Embedding as a Double Commutator in a Type I $AW^*$-Algebra\(^{(1)}\)

by

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1. Introduction. The purpose of this paper is the characterization of those $C^*$-algebras which can be written as their own double commutator in a type I $AW^*$-algebra. In a previous paper [5] the present author considered the module structure induced on a $C^*$-algebra $\mathcal{A}$ by its center $\mathcal{Z}$ which was taken to be a von Neumann algebra. It was shown that $\mathcal{A}$ is a von Neumann algebra if and only if it could be identified with the space of all bounded module homomorphisms into $\mathcal{Z}$ on a normed $\mathcal{Z}$-module. Here, an analogue of this theorem is obtained: a $C^*$-algebra $\mathcal{A}$ whose center is an $AW^*$-algebra $\mathcal{Z}$ can be isomorphically and isometrically embedded as a double commutator in a type I $AW^*$-algebra with center $\mathcal{Z}$ if and only if $\mathcal{A}$ can be written as the set of all bounded module homomorphisms into a normed $\mathcal{Z}$-module $M$. The topology induced on the unit sphere of $\mathcal{A}$ by pointwise convergence on $M$ will be the weak topology on the unit sphere of $\mathcal{A}$. This result can be regarded as a generalization of Sakai's theorem relating to von Neumann algebras [12] and in a certain sense it also illustrates that the generality of such an $AW^*$-algebra $\mathcal{A}$ as compared to a von Neumann algebra lies in its center.

The problem of embedding an $AW^*$-algebra $\mathcal{A}$ in a type I $AW^*$-algebra so as to preserve the sums of orthogonal projections was studied by H. Widom [18]. He found that such an embedding was possible if and only if $\mathcal{A}$ possesses a complete set $\{\phi_n\}$ of positive module homomorphisms into the center $\mathcal{Z}$ which mapped 1 into 1 and were completely additive on projections. He also studied those $AW^*$-algebras $\mathcal{A}$ which were embedded as double commutators in type I algebras and showed that a finite $AW^*$-subalgebra of a type I algebra $\mathcal{B}$ is its own double commutator in $\mathcal{B}$. T. Yen also studied the problem and showed that a type II\(_1\) $AW^*$-algebra with finite trace is its own double commutator in a type I algebra [19].

2. The weak topology. Let $H$ be an $AW^*$-module [10]. For each $x$ and $y$ in $H$ let $w_{x,y}$ and $w_x$ be the functions defined on the algebra $L(H)$ of all bounded linear operators on $H$ by $w_{x,y}(A) = (Ax, y)$ and $w_x(A) = (Ax, x)$ respectively. The weak topology on a $*$-subalgebra $\mathcal{A}$ of $L(H)$ is the weakest topology on $\mathcal{A}$ in

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which each function $A \rightarrow \| w_{x,y}(A) \|$ (x, y ∈ H) or equivalently in which each function $A \rightarrow \| w_x(A) \|$ (x ∈ H) is continuous on $\mathcal{A}$.

**Proposition 1.** Let $H$ be an AW*-module over the commutative AW*-algebra $\mathcal{Z}$. Let $\mathcal{A}$ be a *-subalgebra of $L(H)$ which contains $\mathcal{Z}$ and let $\mathcal{A}_c$ be the algebraic $\mathcal{Z}$-module generated by the functions $w_{x,y}$ (x, y ∈ H) restricted to $\mathcal{A}$. Then $\mathcal{A}_c$ is the set of weakly continuous $\mathcal{Z}$-module homomorphisms of the $\mathcal{Z}$-module $\mathcal{A}$ into $\mathcal{Z}$.

**Proof.** It is sufficient to prove that $\mathcal{A}_c$ contains the set of weakly continuous module homomorphisms because clearly $\mathcal{A}_c$ is contained in the set of weakly continuous $\mathcal{Z}$-module homomorphisms. Let $f$ be weakly continuous. There are elements $x_i$ (1 ≤ i ≤ n) in H such that $\|f(A)\| \leq 1$ whenever $\|w_{x_i}(A)\| \leq 1$ for every $i = 1, 2, \ldots, n$. If $A \in L(H)$, let $\|A\| = (A^*A)^{1/2}$. By setting $p(A) = \sum |w_{x_i}(A)|$ for $A \in L(H)$, we define a function of $L(H)$ into $\mathcal{Z}$ such that $p(A + B) \leq p(A) + p(B)$ and $p(CA) = \|C\|p(A)$ for every $A, B \in L(H)$ and $C \in \mathcal{Z}$. We have that $\|f(A)\| \leq 1$ whenever $p(A) \leq 1$. Therefore, $\|f(A)\| \leq p(A)$ for every $A$ in $\mathcal{A}_c$. Setting $g(A) = (f(A) + f(A^*))/2$, we obtain a function of $\mathcal{A}_c$ into the set of hermitian elements $H(\mathcal{Z})$ which is a module homomorphism when $\mathcal{A}_c$ is considered to be an $\mathcal{Z}$-module. We still have that $g(A) \leq p(A)$ for every $A$ in $\mathcal{A}_c$. There is a module homomorphism $h$ of the $\mathcal{H}(\mathcal{Z})$-module $L(H)$ into $H(\mathcal{Z})$ such that $h(A) = g(A)$ for every $A$ in $\mathcal{A}_c$ and $h(A) \leq p(A)$ for every $A$ in $L(H)$ [17]. Let $k(A) = h(A) - i(h(A))$. Then $k$ is a module homomorphism of $L(H)$ into $\mathcal{Z}$. If $A \in L(H)$ and if $U$ is a partial isometric operator in $\mathcal{Z}$ with $U|k(A)| = k(A)$ [19, Lemma 2.1], then

$$\|k(A)\| = k(U^*A) \leq p(U^*A) \leq p(A).$$

We also have that $k(A) + k(A^*) = f(A) + f(A^*)$ for every $A$ in $\mathcal{A}_c$. However, this means that $k(A) = f(A)$ for every $A$ in $\mathcal{A}_c$. This proves that $k$ is a module homomorphism of $L(H)$ into $\mathcal{Z}$ which coincides with $f$ on $\mathcal{A}_c$ and which satisfies $\|k(A)\| \leq p(A)$.

Now for each $x_i$ (1 ≤ i ≤ n) there is a $C_i$ in $\mathcal{Z}$ and a $y_i$ in H such that $C_i y_i = x_i$ and such that $|y_i|$ is a projection in $\mathcal{Z}$. Let $E_i$ be the abelian projection in $L(H)$ defined by $E_i x = (x, y_i) y_i$ [10, Lemma 13]. We have that

$$k(A(1 - E)) = k((1 - E)A) = 0,$$

where $E$ is the least upper bound of $E_1, E_2, \ldots, E_n$. The projection $E$ is in the closed two-sided ideal $I_a$ of $L(H)$ generated by the abelian projections of $L(H)$ due to the relation

$$\text{lub} \{E_1, E_2\} - E_1 \sim E_2 - \text{glb} \{E_1, E_2\} \quad \text{[8, Theorem 5.4]}$$

and to the fact that $E_2 - \text{glb} \{E_1, E_2\}$ is abelian. There are orthogonal projections $P_1, P_2, \ldots, P_m$ in $\mathcal{Z}$ whose sum $P$ is the central support of $E$ such that each algebra $EL(H)EP_i$ is either zero or homogeneous of degree $i$ (cf. [4, Theorem 2.1]). Since

$$f(A)(1 - P) = k(A(1 - P)) = 0$$
for every $A$ in $\mathcal{A}$, it is sufficient to prove that each function $P_{ij}$ is in $\mathcal{A}$. So we may assume that $EL(H)E$ is homogeneous of degree $m$. There are equivalent orthogonal abelian projections $\{E_i \mid 1 \leq i \leq m\}$ of sum $E$ and partial isometric operators $\{U_{ij} \mid 1 \leq i, j \leq m\}$ such that

1. $U_{ij}U_{kl} = \delta_{il}U_{kj}$;
2. $U_{ij} = U^*_{ji}$; and
3. $U_{ii} = F_i$ for all $i, j, k, l$.

