma' : I]$ may be true in $\Lambda$ but that $[\Gamma : \Lambda] \neq [\Gamma' : I]$ and $[\Gamma' : \Lambda] \neq [\Gamma' : I]$ in $\Lambda$. Further, this example will show that if $\Lambda$ is not reduced, then an invertible $\Lambda$-ideal $I$ may not be a cyclic $\Lambda$-ideal. Thus the condition that $\Lambda$ is reduced is essential in Theorem 5.

Now we will prove the converse to Theorem 5. Define

$$(\Lambda : I)_r = \{x \in \Sigma : xI \subseteq \Lambda\}, \quad (\Lambda : I)_l = \{x \in \Sigma : xI \subseteq \Lambda\}.$$

**Theorem 6.** Let $I$ be a full two-sided ideal of the reduced $\Lambda$-order $\Lambda$ in $\Sigma$. Let $f_1, \ldots, f_k$ be the nonequivalent primitive orthogonal idempotents of $\Lambda$. If there is a permutation $\tau$ of the set $\{1, 2, \ldots, k\}$ such that

(a) $[f_i\Gamma f_j : f_i\Lambda f_j] = [f_{\tau(i)}\Gamma f_j : f_{\tau(i)}f_i\Lambda f_j],$

(b) $[f_{\tau(i)}\Gamma f_j : f_{\tau(i)}f_i\Lambda f_j] = [f_{\tau(i)}\Gamma f_j : f_{\tau(i)}f_i\Lambda f_j],$

(c) $(f_{\tau(i)}f_i\Lambda f_j) = f_{\tau(i)}f_i\Lambda f_j,$

(d) $(f_{\tau(i)}f_i\Lambda f_j) = f_{\tau(i)}f_i\Lambda f_j,$

for all $i$ and $j, i, j = 1, 2, \ldots, k$, then $I$ is invertible by the two-sided $\Lambda$-module $J = (\Lambda : I)_r = (\Lambda : I)_l$.

**Proof.** We consider $f_{\tau(i)}f_i\Lambda f_j$ as a right $f_i\Lambda f_j$-module and $f_{\tau(i)}\Gamma f_i$ as a right $f_i\Gamma f_i$-module.
Since $A$ is a complete discrete rank one valuation ring and since $f_i \Gamma f_i$ has no idempotents other than $f_i$ or 0, we can apply [14, Theorem 7.12, p. 548] and [1, Proposition 2.7, p. 8] to see that $f_{i_0} \Gamma f_{i_0}$ is a free $\Gamma$-module of rank 1. Since $f_i$ is a primitive idempotent, $f_i \Gamma f_i$ has a single maximal ideal $N_i$ which must be the Jacobson radical of $f_i \Gamma f_i$. Further, $f_i \Gamma f_i/N_i$ is a division algebra over $A/(\pi)$.

We claim that $f_{i_0} \Gamma f_{i_0}/f_{i_0} \Gamma f_{i_0} N_{i_0}$ is a free $f_{i_0} \Gamma f_{i_0}/N_{i_0}$-module of rank one. For, if $x_1, \ldots, x_r$ is a basis of $f_{i_0} \Gamma f_{i_0}/f_{i_0} \Gamma f_{i_0} N_{i_0}$ over $f_{i_0} \Gamma f_{i_0}/N_{i_0}$, then let $B$ be the $f_{i_0} \Gamma f_{i_0}$-submodule generated by the set $\{x_1, \ldots, x_r\}$. Then $B + f_{i_0} \Gamma f_{i_0} N_{i_0} = f_{i_0} \Gamma f_{i_0}$. Hence, by Nakayama’s Lemma, $B = f_{i_0} \Gamma f_{i_0}$, and $r = 1$. Thus

$$\langle f_{i_0} \Gamma f_{i_0}/f_{i_0} \Gamma f_{i_0} N_{i_0}; f_{i_0} \Gamma f_{i_0}/N_{i_0} \rangle = 1.$$  

Consider the composed map

$$\theta_1: f_{i_0} \Gamma f_{i_0} \to f_{i_0} \Gamma f_{i_0} \to f_{i_0} \Gamma f_{i_0}/f_{i_0} \Gamma f_{i_0} N_{i_0}.$$