Thus $f(A) = k(A) = k(EAE) = \sum \tau_{F_i}(U_{ij}A)k(U_{ji})$. Here $\tau_{F_i}(B)$ denotes the unique element in $\mathcal{Z}P$ such that $\tau_{F_i}(B)F_j = F_jBF_j$ [8, Lemma 4.7]. Let $z_j$ be an element in $H$ such that $F_jx = (x, z_j)z_j$ [10, Lemma 13]. Then

$$\tau_{F_i}(U_{ij}A) = (U_{ij}A z_j, z_j) = (Az_j, U_{ji}z_j).$$

This proves that $f \in \mathcal{A}$. Q.E.D.

Let $M$ be a normed vector space which is also an algebraic module over a commutative $AW^*$-algebra $\mathcal{Z}$; then $M$ is said to be a normed $\mathcal{Z}$-module if $\|Ax\| \leq \|A\|\|x\|$ for every $A \in \mathcal{Z}$ and $x \in M$. A bounded module homomorphism of $M$ into $\mathcal{Z}$ will be called a functional of the module $M$. By defining operations in a pointwise fashion, we obtain an algebraic $\mathcal{Z}$-module structure on the set of all functionals of the module $M$. The function

$$\|\phi\| = \text{lub} \{\|\phi(x)\| \mid x \in M, \|x\| \leq 1\}$$

defines a norm on the $\mathcal{Z}$-module of functionals. With this norm the module becomes a normed $\mathcal{Z}$-module. We call this module the dual of $M$ and denote it by $M^\sim$.

**Theorem 2.** Let $H$ be an $AW^*$-module over the commutative $AW^*$-algebra $\mathcal{Z}$ and $\mathcal{A}$ be a $\ast$-subalgebra of the algebra $L(H)$ of all bounded linear operators on $H$ such that $\mathcal{A}$ is equal to its own second commutator in $L(H)$. For each $A$ in $\mathcal{A}$ let $F_A$ be the function defined on the $\mathcal{Z}$-module $\mathcal{A}_-$ (considered as a submodule of $\mathcal{A}$) of weakly continuous module homomorphisms of $\mathcal{A}$ into $\mathcal{Z}$ by $F_A(\phi) = \phi(A)$. Then $A \rightarrow F_A$ defines an isometric isomorphism of $\mathcal{A}$ onto the dual of $\mathcal{A}_-$.

**Proof.** First let $\mathcal{A} = L(H)$. If $\Phi \in (\mathcal{A}_-)\sim$, then $\Phi(w_{x,y}) = \langle x, y \rangle$ defines a $\mathcal{Z}$-valued hermitian form on $H$ such that

$$\|\langle x, y \rangle\| \leq \|\Phi\| \|w_{x,y}\| \leq \|\Phi\| \|x\| \|y\|.$$ 

The function $x \rightarrow \langle x, y \rangle$ is a bounded $\mathcal{Z}$-linear function of $H$ into $\mathcal{Z}$. Therefore, there is a unique element $A_y$ in $H$ with $\langle x, y \rangle = (x, A_y)$ for every $x$ in $H$ [10, Theorem 5]. We have that $\|A_y\| \leq \|\Phi\| \|y\|$. From the uniqueness of $A_y$ we conclude that there is an $A$ in $L(H)$ such that $A_y = A_y$ for every $y$ in $H$. Thus, $\Phi(w_{x,y}) = w_{x,y}(A)$ for every $w_{x,y}$. Since functions of the form $w_{x,y}$ generate $\mathcal{A}_-$, we have that $\Phi(\phi) = \phi(A)$ for every $\phi \in \mathcal{A}_-$. 

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Now we have that \( A \to F_A \) defines a \( \mathfrak{L} \)-linear function of \( \mathcal{A} \) into \( (\mathcal{A}_*)^\sim \). We have that \( \|F_A\| \leq \|A\| \) since \( \|\phi(A)\| \leq \|\phi\| \|A\| \) for every \( \phi \in \mathcal{A}_* \). But
\[
\|A\| = \text{lub} \{\|w_{x,y}(A)\| \mid \|w_{x,y}\| \leq 1\}
\]
and so \( \|A\| = \|F_A\| \). Thus \( A \to F_A \) is an isometric isomorphism of \( \mathcal{A} \) into \( (\mathcal{A}_*)^\sim \).

The preceding paragraph allows us to conclude that \( A \to F_A \) is onto \( (\mathcal{A}_*)^\sim \).

Now assume that \( \mathcal{A} \) is an arbitrary \( * \)-subalgebra of \( L(H) \) which is equal to its own double commutator. Let \( G \) be the bounded \( \mathfrak{L} \)-linear map which takes an element in \( L(H)_* \) onto its restriction to \( \mathcal{A} \). Then \( G \) is a map of \( L(H)_* \) onto \( \mathcal{A}_* \) (Proposition 1). If \( \Phi \) is an element of \( (\mathcal{A}_*)^\sim \), then \( \Phi \cdot G \) defines an element of \( (L(H)_*)^\sim \). By the first part of this proof we may find an \( A \) in \( L(H) \) with \( \Phi \cdot G(\phi) = \phi(A) \) for every \( \phi \) in \( L(H)_* \). If \( A \) is not in \( \mathcal{A} \), there is a unitary operator \( U \) in the commutator of \( \mathcal{A} \) such that \( U^*AU \neq A \). Then there is an \( x \) in \( H \) with \( w_x(A) - w_{UX}(A) \neq 0 \). But \( \phi = w_x - w_{UX} \) vanishes on \( \mathcal{A} \) and so \( \phi(A) = \Phi(G(\phi)) = 0 \). This is a contradiction. Thus \( A \) is in \( \mathcal{A} \). Since every \( \phi \) in \( \mathcal{A}_* \) has an extension to a function in \( L(H)_* \), we conclude that \( \Phi(\phi) = \phi(A) \) for every \( \phi \) in \( \mathcal{A}_* \). Thus we may apply the arguments of the preceding paragraph in order to show that \( A \to F_A \) is an isometric isomorphism of \( \mathcal{A} \) onto \( (\mathcal{A}_*)^\sim \). Q.E.D.

Remark. The algebra \( \mathcal{A} \) in the preceding theorem is expressed as the dual of a module whose ring of multipliers is a subalgebra of the center of \( \mathcal{A} \). This pathological feature can be removed by the following additional argument. Let \( \mathfrak{L}_0 \) be the center of \( \mathcal{A} \). The commutator \( \mathfrak{L}_0' \) of \( \mathfrak{L}_0 \) on \( H \) is a type I algebra by a proof that is entirely similar to the corresponding proof for von Neumann algebras (cf. [1, I, §2, Proposition 1 and §6, Problem 5]). The center of \( \mathfrak{L}_0' \) is \( \mathfrak{L}_0 = \mathfrak{L}_0' \). Since \( \mathfrak{L}_0 \) is the algebra of all bounded linear operators on an \( AW^* \)-module over \( \mathfrak{L}_0 \) [10, Theorem 8] and since \( \mathcal{A} \) is its own double commutator in \( \mathfrak{L}_0 \), we may conclude that \( \mathcal{A} \) is the dual of \( \mathfrak{L}_0 \)-module by Theorem 2.