Let $z_i$ be an element of $f_{i_0} \Gamma f_{i_0}$ which does not lie in a kernel of $\theta_1$. The images from $f_{i_0} \Gamma f_{i_0}$ generate $f_{i_0} \Gamma f_{i_0}/f_{i_0} \Gamma f_{i_0} N_{i_0}$ over $f_{i_0} \Gamma f_{i_0}/N_{i_0}$, and so $z_i$ freely generates $f_{i_0} \Gamma f_{i_0}/f_{i_0} \Gamma f_{i_0} N_{i_0}$ over $f_{i_0} \Gamma f_{i_0}/N_{i_0}$. Hence

$$z_1 + f_{i_0} \Gamma f_{i_0} N_{i_0} = f_{i_0} \Gamma f_{i_0},$$

and by Nakayama’s Lemma,

$$z_1 f_{i_0} = f_{i_0} \Gamma f_{i_0}.$$  

Therefore, we have shown that $f_{i_0} \Gamma f_{i_0}$ is freely generated over $f_{i_0} \Gamma f_{i_0}$ by an element $z_i$ of $f_{i_0} \Gamma f_{i_0}$. Certainly, we can find such $z_i$ for all $i$.

From the sequence

$$f_i \Gamma f_i \simeq z_i f_i \Gamma f_i = f_{i_0} \Gamma f_{i_0} \simeq f_{i_0} \Gamma f_{i_0} \simeq z_i f_i \Lambda f_i \simeq f_i \Lambda f_i$$

and by hypothesis (a), we have that

$$f_{i_0} \Gamma f_{i_0} = z_i f_i \Lambda f_i$$

for all $i$, $i = 1, \ldots, k$.

Set $z = z_1 + \cdots + z_k$, and we will show that $z \Lambda = I$. Note that $(f_i \Lambda f_j)(f_i \Lambda f_j) = f_i \Lambda f_j$ for all $i$ and $j$. Thus,

$$z \Lambda = \sum_{i=1}^k z_i \Lambda = \sum_{i,j} z_i f_i \Lambda f_j = \sum_{i,j} z_i (f_i \Lambda f_j)(f_i \Lambda f_j) = \sum_{i,j} (f_{i_0} \Gamma f_i)(f_i \Lambda f_j) = f_{i_0} \Gamma f_i = I$$

(using hypothesis (c) and the fact that $\tau$ is a permutation).

Now we consider $f_{i_0} \Gamma f_i$ as a left $f_{i_0} \Lambda f_{i_0}$-module and $f_{i_0} \Gamma f_i$ as a left $f_{i_0} \Gamma f_{i_0}$-module. Hence, by a proof symmetric to the above, we can pick an element $z'_i$ in $f_{i_0} \Gamma f_i$ which does not lie in the kernel of the composed map

$$\theta_i : f_{i_0} \Gamma f_i \to f_{i_0} \Gamma f_i \to f_{i_0} \Gamma f_i/N_{i_0} f_{i_0} \Gamma f_i.$$
We claim that \( z_i \) can be chosen to be \( z'_i \). Certainly, this is true if \( f_{(i)}f_i \) is not contained in the union of \( f_{(i)}f_i N_i \) and \( N_{(i)}f_{(i)}f_i \) (considered as subsets of \( T \)). Assume that \( f_{(i)}f_i \) is contained in this union. It is known that \( z_i \) is not in \( f_{(i)}f_i N_i \), so it must be in \( N_{(i)}f_{(i)}f_i \). Also, \( z'_i \) is not in \( N_{(i)}f_{(i)}f_i \), so it must be in \( f_{(i)}f_i N_i \).

Consider \( z_i + z'_i \in f_{(i)}f_i \). If \( z_i + z'_i \) is in \( f_{(i)}f_i N_i \), then \( z_i \) must be in \( f_{(i)}f_i N_i \), a contradiction. Assuming \( z_i + z'_i \) is in \( N_{(i)}f_{(i)}f_i \) implies that \( z'_i \) is in \( N_{(i)}f_{(i)}f_i \), again a contradiction. Therefore, \( f_{(i)}f_i \) is not contained in the above union.

Hence, by proofs completely symmetric to the previous ones,

\[
f_{(i)}f_i = f_{(i)}\Lambda f_{(i)}z_i
\]

(assuming index condition (b)), and \( \Lambda z = I \) (using condition (d)).

Since \( I \) is full, \( z\Sigma = \Sigma \) and \( \Sigma z = \Sigma \). Thus, let \( w \) be in \( \Sigma \) such that \( zw = 1 = wz \). Set \( J = \Lambda w \Lambda \). Then

\[
JI = \Lambda w \Lambda z \Lambda = \Lambda w z \Lambda = \Lambda, \quad \text{and} \quad IJ = \Lambda z \Lambda w \Lambda = \Lambda z w \Lambda = \Lambda.
\]

Thus \( I \) is invertible.

Since \( J \subseteq (\Lambda : I) \), and \( (\Lambda : I) \) is \( IJ(\Lambda : I) \) \( J \Lambda = I \), we have that \( J = (\Lambda : I) \). Similarly, \( J = (\Lambda : I) \). Q.E.D.

It is appropriate to note here that in the commutative case every order is reduced, so Theorems 5 and 6 generalize Fröhlich’s Theorem to the noncommutative case.

**Theorem 7.** Let \( I \) be a full two-sided ideal in the reduced \( A \)-order \( \Lambda \). Let \( f_1, \ldots, f_k \) be a set of reduced idempotents for \( \Lambda \). Then \( I \) is \( \Lambda \)-invertible if and only if the following conditions are satisfied:

\[
\begin{align*}
(a) \quad & [f_i \Gamma f_i^2 : f_i \Gamma f_i^2] = [f_i \Gamma f_i^2 : f_i \Gamma f_i^2], \\
(b) \quad & [f_i \Gamma f_i^2 : f_i \Gamma f_i^2] = [f_i \Gamma f_i^2 : f_i \Gamma f_i^2], \\
(c) \quad & [f_i \Lambda f_i : f_i \Lambda f_i] = [f_i \Lambda f_i : f_i \Lambda f_i]^2, \\
(d) \quad & (f_i \Lambda f_i)(f_i \Lambda f_i) = f_i \Lambda f_i, \\
(e) \quad & (f_i \Lambda f_i)(f_i \Lambda f_i) = f_i \Lambda f_i.
\end{align*}
\]

**Proof.** Assume that \( I \) is \( \Lambda \)-invertible. In Theorem 5, we showed that there was a permutation \( \sigma \) of 1, 2, \ldots, \( k \) and that there were elements \( x_i \) in \( f_i f_{\sigma(i)} \) such that

\[
x_{\sigma^{-1}(i)} f_i \Lambda f_i = f_{\sigma^{-1}(i)} f_{\sigma(i)}
\]

and such that for \( x = x_1 + \cdots + x_k \), we have \( \Lambda x = x \Lambda = I \). Now,

\[
\begin{align*}
x^2 &= (x_1 + \cdots + x_k)^2 = \sum_{i=1}^{k} x_i x_{\sigma(i)}, \\
x^3 &= (x_1 + \cdots + x_k)^3 = \sum_{i=1}^{k} x_i x_{\sigma(i)} x_{\sigma^2(i)}, \\
x^n &= (x_1 + \cdots + x_k)^n = \sum_{i=1}^{k} x_i x_{\sigma(i)} \cdots x_{\sigma^n(i)}.
\end{align*}
\]
So since $a^k=1$, $x^k=z_1 + \cdots + z_k$ with $z_i$ an element of $f_iI^k\gamma_i$. Set $z=z_1 + \cdots + z_k$. Then $I^k = z\Lambda = \Lambda z$. Hence we have

$$I^k = (z_1 + \cdots + z_k)\Lambda = (z_1\Lambda f_1 + \cdots + z_k\Lambda f_k).$$