3. The dual of a \( \mathfrak{L} \)-module. Let \( \mathcal{A} \) be a \( C^* \)-algebra whose center \( \mathfrak{L} \) is an \( AW^* \)-algebra. Then \( \mathcal{A} \) with its norm is a normed \( \mathfrak{L} \)-module. In this section whenever we talk about the module \( \mathcal{A} \), we shall have this particular module structure in mind. If \( \phi \in \mathcal{A}_* \) and \( A \in \mathcal{A} \), the functional \( (A \cdot \phi)(B) = \phi(AB) \) is in \( \mathcal{A}_* \). This defines a right multiplication of elements of \( \mathcal{A}_* \) by \( \mathcal{A} \). Similarly, a left multiplication is defined by \( (\phi \cdot A)(B) = \phi(BA) \). A functional \( \phi \) in \( \mathcal{A}_* \) is said to be positive if \( \phi(A^*A) \geq 0 \) for every \( A \) in \( \mathcal{A} \). Then \( \phi \) is positive if \( \phi(1) \geq 0 \) and \( \|\phi(1)P\| = \|P \cdot \phi\| \) for every projection \( P \) in \( \mathfrak{L} \). Indeed, if \( \|\phi(1)P\| = \|P \cdot \phi\| \) for every projection \( P \) in \( \mathfrak{L} \), then for every \( \zeta \) in the spectrum of \( \mathfrak{L} \) the relation \( |\phi_\zeta(1)| = \|\phi_\zeta\| \) is seen to be true. Here \( \phi_\zeta(A) = \phi(A) \zeta \) where \( B^\zeta \) denotes the Gelfand transform of \( B \in \mathfrak{L} \). This means that \( \phi_\zeta(A^*A) \geq 0 \) for every \( \zeta [2, 2.1.9] \). Therefore the functional \( \phi \) is positive.

Suppose now that the module \( \mathcal{A} \) is the dual of a normed \( \mathfrak{L} \)-module \( M \). Since \( \|A(\phi)\| \leq \|A\| \|\phi\| \) for every \( \phi \in M \) and \( A \in \mathcal{A} \) and since \( (C_1A_1 + C_2A_2)(\phi) = C_1A_1(\phi) + C_2A_2(\phi) \) for every \( C_1, C_2 \) in \( \mathfrak{L} \) and \( A_1, A_2 \) in \( \mathcal{A} \), the function \( \phi \to \phi' \)
of \( M \) into \( \mathcal{A}^- \), where \( \phi' \) is defined by \( \phi'(A) = A(\phi) \), is a norm-decreasing \( \mathcal{Z} \)-module homomorphism of \( M \) into a submodule \( N \) of \( \mathcal{A}^- \). We have that

\[
\| A \| = \text{lub} \{ \| A(\phi) \| \mid \phi \in M, \| \phi \| \leq 1 \}
\]

\[
\leq \text{lub} \{ \| \phi(A) \| \mid \phi \in N, \| \phi \| \leq 1 \} \leq \| A \|
\]

and so we have that \( \| A \| = \text{lub} \{ \| \phi(A) \| \mid \phi \in N, \| \phi \| \leq 1 \} \). Actually, the module \( \mathcal{A} \) is identified with the dual of \( N \). Indeed, if \( \Phi \in N^- \), then \( \phi \mapsto \Phi(\phi') \) defines an element of \( M^- \). There is a unique element \( A_\phi = A \) in \( \mathcal{A} \) such that \( \Phi(\phi') = A(\phi') = \phi'(A) \) for every \( \phi \in M \). The function \( \Phi \mapsto A_\phi \) of \( N^- \) into \( \mathcal{A} \) is easily seen to be an isometric isomorphism of the \( \mathcal{Z} \)-module \( N^- \) onto the module \( \mathcal{A} \). Since we are interested in the topology on \( \mathcal{A} \) induced by pointwise convergence on \( M \), we may assume that \( M \) is embedded in \( \mathcal{A}^- \). We call this topology of pointwise convergence on \( M \) the \( \sigma(\mathcal{A}, M) \)-topology of \( \mathcal{A} \).

Let \( M \) be a submodule of \( \mathcal{A}^- \). For each bounded subset \( \{ \phi_i \} \) of \( M \) and each set \( \{ P_i \} \) of mutually orthogonal projections in \( \mathcal{Z} \) of sum 1, there is a unique \( \phi = \sum P_i \phi_i \) in \( \mathcal{A}^- \) satisfying the relation \( P_i \phi = P_i \phi_i \) for each \( P_i \). Let \( N \) be the smallest algebraic submodule of \( \mathcal{A}^- \) which contains \( M \) and is closed under the formation of such sums. Then every element \( \phi \in N \) is of the form \( \phi = \sum P_i \phi_i \) where \( \{ \phi_i \} \) is a bounded subset of \( M \) and \( \{ P_i \} \) is a set of mutually orthogonal projections in \( \mathcal{Z} \) of sum 1. The \( \mathcal{Z} \)-module \( N \) will be called the module generated by \( M \) in \( \mathcal{A}^- \).

**Proposition 3.** Let \( \mathcal{A} \) be a C*-algebra whose center \( \mathcal{Z} \) is an AW*-algebra. Let \( M \) be a normed \( \mathcal{Z} \)-module whose dual is the module \( \mathcal{A} \); let \( N \) be the module generated by \( M \) in \( \mathcal{A}^- \). Then the dual of the module \( N \) is also equal to \( \mathcal{A} \).

**Proof.** Let \( \Phi \) be a functional in \( N^- \). Then the restriction \( \Psi \) of \( \Phi \) to \( M \) is a bounded functional of the module \( M \). There is an \( A = A_\phi \) in \( \mathcal{A} \) such that \( \Psi(\phi) = \phi(A) \) for every \( \phi \in M \). Let \( \phi \in N \); there is a bounded subset \( \{ \phi_i \} \) of \( M \) and a set \( \{ P_i \} \) of mutually orthogonal projections in \( \mathcal{Z} \) of sum 1 such that \( P_i \phi = P_i \phi_i \) for each \( P_i \). Then

\[
P_i \Phi(\phi) = \Phi(P_i \phi_i) = \Psi(P_i \phi_i) = P_i \phi_i(A) = P_i \phi(A)
\]

for each \( P_i \). This means that \( \Phi(\phi) = \phi(A) \). Suppose there is a second element \( A' \) in \( \mathcal{A} \) such that \( \Phi(\phi) = \phi(A') \) for every \( \phi \in N \). Then every element of \( M \) vanishes on \( A' - A \). Because \( \mathcal{A} \) is the dual of \( M \), we have that \( A' = A \). This means that \( \Phi \mapsto A_\phi \) is a module isomorphism of \( N^- \) onto \( \mathcal{A} \). We have that

\[
\| \Phi \| = \text{lub} \{ \| \Phi(\phi) \| \mid \phi \in N, \| \phi \| \leq 1 \}
\]

\[
\leq \| A_\phi \| = \text{lub} \{ \| \phi(A_\phi) \| \mid \phi \in M, \| \phi \| \leq 1 \} \leq \| \Phi \|
\]

for every \( \Phi \in N^- \). Therefore, the map \( \Phi \mapsto A_\phi \) is an isometric isomorphism of the module \( N^- \) onto the module \( \mathcal{A} \). Q.E.D.

We need the following lemma which is known for \( \sigma \)-weakly continuous functionals on a von Neumann algebra (cf. [2, 12.2.3]).
**Lemma 4.** Let $\mathcal{A}$ be a C*-algebra, $E$ a projection in $\mathcal{A}$ and $f$ a bounded linear functional on $\mathcal{A}$. If the norm of the function $g(A) = f(EA)$ on $\mathcal{A}$ is equal to that of $f$, then $g = f$.

**Proof.** Let $\mathcal{B}$ be the enveloping von Neumann algebra of $\mathcal{A}$. We may consider $\mathcal{A}$ as a weakly dense subset of $\mathcal{B}$. The functionals $f$ and $g$ on $\mathcal{A}$ have unique extensions to weakly continuous functionals $f'$ and $g'$ respectively on $\mathcal{B}$. By the uniqueness of the extension we have that $g'(A) = f'(EA)$ for every $A$ in $\mathcal{B}$. Since the unit sphere of $\mathcal{A}$ is weakly dense in that of $\mathcal{B}$ [7], we have that $\|f'\| = \|f\| = \|g\| = \|g'\|$. Therefore, $f' = g'$ and so $f = g$. Q.E.D.

We now prove the existence of a polar decomposition.

**Proposition 5.** Let $\mathcal{A}$ be a C*-algebra whose center $\mathcal{Z}$ is an AW*-algebra. Suppose that $\mathcal{A}$ is the dual of a normed $\mathcal{Z}$-module $M$. Then given $\phi$ in $M$, there is a partial isometric operator $U$ in $\mathcal{A}$ such that $\theta = U \cdot \phi$ is a positive functional of the module $\mathcal{A}$ and such that the functional $U^* \cdot \theta$ is equal to $\phi$.

**Proof.** Let $\mathcal{A}_1$ be the unit sphere of $\mathcal{A}$. For each $\phi$ in $M$ let

$$S(\phi) = \{ |\phi(A)| \mid A \in \mathcal{A}_1 \}.$$

If $|\phi(A_1)|$ and $|\phi(A_2)|$ are in $S(\phi)$, there are partial isometric operators $V_1$ and $V_2$ in $\mathcal{Z}$ such that $V_i \phi(A_i) = |\phi(A_i)|$ $(i = 1, 2)$. There is a projection $P$ in $\mathcal{Z}$ such that

$$\text{lub} \{ |\phi(A_1)|, |\phi(A_2)| \} = P |\phi(A_1)| + (1 - P) |\phi(A_2)| = \phi(PV_1 A_1 + (1 - P)V_2 A_2) = |\phi(PV_1 A_1 + (1 - P)V_2 A_2)|.$$

This proves that $S(\phi)$ is monotonically increasing in $\mathcal{Z}$. Since $\mathcal{Z}$ is an AW*-algebra and since $S(\phi)$ is bounded above by $\|\phi\|$, the set $S(\phi)$ has a least upper bound $|\phi|$. Actually, we have that $\|\phi\| = \||\phi||$ for given $e > 0$, there is an $A$ in $\mathcal{A}_1$ such that

$$\|\phi\| - e \leq \|\phi(A)\| \leq \||\phi(A)|| \leq \|\phi\|.$$ 

Since $e > 0$ is arbitrary we have that $\|\phi\| = \||\phi||$. Now it is clear from the definition of $|\phi|$ that $|\phi|$ is a $\mathcal{Z}$-valued seminorm on $M$, i.e. $|\phi|$ is a map of $M$ into $\mathcal{Z}^*$ such that

$$|\phi + \psi| \leq |\phi| + |\psi| \quad \text{and} \quad |C\phi| = |C| |\phi|$$ 

for every $\phi, \psi$ in $M$ and $C$ in $\mathcal{Z}$.

Let $\phi$ be a given element in $M$. By considering $M$ as a module over the hermitian elements $H(\mathcal{Z})$ of $\mathcal{Z}$, we can construct, by using the generalized Hahn-Banach Theorem [17], an $H(\mathcal{Z})$-module homomorphism $F$ of $M$ into $H(\mathcal{Z})$ such that

1. $F(\phi) = |\phi|$,  
2. $F(\phi) \leq |\phi|$ for every $\phi$ in $M$, and such that
3. $\alpha F_1 + (1 - \alpha) F_2 = F$ implies $F_1 = F_2 = F$

whenever $F_1$ and $F_2$ are $H(\mathcal{Z})$-module homomorphisms satisfying (1) and (2) and $\alpha$ is a real number between 0 and 1. Setting $G(\phi) = F(\phi) - iF(\phi)$ for every $\phi$.
in \( M \), we obtain a \( \mathcal{Z} \)-module homomorphism of \( M \) into \( \mathcal{Z} \). For every \( \psi \) in \( M \) there is a partial isometric operator \( V \) in \( \mathcal{Z} \) such that \( V G(\psi) = |G(\psi)| \). Thus we have that \( |G(\psi)| = G(V \psi) = F(V \psi) \leq |V \psi| \leq |\psi| \). Since \( \|\psi\| = \|\psi\| \) for every \( \psi \) in \( M \), the functional \( G \) is an element of \( M^\sim \); and consequently there is an element \( U \) in \( \mathcal{A} \) such that \( G(\psi) = \psi(U) \) for every \( \psi \) in \( M \). In particular we have that \( \phi(U) = |\psi| \).

Since \( \|G\| \leq 1 \), we have that \( \|U\| \leq 1 \). Let \( \theta \) be the functional in \( \mathcal{A}^\sim \) defined by \( \theta(A) = \phi(UA) \) for every \( A \) in \( \mathcal{A} \). The functional \( \theta \) is positive since

\[
\|P\theta\| \leq \|P\phi\| = \|P\phi\| = \|P\psi(U)\| = \|P\theta(1)\| \leq \|P\theta\|
\]

for every projection \( P \) in \( \mathcal{Z} \). We show that \( U \) is an extreme point of \( \mathcal{A} \). Indeed, if there are \( A_1 \) and \( A_2 \) in \( \mathcal{A} \) and \( 0 < \alpha < 1 \) that satisfy \( \alpha A_1 + (1-\alpha)A_2 = U \), then

\[
\alpha \psi(A_1) + (1-\alpha)\psi(A_2) = \psi(U)
\]

for every \( \psi \) in \( M \). Setting \( F_j(\psi) = (\psi(A_j) + \psi(A_j)^*)/2 \) \( (j = 1, 2) \), we obtain an \( H(\mathcal{Z}) \)-module homomorphism of \( M \) into \( H(\mathcal{Z}) \). We have that \( F_j(\psi) \leq |\psi(A_j)| \leq |\psi| \) for each \( \psi \) in \( M \). Also

\[
\alpha F_1(\psi) + (1-\alpha)F_2(\psi) = F(\phi) = |\phi|.
\]

So \( F_1(\phi) = F_2(\phi) = |\phi| \). Since \( F \) is an extreme point (relation (3)), we have that \( F_1 = F_2 = F \). Then

\[
(\psi(A_j) + \psi(A_j)^*)/2 = F(\psi)
\]

and

\[
(i\psi(A_j) + (i\psi(A_j))^*)/2 = F(i\psi)
\]

for every \( \psi \) implies \( \psi(A_j) = F(\psi) - iF(i\psi) = \psi(U) \) for every \( \psi \) in \( M \). This means that \( A_1 = A_2 = U \). Hence \( U \) is an extreme point of \( \mathcal{A} \). Therefore, \( U \) is a partial isometric operator in \( \mathcal{A} \) [6].