So $f_iI^k\beta_i = z_i f_i \Lambda f_i$. Similarly, $f_iI^k\beta_i = f_i \Lambda f_i z_i$. We note further that

$$z_i f_i I^k\beta_i = z_i f_i \Lambda f_i z_i,$$

$$z_i f_i I^k\beta_i = z_if_i \Lambda f_i z_i f_i \Lambda f_i = z_if_i \Lambda f_i z_i.$$  

Since we have $f_iI^k\beta_i = z_i f_i \Lambda f_i z_i$, then $f_iI^k\beta_i$ is $\Lambda$-isomorphic to $f_iI^k\gamma_i$ by an isomorphism which extends to an isomorphism from $f_iI^k\Gamma_i$ to $f_iI^k\gamma_i$. Thus the index condition (b) holds. By similar arguments, it is clear that the index condition (a) holds.

Now,

$$[f_i\Lambda f_i : f_iI^k\beta_i] = [f_i\Lambda f_i : f_iI^k\gamma_i][f_iI^k\beta_i : f_iI^k\gamma_i]$$

$$= [f_i\Lambda f_i : f_iI^k\gamma_i][f_i\Lambda f_i z_i : f_i\Lambda f_i z_i]$$

$$= [f_i\Lambda f_i : f_iI^k\gamma_i][f_i\Lambda f_i : f_iI^k\beta_i]$$

$$= [f_i\Lambda f_i : f_iI^k\beta_i]^2.$$

Hence index condition (c) is satisfied.

Finally,

$$(f_iI^k\beta_i)(f_i\Lambda f_i) = z_i f_i \Lambda f_i f_i \Lambda f_i$$

$$= z_i f_i \Lambda f_i$$

$$= z_iI^k\beta_i.$$  

So the condition (d) is true, and by similar arguments condition (e) is true.

Now we will assume the conditions (a) through (e). In Theorem 6, we saw that we could find elements $z_i$ in $f_iI^k\beta_i$ which freely generate $f_iI^k\gamma_i$ over $f_i\Gamma_i$ as right $f_i\Gamma_i$-modules and which freely generate $f_i\Gamma_i^k\gamma_i$ over $f_i\Gamma_i$ as left $f_i\Gamma_i$-modules.

Then

$$[f_i\Lambda f_i : f_iI^k\gamma_i] = [f_i\Lambda f_i : f_iI^k\gamma_i][f_i\Gamma_i^k\gamma_i : f_iI^k\gamma_i]$$

$$= [f_i\Lambda f_i : f_iI^k\gamma_i][f_i\Gamma_i z_i : f_i\Gamma_i z_i]$$

$$= [f_i\Lambda f_i : f_iI^k\gamma_i][f_i\Gamma_i z_i : f_i\Gamma_i^k\gamma_i]$$

$$= [f_i\Lambda f_i : f_iI^k\gamma_i][f_i\Gamma_i z_i : f_i\Gamma_i^k\gamma_i]$$

$$= [f_i\Lambda f_i : f_iI^k\gamma_i][f_i\Gamma_i^k\gamma_i : f_i\Gamma_i^k\gamma_i],$$

since conditions (c) hold for all $i$. We must have

$$[f_i\Lambda f_i : f_iI^k\gamma_i] = [f_i\Gamma_i^k : f_iI^k\gamma_i],$$

or

$$[f_i\Gamma_i^k : f_i\Lambda f_i] = [f_i\Gamma_i^k : f_iI^k\gamma_i].$$
In a similar fashion we have

\[ [f_i \Gamma f_i : f_i \Lambda f_i] = [f_i \Lambda^k f_i : f_i \Lambda^k f_i] \]

Thus in view of conditions (d) and (e), we have from Theorem 6 that \( I^k \) is \( \Lambda \)-invertible, i.e., \( I \) is \( \Lambda \)-invertible. Q.E.D.