We complete the proof by showing that \( \theta(U^*A) = \theta(A) \) for every \( A \) in \( \mathcal{A} \). For \( \zeta \) in the spectrum of \( \mathcal{Z} \) and \( \psi \) in \( \mathcal{A}^\sim \) let \( \psi_\zeta \) be the bounded linear functional on \( \mathcal{A} \) defined by \( \psi_\zeta(A) = \psi(A)^*(\zeta) \); for \( B \) in \( \mathcal{A} \) let \( B \cdot \psi_\zeta \) be defined by \( B \cdot \psi_\zeta(A) = \psi(BA) \). Notice that \( \|B \cdot \psi_\zeta\| \leq \|B\| \|\psi_\zeta\| \). We have that \( \|\psi_\zeta\| \leq \text{glb} \{\|P\phi\| \ P \text{ a projection in } \mathcal{Z} \text{ with } \psi_\zeta(P) = 1\} \leq \text{glb} \{|\psi_\zeta| \ |\psi_\zeta| = \|\psi_\zeta\| \}. \) Indeed, given \( \varepsilon > 0 \) there is a projection \( P \) in \( \mathcal{Z} \) with \( \psi_\zeta(P) = 1 \) and \( \|P\phi(U)\| \leq |\psi(U)^*(\zeta)| + \varepsilon \). Thus \( \|\theta_\zeta\| = \|\phi_\zeta\| \). However, we also have that

\[
\|\theta_\zeta\| = \|(UU^*U) \cdot \phi_\zeta\| \leq \|(UU^*) \cdot \phi_\zeta\| \leq \|\phi_\zeta\| = \|\theta_\zeta\|.
\]

By Lemma 4 we conclude that \( (UU^*) \cdot \phi_\zeta = \phi_\zeta \). Since \( \zeta \) is arbitrary we have that \( \phi(A) = \theta(U^*A) \) for all \( A \) in \( \mathcal{A} \). Q.E.D.

Let \( \mathcal{A} \) be a \( C^* \)-algebra whose center \( \mathcal{Z} \) is an \( AW^* \)-algebra. Let \( \phi \) be a positive functional in \( \mathcal{A}^\sim \). There is a set \( \{\mathcal{C}_i\} \) of positive elements in \( \mathcal{Z} \) and a set \( \{P_i\} \) of mutually orthogonal projections in \( \mathcal{Z} \) of sum \( P \) such that

\[
P_i \mathcal{C}_i(1) = P_i \text{ and } (1-P_i) \phi(1) = 0.
\]
Then setting $\phi(A) = \sum C_i P_i \phi(A)$ for $A$ in $\mathcal{A}$, we obtain a positive functional $\phi$ of the module $\mathcal{A}$ such that $\phi(1) = 1$. Due to the general Hahn-Banach theorem there is a positive functional $\phi_0$ of the module $\mathcal{A}$ such that $\phi_0(1) = 1$. So every positive functional in $\mathcal{A}^*$ can be decomposed into the product of a state of the module $\mathcal{A}$ (i.e. a positive functional taking 1 into 1) and an element in $\mathcal{Z}^+ [11], [15].$

Let $\phi$ be a positive functional in $\mathcal{A}^*$ such that $\phi(1)$ is a projection. Let $L_\phi$ be the left ideal defined by $L_\phi = \{ A \in \mathcal{A} \mid \phi(A^*A) = 0 \}$ and let $\mathcal{A}/L_\phi$ be the $\mathcal{A}$-module reduced modulo $L_\phi$. Setting $(A-L_\phi, B-L_\phi) = \phi(B^*A)$ for $A$ and $B$ in $\mathcal{A}$, we introduce a $\mathcal{Z}$-valued hermitian form on $\mathcal{A}/L_\phi$ and then using this form and the norm of $\mathcal{Z}$, we introduce a norm on $\mathcal{A}/L_\phi$. Let $H_\phi$ be the set of all pairs $((x_i), (P_i)) = x$ where $\{x_i\}$ is a bounded subset of $\mathcal{A}/L_\phi$ and $\{P_i\}$ is a set of mutually orthogonal projections in $\mathcal{Z}$ of sum 1. If $y = ((y_i), (Q_i)) = y_0$ then set $y = x$ if and only if $y_i Q_i P_i = x_i Q_i P_i$ for all $i$ and $j$. The hermitian form on $\mathcal{A}/L_\phi$ has a unique extension to $H_\phi$. The completion $H_\phi$ of $H_\phi$ in the norm induced by the hermitian form is an $\text{AW}^*$-module over $\mathcal{Z}$ with inner product induced by the hermitian form on $H_\phi$. Actually, the module $H_\phi$ is not faithful over $\mathcal{Z}$ but it is faithful over $\mathcal{Z}^\phi(1)$. The representation of $\mathcal{A}$ on $\mathcal{A}/L_\phi$ by left multiplication has a unique extension to a representation $\pi_\phi$ of $\mathcal{A}$ as bounded linear operators on $H_\phi$. The map $\pi_\phi$ is seen to be a module homomorphism as well as a $*$-algebra homomorphism. This map is called the canonical representation induced by $\phi$ of $\mathcal{A}$ on $H_\phi$ [18, §§2–3].

Now let $\mathcal{A}$ be an $\text{AW}^*$-algebra with center $\mathcal{Z}$. Suppose $\mathcal{A}$ is a subalgebra of the algebra $L(H)$ of all bounded linear operators on an $\text{AW}^*$-module $H$ over $\mathcal{Z}$. Let $\{A_i\}$ be a bounded subset of $\mathcal{A}$ and let $\{P_i\}$ be a set of orthogonal projections in $\mathcal{Z}$ of least upper bound 1. It is immaterial whether $\mathcal{Z}$ is considered as a subalgebra of $\mathcal{A}$ or of $L(H)$ in order to evaluate this least upper bound. Then there is a unique $A$ in $\mathcal{A}$ (respectively $B$ in $L(H)$) such that $P_i A = A_i P_i$ (respectively, $P_i B = A_i P_i$) for each $P_i$. This means that $A = B$. This remark plus I. Kaplansky's matrix method for passing from the hermitian to the nonhermitian case ([7]; also cf. [1, I, §3, Theorem 3]) gives the following version of H. Widom's lemma [18, Lemma 4.2].

**Lemma.** Let $H$ be an $\text{AW}^*$-module over the commutative $\text{AW}^*$-algebra $\mathcal{Z}$. Let $\mathcal{A}$ be an $\text{AW}^*$-algebra with center $\mathcal{Z}$ and let $\mathcal{A}$ be a subalgebra of $L(H)$. Given any $B$ in the double commutator of $\mathcal{A}$ on $H$, any $x_1, x_2, \ldots, x_n$ in $H$, and any $\epsilon > 0$, there is an $A$ in $\mathcal{A}$ whose norm is majorized by that of $B$ such that $\|(A - B)x_i\| < \epsilon$ for every $i = 1, 2, \ldots, n$.

**Proposition 6.** Let $\mathcal{A}$ be a $C^*$-algebra whose center $\mathcal{Z}$ is an $\text{AW}^*$-algebra. Suppose $\mathcal{A}$ is the dual of a normed $\mathcal{Z}$-module $M$. Then $\mathcal{A}$ is an $\text{AW}^*$-algebra. Furthermore, let $N$ be the smallest $\mathcal{Z}$-module in $\mathcal{A}^*$ which contains $M$ and is closed under
left and right multiplication by elements of \( \mathcal{A} \). Then the module \( \mathcal{A} \) is the dual of the module \( N \).

**Proof.** Let \( S \) be the set of all states in \( \mathcal{A} \). For each \( \phi \in S \), let \( \pi_\phi \) be the canonical representation of \( \mathcal{A} \) on the \( AW*-\)module \( H_\phi \) over \( L \) which is induced by \( \phi \). Let \( H = \sum \oplus H_\phi \) and let \( \pi = \sum \oplus \pi_\phi \) [10, §5]. Then \( \pi \) is a \( L \)-linear, norm-decreasing, \(*\)-homomorphism of the algebra \( \mathcal{A} \) into \( L(H) \). Now, we have that

\[
\| A \| = \text{lub} \{ \| \phi(A) \| : \phi \in M, \| \phi \| \leq 1 \}.
\]

Let \( \epsilon > 0 \) be given; there is a \( \phi \) in the unit sphere of \( M \) such that \( \| \phi(A) \| \geq \| A \| - \epsilon \). There is a partial isometry \( V \) in \( \mathcal{A} \) such that \( V \cdot \phi \) is a positive functional on the module \( \mathcal{A} \) and \( (VV^*) \cdot \phi = \phi \) (Proposition 5). Then we have that \( \| V \cdot \phi \| = \| \phi \| \). There is a \( C \) in \( L^+ \) and a state \( \psi \) on the module \( \mathcal{A} \) such that \( C \psi = V \cdot \phi \). Then \( \| C \| = \| \phi(V) \| = \| \phi \| \leq 1 \). We have that \( V - L_\psi \) has norm not exceeding one in \( H_\psi \). Thus

\[
\| \pi_\phi(A)(1 - L_\psi), V - L_\psi \| = \| \psi(V^*A) \| \geq \| \phi(A) \| \geq \| A \| - \epsilon.
\]

Since \( \epsilon > 0 \) is arbitrary we have that \( \| \pi(A) \| = \| A \| \). So \( \pi \) is an isometric isomorphism of \( \mathcal{A} \) into a \(*\)-subalgebra of \( L(H) \).

We show that the double commutator \( \mathfrak{B} \) of \( \pi(\mathcal{A}) \) on \( H \) is isometrically isomorphic to the second dual \( \mathcal{A}^{**} \) of the module \( \mathcal{A} \). Let \( \phi \in \mathcal{A} \); then \( \phi \) may be written as a linear combination of four positive functionals \( \phi_i \) (1 \( \leq i \leq 4 \)) of the module \( \mathcal{A} \) [11], [15]. There are positive elements \( C_i \) (1 \( \leq i \leq 4 \)) in \( \mathcal{A} \) and states \( \psi_i \) (1 \( \leq i \leq 4 \)) of the module \( \mathcal{A} \) such that \( C_i \psi_i = \phi_i \) (1 \( \leq i \leq 4 \)). There are \( x_i \) (1 \( \leq i \leq 4 \)) in \( H \) such that

\[
\psi_i(A) = \langle \pi(A)x_i, x_i \rangle \quad (1 \leq i \leq 4)
\]

for every \( A \in \mathcal{A} \). Thus there is an element \( \phi' \) in \( \mathfrak{B}_- \) such that \( \phi'(\pi(A)) = \phi(A) \) for every \( A \in \mathcal{A} \). Because \( \pi(\mathcal{A}) \) is weakly dense in \( \mathfrak{B} \) (H. Widom's lemma), there is only one functional \( \phi' \) in \( \mathfrak{B}_- \) such that \( \phi'(\pi(A)) = \phi(A) \) for every \( A \in \mathcal{A} \). This proves that the relation \( \phi \rightarrow \phi' \) is a function of \( \mathcal{A}^- \) into \( \mathfrak{B}_- \). It is easily seen to be \( L \)-linear. For each \( \psi \in \mathfrak{B}_- \) the relation \( \phi(A) = \psi(\pi(A)) \) defines a bounded functional \( \phi \) of the module \( \mathcal{A} \) such that \( \phi' = \psi \). So the map \( \phi \rightarrow \phi' \) is onto \( \mathfrak{B}_- \). Furthermore, for each \( \phi \) in \( \mathcal{A}^- \) we have that

\[
\| \phi \| = \text{lub} \{ \| \phi(A) \| : \| A \| \leq 1 \} = \text{lub} \{ \| \phi'(A) \| : A \in \pi(\mathcal{A}), \| A \| \leq 1 \} = \| \phi' \|
\]

since the unit sphere of \( \pi(\mathcal{A}) \) is weakly dense in the unit spheres of \( \mathfrak{B} \) (H. Widom's lemma). This proves that \( \mathcal{A}^- \) is isometrically isomorphic with \( \mathfrak{B}_- \) and thus that \( \mathcal{A}^{**} \) is isometrically isomorphic with \( \mathfrak{B} \) (Remark, Theorem 2).

Let \( \rho \) be the transpose of the identity map of \( M \) into \( \mathcal{A} \), i.e. let \( \rho \) be the map of \( \mathfrak{B} \) into \( \mathcal{A} \) given by \( \phi(\rho(A)) = \phi'(A) \) for every \( A \in \mathfrak{B} \) and \( \phi \) in \( M \). Then we have that

\[
\phi(A) = \phi'(\pi(A)) = \phi(\rho(\pi(A)))
\]

for every \( \phi \) in \( M \) and \( A \) in \( \mathcal{A} \). This means that \( \rho(\pi(A)) = A \) and that \( \pi \cdot \rho(\pi(A)) \)