4. Reduction from a Dedekind domain. In this section, we will let \( D \) be a Dedekind domain with quotient field \( K \). Assume \( \Sigma \) is a finite dimensional separable \( K \)-algebra, \( \Lambda \) a \( D \)-order in \( \Sigma \) and \( \Gamma \) a maximal \( D \)-order in \( \Sigma \) containing \( \Lambda \). Let \( P \) be a maximal prime ideal in \( D \). Let \( K(P) \) denote the completion of \( K \) at \( P \), and let \( D(P) \) be the complete discrete valuation ring in \( K(P) \). Let \( \Sigma(P) = \Sigma \otimes_D D(P), \Lambda(P) = \Lambda \otimes_D D(P), I(P) = I \otimes_D D(P) \), etc. There are well-known canonical embeddings of \( \Gamma(P), \Lambda(P), I(P) \) into \( \Sigma(P) \) \[5\]. For the \( D \)-order \( \Lambda \) and a maximal prime ideal \( P \) of \( D \), we will let \( e_1, e_2, \ldots, e_{k_P} \) be the primitive orthogonal idempotents of \( \Lambda(P) \) and assume that they are primitive in \( \Gamma(P) \). Let \( f_1, f_2, \ldots, f_{k_P} \) be representatives of the distinct equivalence classes of idempotents in \( \Lambda(P), f = f_1 + \cdots + f_{k_P} \) and let \( \tilde{\Lambda}(P) = f\Lambda(P), \tilde{I}(P) = fI(P), f \) for \( I \) a two-sided \( \Lambda \)-module.

We will now give “global” versions of the theorems of the preceding section.

**Theorem 8.** Let \( I \) be an invertible two-sided ideal in the \( D \)-order \( \Lambda \). Let \( \{P\} \) denote the set of all maximal prime ideals of \( D \). Then there are permutations \( \sigma \) of the sets \( \{1, 2, \ldots, k_P\} \) such that for \( i \) and \( j \) and all primes \( P \) of \( D \),

\begin{align*}
(a) \quad & [f_i \Gamma f_i : f_i \Lambda f_i] = [f_i \Lambda^{-1} \Gamma f_i : f_i \Lambda^{-1} \Lambda f_i], \\
(b) \quad & [f_i \Gamma f_i : f_i \Lambda f_i] = [f_i \Gamma f_i \sigma(\Gamma(f_i) f_i) : f_i \Gamma f_i \sigma(f_i)], \\
(c) \quad & (f_i^{-1} \Gamma f_i f_i)(f_i \Lambda f_i) = f_i^{-1} (f_i \Lambda f_i) f_i, \\
(d) \quad & (f_i \Lambda(f_i f_i))(f_i \Lambda(f_i f_i)) = f_i \Lambda(f_i f_i),
\end{align*}

where \( f_1, \ldots, f_{k_P} \) are the reduced idempotents of \( \tilde{\Lambda}(P) \).

**Proof.** Since \( D \) is a Dedekind domain, \( I \) is reflexive and \( I = \bigcap P I(P) \). Also, \( I(P) = \Lambda(P) \) for almost all \( P \). So if each \( I(P) \) is invertible by \( J(P) \) in \( \Lambda(P) \), set \( J = \bigcap P J(P) \), and consider \( J(J(P)) = \Lambda(P) \) for all \( P \), so \( J = \Lambda \) since \( \Lambda = \bigcap P \Lambda(P) = J \). Certainly if \( I \) is invertible by \( J \) in \( \Lambda \), then \( J(J(P)) = I(P)J(P) = \Lambda(P) \), so \( I(P) \) is invertible in \( \Lambda(P) \) for all \( P \). Hence by \[5\, ,\, \text{Theorem 2, p. 204}\], we need only prove the theorem for each \( D(P) \).

Let \( I(P) \) be a two-sided \( \Lambda(P) \)-ideal which is invertible in \( \Lambda(P) \) by \( J(P) \). Setting \( \tilde{I}(P) = fI(P) \) and \( \tilde{J}(P) = fJ(P) \), we have by Lemma 4 that \( \tilde{I}(P) \) is a two-sided \( \tilde{\Lambda}(P) \)-ideal which is invertible in \( \tilde{\Lambda}(P) \) by \( f \tilde{I}(P)J(P) = (fI(P)J(P)) = fI(P)J(P) = f\Lambda(P) = \tilde{I}(P) \).

Since \( \tilde{\Lambda}(P) \) is reduced, we can apply Theorem 5 to obtain a permutation \( \sigma \) of the set \( \{1, 2, \ldots, k_P\} \) such that the conclusions (a), (b), (c) and (d) are true. Q.E.D.