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\[ \eta(A) = \rho(A) \quad \text{for every } A \in \mathcal{A}. \] Therefore the map \( \eta = \pi \cdot \rho \) is a projection of \( \mathcal{B} \) onto \( \pi(\mathcal{A}) \). We have that
\[
\|\eta(A)\| = \|\rho(A)\| = \text{lub} \{\|\phi(\rho(A))\| \mid \phi \in M, \|\phi\| \leq 1\}
\leq \text{lub} \{\|\phi'(A)\| \mid \phi \in \mathcal{A}^*, \|\phi'\| \leq 1\} \leq \|A\|
\]
for every \( A \in \mathcal{B} \). Thus the function \( \eta \) is a projection of norm 1. This proves that \( \mathcal{A} \) is an \( AW^* \)-algebra due to a result of Tomiyama [16, Theorem 5]. Also following Tomiyama, we can show that the kernel \( K \) of \( \eta \) is an ideal in \( \mathcal{B} \). Indeed, if \( A \) and \( C \) are in \( \pi(\mathcal{A}) \) and if \( \eta(B) = 0 \), then \( \eta(ABC) = A \eta(B)C = 0 \). Now if \( A \) is in \( \mathcal{B} \), then \( A \) is the weak limit of a net \( \{A_n\} \) in \( \pi(\mathcal{A}) \). This means that \[ \eta(A) = \rho(A) = \lim \phi(A_nB) = \lim \phi(A_n\eta(B)) = 0 \]
for every \( \phi \in M \). This proves that \( \rho(AB) = 0 \) and that \( \eta(AB) = 0 \), and therefore that \( \eta \) is the left ideal. Similarly, we obtain that \( K \) is a right ideal and therefore that \( K \) is a two-sided ideal. By the same reasoning we see that \( K \) is weakly closed. Let \( \{E_n \mid n \in D\} \) be a maximal set of mutually orthogonal nonzero projections in \( K \). Let \( F \) be the family of finite subsets of \( D \). For each \( s \) in \( F \) let \( E_s = \sum \{E_n \mid n \in s\} \). Let \( E \) be the least upper bound of \( \{E_s \} \) in \( \mathcal{B} \). Now given an element \( \phi \) in \( M \), a nonzero projection \( P \) in \( \mathcal{B}^* \), and an \( \epsilon > 0 \), there is an \( s_0 \) in \( F \) and a nonzero projection \( Q \) in \( \mathcal{B}^P \) such that \( \|\phi'(E_s - E)Q\| \leq \epsilon \) whenever \( s \supseteq s_0 \) ([3, Lemma 4.2] and [18, Lemma 1.4]). Since
\[ \phi'(E_s) = \sum \{\phi'(\eta(E_n)) \mid n \in s\} = 0, \]
we have that \( \|\phi'(E)Q\| \leq \epsilon \). Let \( \{Q_n\} \) be a maximal set of mutually orthogonal nonzero projections in \( \mathcal{B}^* \) such that \( \|\phi'(E)Q_n\| \leq \epsilon \) for every \( Q_n \). It is evident that \( \sum Q_n = 1 \) and hence that \( \|\phi'(E)\| \leq \epsilon \). Since \( \epsilon > 0 \) is arbitrary we see that \( \phi'(\rho(E)) = \phi'(E) = 0 \). Since \( \phi \) is arbitrary, we have that \( \rho(E) = 0 \); and therefore, we have that \( E \in K \). Because \( K \) is generated in the uniform topology by its projections, we have that \( AE = EA = A \) for every \( A \) in \( K \). This means that \( E \) is a projection in the center of \( \mathcal{B} \) and that \( \mathcal{B} E = K \). This proves that \( \eta \) is an isomorphism of the algebra \( \mathcal{B}(1 - E) \) onto the algebra \( \pi(\mathcal{A}) \). The map \( \eta \) is also a module isomorphism.

Let \( N \) be the smallest \( \mathcal{B} \)-module in \( \mathcal{A}^- \) which contains \( M \) and is closed under right and left multiplication by elements of \( \mathcal{A} \). We show that \( N^- \) is isometric isomorphic to \( \mathcal{A} \). Let \( \Phi \) be a bounded functional of the module \( N \). There is a functional \( \Psi \) of the module \( \mathcal{A}^- \) such that \( \Psi(\phi) = \Phi(\phi) \) for all \( \phi \) in \( N \) and such that \( \|\Psi\| = \|\Phi\| \) [11], [15]. There is an element \( B \) in \( \mathcal{B} \) such that \( \Psi(\phi) = \phi'(B) \) for all \( \phi \in \mathcal{A}^- \). If \( \Phi \in M \) and if \( A \in \mathcal{A} \), we have that
\[ \Phi(A \cdot \phi) = \phi'(\rho(AB)) = \phi'(\eta(B)) = \phi'(\rho(A\eta(B))) = \phi(A \rho(B)) = (A \cdot \phi)(\rho(B)).\]
Similarly, we have that \( \Phi(\phi \cdot A) = (\phi \cdot A)(\rho(B)) \). So there is a \( B_\phi \) in \( \mathcal{A} \) such that \( \Phi(\phi) = \phi(B_\phi) \) for every \( \phi \) in \( N \). If \( B_\phi \in \mathcal{A} \) and if \( \Phi(\phi) = \phi(B_\phi) \) for every \( \phi \) in \( N \), then \( B_\phi \) is equal to \( B_\phi' \). Hence, there is a unique \( B_\phi \) in \( \mathcal{A} \) such that \( \Phi(\phi) = \phi(B_\phi) \) for every \( \phi \in N \). The function \( \Phi \rightarrow B_\phi \) is obviously a module isomorphism of the module \( N^- \) onto the module \( \mathcal{A} \). Finally we have that

\[
\|B_\phi\| = \text{lub}\{\|\phi(B_\phi)\| \mid \phi \in M, \quad \|\phi\| \leq 1\}
\leq \text{lub}\{\|\phi(B_\phi)\| \mid \phi \in N, \quad \|\phi\| \leq 1\} = \|\Phi\| \leq \|B_\phi\|.
\]

So \( \Phi \rightarrow B_\phi \) is an isometric isomorphism of \( N^- \) onto \( \mathcal{A} \). Q.E.D.

**Theorem 7.** Let \( \mathcal{A} \) be a \( C^* \)-algebra whose center is an \( AW^* \)-algebra \( \mathcal{Z} \). Suppose \( \mathcal{A} \) is the dual of a \( \mathcal{Z} \)-module \( M \). Let \( N' \) be the smallest \( \mathcal{Z} \)-module in the dual of the module \( \mathcal{A} \) which contains \( M \) and is closed under left and right multiplication by elements of \( \mathcal{A} \), and let \( N \) be the module generated by \( N' \) in \( \mathcal{A}^- \). Then \( \mathcal{A} \) may be embedded as a double commutator in the algebra of all bounded linear operators on an \( AW^* \)-module over \( \mathcal{Z} \) so that the weak topology and the \( \sigma(\mathcal{A}, N) \)-topology coincide on the unit sphere of \( \mathcal{A} \).

**Proof.** Let \( S \) be the set of all positive functionals \( \phi \) in \( N \) such that \( \phi(1) \) is a projection in \( \mathcal{Z} \). For each \( \phi \) in \( S \) let \( \pi_\phi \) be the canonical representation of \( \mathcal{A} \) on the \( AW^* \)-module \( H_\phi \) over \( \mathcal{Z} \phi(1) \) which is induced by \( \phi \). Then \( H_\phi \) may be considered as an \( AW^* \)-module over \( \mathcal{Z} \). Let \( H = \sum \oplus \{ H_\phi \mid \phi \in S \} \) and let \( \pi = \sum \oplus \{ \pi_\phi \mid \phi \in S \} \).

The \( AW^* \)-module \( H \) is a faithful \( AW^* \)-module over \( \mathcal{Z} \). Indeed, if \( P \) is a nonzero projection in \( \mathcal{Z} \), then

\[
\text{lub}\{\|\phi(P)\| \mid \phi \in M, \quad \|\phi\| \leq 1\} = 1.
\]

So \( \|\phi(P)\| \neq 0 \) for some \( \phi \) in the unit sphere of \( M \). Let \( V \) be a partial isometry in \( \mathcal{A} \) such that \( V \cdot \phi \) is a positive functional and such that \( VV^* \cdot \phi = \phi \) (Proposition 5). Then \( (V \cdot \phi)(P) \neq 0 \) because \( |\phi(P)|^2 = |\phi(VV^*P)|^2 \leq \phi(V)\phi(VP) \). There is a \( C \) in \( \mathcal{Z}^+ \) such that \( \phi(CV) \) is a nonzero projection in \( \mathcal{Z} \) majorized by \( P \). Setting \( \psi = CV \cdot \phi \), we obtain an element \( \psi \) in \( S \) such that \( P(1 - L_\psi) \neq 0 \). Hence \( H \) is a faithful \( AW^* \)-module over \( \mathcal{Z} \).