**Theorem 9.** If \( I \) is a full two-sided ideal of the \( D \)-order \( \Lambda \) such that for all \( P \) there is a permutation \( \sigma \) of the \( \{1, 2, \ldots, k_P\} \) having the properties

\begin{align*}
(a) \quad & [f_i \Gamma f_i : f_i \Lambda f_i] = [f_i \Lambda^{-1} \Gamma f_i : f_i \Lambda^{-1} \Lambda f_i], \\
(b) \quad & [f_i \Gamma f_i : f_i \Lambda f_i] = [f_i \Gamma f_i \sigma(\Gamma(f_i) f_i) : f_i \Gamma f_i \sigma(f_i)], \\
(c) \quad & (f_i^{-1} \Gamma f_i f_i)(f_i \Lambda f_i) = f_i^{-1} (f_i \Lambda f_i) f_i, \\
(d) \quad & (f_i \Lambda(f_i f_i))(f_i \Lambda(f_i f_i)) = f_i \Lambda(f_i f_i),
\end{align*}

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
\[ \begin{align*}
\text{(b) } & [f_0^{-1}I_0 \Gamma_{P} f_0^{-1}I_0 : f_0^{-1}I_0 \Lambda_{P} f_0^{-1}I_0] = [f_0^{-1}I_0 \Gamma_{P} f_0^{-1}I_0 : f_0^{-1}I_0 \Lambda_{P} f_0^{-1}I_0], \\
\text{(c) } & [f_0^{-1}I_0 \Lambda_{P} f_0^{-1}I_0](f_i \Lambda_{P} f_i) = f_0^{-1}I_0 \Lambda_{P} f_0^{-1}I_0, \\
\text{(d) } & (f_i \Lambda_{P} f_i)(f_i \Lambda_{P} f_i) = (f_i \Lambda_{P} f_i). \\
\end{align*} \]

for all \( i \) and \( j \), where \( f_1, \ldots, f_k \) denote the reduced idempotents of \( \Lambda_{P} \). Then \( I \) is invertible in \( \Lambda \).

**Proof.** Just as in the proof of Theorem 8, to prove that \( I \) is invertible in \( \Lambda \), it is sufficient to prove that for all \( P \), \( I_{(P)} \) is invertible in \( \Lambda_{(P)} \). Further, in Theorem 8, we showed that if \( I_{(P)} \) was \( \Lambda_{(P)} \)-invertible, then \( I_{(P)} \) was \( \Lambda_{(P)} \)-invertible. Note that if \( I_{(P)} \) is \( \Lambda_{(P)} \)-invertible by \( J_{(P)} \), then

\[ A_{(P)} = I(P)A(P)/I(P)A(P). \]

Thus,

\[ A_{(P)} = fI(P)A(P)/fI(P)A(P). \]

So \( I_{(P)}J_{(P)} \) maps to \( \Lambda_{(P)} \) in the map of \( \Lambda_{(P)} \)-modules defined in Lemma 4. Hence \( I_{(P)}J_{(P)} = \Lambda_{(P)} \). Thus, \( I_{(P)} \) is \( \Lambda_{(P)} \)-invertible if and only if \( I_{(P)} \) is \( \Lambda_{(P)} \)-invertible. Therefore, it suffices to prove the result for reduced orders over complete discrete valuation rings, and by Theorem 6, the proof is completed. Q.E.D.

Finally we give the “global version” of Theorem 7.

**Theorem 10.** Let \( I \) be a full two-sided ideal in the \( D \)-order \( \Lambda \). Let \( P \) denote a maximal prime in \( D \) and let \( f_1, \ldots, f_k \) then \( I \) is \( \Lambda \)-invertible if and only if the following conditions hold for every maximal prime \( P \) and for all \( i \) and \( j \):

\[ \begin{align*}
\text{(a) } & [f_i \Gamma_{P} f_i : f_i \Gamma_{P} f_i] = [f_i \Gamma_{P} f_i : f_i \Gamma_{P} f_i], \\
\text{(b) } & [f_i \Lambda_{P} f_i : f_i \Lambda_{P} f_i] = [f_i \Lambda_{P} f_i : f_i \Lambda_{P} f_i], \\
\text{(c) } & (f_i \Lambda_{P} f_i)(f_i \Lambda_{P} f_i) = (f_i \Lambda_{P} f_i), \\
\text{(d) } & (f_i \Lambda_{P} f_i)(f_i \Lambda_{P} f_i) = (f_i \Lambda_{P} f_i). \\
\end{align*} \]

**Proof.** It was proved in Theorems 8 and 9 that \( I \) is \( \Lambda \)-invertible if and only if \( I_{(P)} \) is \( \Lambda_{(P)} \)-invertible for every maximal prime \( P \) in \( D \), and \( I_{(P)} \) is \( \Lambda_{(P)} \)-invertible if and only if \( I_{(P)} \) is \( \Lambda_{(P)} \)-invertible. Hence this result follows from Theorem 7. Q.E.D.

5. **Examples.** This section will give examples and counterexamples relevant to the preceding sections. Let \( A \) be a discrete valuation ring with quotient field \( K \) and prime element \( \pi \). Let \( \Sigma_n \) be the \( K \)-algebra of \( n \times n \) matrices. Let \( \Lambda = [\pi^{uj}A] \) denote an \( A \)-order in \( \Sigma_n \). This notation means that the \((i,j)\) positions of \( \Lambda \) has elements in the form \( \pi^{uj}a \) with \( r_{ij} \) an integer and \( a \) any element of \( A \).
We note the following lemma.

**Lemma 11.** Let $\Lambda = [\pi^n A]$, $I = [\pi^n A]$ and $(\Lambda : I) = \{x : Ix \subseteq \Lambda\} = [\pi^n A]$. Then $I$ is left $\Lambda$-projective if and only if for every $P, P = 1, \ldots, n$, there is a $k_P, 1 \leq k_P \leq n$ such that $m_{P,k} = -s_{kP}$.

The lemma is actually a restatement in the language of matrices of a classical result of Cartan and Eilenberg [3, p. 132].

**Example 12.** In $\Sigma_2$, the fact that a full left ideal is $\Lambda$-projective does not necessarily imply that $[\Gamma : I] = [\Gamma : \Lambda]$. Note that this example shows that one part of Fröhlich’s Theorem does not extend to all finite dimensional central simple algebras.

Set $\Gamma = \Lambda_2$, and let $\Lambda$ be the $A$-order contained in $\Gamma$ of the form

$$\Lambda = \begin{bmatrix} A & \pi^2 A \\ \pi A & A \end{bmatrix}.$$ 

Let $I$ be the left ideal of the form

$$I = \begin{bmatrix} \pi^4 A & \pi^3 A \\ \pi^2 A & \pi A \end{bmatrix}.$$ 

Then $\Gamma I$ has the form

$$\Gamma I = \begin{bmatrix} \pi^5 A & \pi^4 A \\ \pi^4 A & \pi^3 A \end{bmatrix}.$$ 

Then $[\Gamma : \Lambda] = \pi^3 A$ and $[\Gamma I : I] = \pi^4 A$, and $[\Gamma I : I] \neq [\Gamma : \Lambda]$. Using Lemma 11, it is easy to check that $I$ is left $\Lambda$-projective.

**Example 13.** In $\Sigma_2$, the fact that $[\Gamma I : I] = [\Gamma : \Lambda]$ does not imply that a full left ideal is $\Lambda$-projective. Therefore, this example along with Example 11 shows that Fröhlich’s Theorem has no direct extension to central simple algebras.

Set $\Gamma = \Lambda_2$ and let $\Lambda$ be as in Example 11. Let $I$ be the left $\Lambda$-ideal of the form

$$\begin{bmatrix} \pi^3 A & \pi^3 A \\ \pi^2 A & \pi A \end{bmatrix}.$$ 

Then $\Gamma I$ is of the form

$$\begin{bmatrix} \pi^2 A & \pi A \\ \pi^2 A & \pi A \end{bmatrix}.$$ 

Then $[\Gamma : \Lambda] = \pi^3 A = [\Gamma I : I]$. It is easy to check that $I$ is not $\Lambda$-projective by using Lemma 11.

One of the conclusions of Theorem 5 is that $[\Gamma : \Lambda] = [\Gamma I : I] = [\Gamma I : I]$ for $I$ an invertible two-sided $\Lambda$-ideal. Therefore it is reasonable to ask if this condition is strong enough to imply invertibility. The following example shows that the condition is not strong enough even if $I$ is assumed to be both right and left $\Lambda$-projective.

**Example 14.** In $\Sigma_2$ there is a two-sided $\Lambda$-deal which is right and left projective and such that $[\Gamma I : I] = [\Gamma I : I] = [\Gamma : \Lambda]$ but is not invertible in $\Lambda$. 

Let
\[ \Gamma = \Lambda_2, \quad \Lambda = \begin{bmatrix} A & \pi A \\ \pi A & A \end{bmatrix}, \]
and
\[ I = \begin{bmatrix} \pi^2 A & \pi A \\ \pi A & A \end{bmatrix}. \]

Then \( I \) is a two-sided \( \Lambda \)-ideal and
\[ [\Gamma : I] = [I : \Gamma] = [\Gamma : \Lambda] = \pi^2 A. \]

It is seen by Lemma 11 and its symmetric counterpart for right ideals that \( I \) is both right and left \( \Lambda \)-projective. \( I \) is not \( \Lambda \)-invertible since it is not cyclic and \( \Lambda \) is reduced.

**Example 15.** This example shows that if \( \Lambda \) is not a reduced \( A \)-order, then an invertible ideal \( I \) may not be cyclic. Further, \([\Gamma : I] \neq [\Gamma : \Lambda]\) and \([I : \Gamma] \neq [\Gamma : \Lambda]\), but \([I : \Gamma] = [\Gamma : \Lambda] = [\Gamma : I]\). So, in particular, the condition that \( \Lambda \) be a reduced \( A \)-order is essential in Theorem 5.

Let
\[ \Lambda = \begin{bmatrix} A & A & A \\ \pi A & A & \pi A \\ A & A & A \end{bmatrix}, \]
and
\[ I = \begin{bmatrix} \pi A & A & \pi A \\ \pi A & A & \pi A \\ \pi A & A & \pi A \end{bmatrix}. \]

\( \Lambda \) is not reduced since \( e_{11} \sim e_{33} \). \( I \) is \( \Lambda \)-invertible by
\[ J = \begin{bmatrix} A & \pi^{-1} A & A \\ A & A & A \\ A & \pi^{-1} A & A \end{bmatrix}. \]

If \( \Gamma = \Lambda_3 \), then
\[ [\Gamma : \Lambda] = \pi^2 A, \quad [\Gamma : I] = \pi^3 A, \quad [I : \Gamma] = \pi^3 A. \]

To see that \( I \) is not cyclic, assume the contrary. On noting that \( I^2 = \pi \Lambda \), we see that there are elements \( a, b \) in \( \Lambda \) such that \( x^2 = \pi a, \ xbx = \pi \). Then \((\det a)(\det b) = 1\), and \((\det x)^2 = \pi^3(\det b)\). Let \( \det x = \pi^s \). Then \( 2s = 3 \), a contradiction.

Now set \( f_1 = e_{11}, f_2 = e_{22} \), and \( f = f_1 + f_2 \). Then
\[ \tilde{\Lambda} = f \Lambda f = \begin{bmatrix} A & A \\ \pi A & A \end{bmatrix}, \quad \tilde{I} = f I f = \begin{bmatrix} \pi A & A \\ \pi A & \pi A \end{bmatrix}. \]

Set
\[ x = \begin{bmatrix} 0 & 1 \\ \pi & 0 \end{bmatrix}. \]
Then, $I = \tilde{\Lambda} \tilde{x} = \tilde{x} \tilde{\Lambda}$, and $\Lambda = \Lambda_2$, so

$$[\tilde{f} : \tilde{\Lambda}] = [\tilde{f} : I] = \pi A.$$ 

**References**


**South Dakota School of Mines and Technology,**
**Rapid City, South Dakota**