We show that the map \( \pi \) is an isometry. Let \( A \) be a nonzero positive element in \( \mathcal{A} \). It is enough to show that \( \|\pi(A)\| = \|A\| \). Let \( \epsilon > 0 \) be an arbitrary number less than \( \|A\| \). There is a \( \phi \) in the unit sphere of \( N \) such that \( \|\phi(A)\| > \|A\| - \epsilon \) and thus there is a nonzero projection \( P \) in \( \mathcal{Z} \) such that

\[
|P\phi(A)| \geq (\|A\| - \epsilon)P.
\]

There is a partial isometry \( V \) in \( \mathcal{A} \) such that \( V \cdot \phi \) is a positive functional and such that \( VV^* \cdot \phi = \phi \) (Proposition 5). Then we have that

\[
(\|A\| - \epsilon)^2P \leq |\phi(A)|^2 \leq (\|A\| - \epsilon)^2P.
\]

So there is a positive element \( C \) in \( \mathcal{Z} \) such that \( CV \cdot \phi = \psi \) is in \( S \) and such that
Since \( PV \cdot \phi(1) \leq P \), we see that \( CP \geq P \). Hence, we have that

\[
\begin{align*}

\| A - e \|^2 P & \leq \| P \phi(V) \phi(V^A) \| \leq \| A \| \| P \phi(V) \phi(V^A) \| \\
& \leq \| A \| \| P \phi(1) \phi(A) \| \leq \| A \| \| \phi(A) \|.
\end{align*}
\]

This proves that \( \text{lub} \{ \| \phi(A) \| : \phi \in \mathcal{S} \} = \| A \| \) and that \( \| \pi(A) \| = \| A \| \).

We show that \( \pi(\mathcal{A}) \) is equal to its double commutator \( \mathcal{B} \) on \( H \). Let \( B \) be an element in \( \mathcal{B} \). There is a net \( \{ A_n \} \) in the sphere of \( \mathcal{A} \) about the origin of radius \( \| B \| \) such that \( \lim \pi(A_n) = B \) weakly in \( L(H) \) because \( \pi(\mathcal{A}) \) is an \( AW^* \)-algebra with center \( \mathcal{Z} \) (Proposition 6) and thus Widom's lemma may be employed. Let \( \phi \in N \) and let \( V \cdot \phi \) be the polar decomposition of \( \phi \). There is a sequence \( \{ P_m \} \) of orthogonal projections in \( \mathcal{Z} \) of sum \( P \) such that \( P_m \phi(V) \) has inverse \( C_m \) in \( \mathcal{Z} P_m \) and such that \( (1 - P) \phi(V) = 0 \). By the hypothesis on \( N \), we see that \( \psi = \sum P_m (C_m V \cdot \phi) \) is in \( N \) and therefore in \( S \) and that \( \phi(V) \psi = V \cdot \phi \). Setting \( x = 1 - L \psi \), we have that

\[
\lim (\pi(A_n)x, \pi(V)x) = (Bx, \pi(V)x)
\]

uniformly in \( \mathcal{Z} \). This means that \( \{ \phi(A_n) \} \) is a Cauchy net in the uniform topology of \( \mathcal{Z} \) and therefore \( \{ \phi(A_n) \} \) converges uniformly to an element \( \Phi(\phi) \) in the sphere of radius \( \| \phi \| \| B \| \) about the origin. Hence, we see that \( \phi \rightarrow \Phi(\phi) \) defines an element \( \Phi \) in \( N^* \) and therefore we have an element \( A_0 \) in \( \mathcal{A} \) such that \( \Phi(\phi) = \phi(A_0) \) for every \( \phi \) in \( N \). Now for arbitrary \( \psi \) in \( S \) we have that \( \psi \cdot C \) is in \( N \) and therefore that

\[
(\pi(A_0)\pi(C)x, \pi(A)^*x) = \lim (A \cdot \psi \cdot C)(A_0)
= \lim (\pi(A_0)\pi(C)x, \pi(A)^*x) = (B\pi(C)x, \pi(A)^*x)
\]

where \( x = 1 - L \psi \). Therefore, we have proved that \( ((\pi(A_0) - B)y, z) = 0 \) for all \( y, z \) in \( K = \{ \pi(A)x : A \in \mathcal{A} \} \). Now given \( A \) in \( \mathcal{A} \), there is a net \( \{ C_n \} \) in \( \mathcal{A} \) such that \( \{ \pi(C_n) \} \) converges weakly to \( (\pi(A_0) - B)\pi(A) \) since \( (\pi(A_0) - B)\pi(A) \) is in \( \mathcal{B} \). Therefore,

\[
\| (\pi(A_0) - B)\pi(A) \| \leq \lim \| (\pi(A_0) - B)\pi(A) \| x, \pi(C_n)x = 0.
\]

This means that \( \pi(A_0) - B \) vanishes on \( K \) and therefore on \( H_\psi \). Since \( \psi \) is arbitrary, we conclude that \( \pi(A_0) = B \) and therefore that \( \pi(\mathcal{A}) = \mathcal{B} \).

We now identify \( \mathcal{A} \) with \( \mathcal{B} \) and we show that the \( \sigma(\mathcal{A}, N) \)-topology and the \( \sigma(\mathcal{A}, \mathcal{A}_0) \)-topology coincide on the unit sphere of \( \mathcal{A} \). For each \( \psi \in S \) let \( E_\psi \) be the projection of \( H \) on \( H_\psi \) [10, §6]. By the definition of \( H \) we have that the least upper bound of the family \( \{ E_\psi : \psi \in S \} \) is 1. Let \( x \) be in \( H \) and let \( e \) be a strictly positive number. There is a set \( \{ P_i \} \) of mutually orthogonal projections in \( \mathcal{Z} \) of sum 1 such that for each \( i \) there is a finite subset \( n(i) \) of \( S \) with

\[
P_i(|x|^2 - \sum \{ |E_\psi x|^2 : \psi \in n(i) \}) < e^2 P_i
\]

since

\[
\sum \{ |E_\psi x|^2 : \psi \in S \} = |x|^2
\]

[3, Lemma 4.2]. Let \( i \) be fixed and let \( n(i) = \{ \psi_1, \ldots, \psi_k \} \). Let \( E_{\psi_k} = E_k \) and let
There is a set \( \{Q_i\} \), not depending on \( k \), of mutually orthogonal projections in \( \mathcal{Z} \) of sum \( P \), such that for each \( k = 1, 2, \ldots, n \) there is a set \( \{A_k\} \) in \( \mathcal{A} \) with \( \{\phi_k(A_k, A_k)\} \) bounded and

\[
\left\| \sum_j Q_j A_k x_k - E_k x \right\| < \varepsilon n^{-1}.
\]

Let \( y_j = \sum_k Q_j A_k x_k \). Then we have that

\[
|Q_j(y_j - x)| \leq |Q_j(y_j - \sum E_k x)| + |Q_j(1 - \sum E_k) x| \leq \sum_k |Q_j(A_k x_k - E_k x)| + |Q_j(1 - \sum E_k) x| \leq 2\varepsilon,
\]

and

\[
|y_j| \leq |Q_j(y_j - \sum E_k x)| + |Q_j \sum E_k x| \leq (\varepsilon + (\sum E_k x)^2)^{1/2}) Q_j \leq (\varepsilon + \|x\|^2) Q_j.
\]

Then setting

\[
\phi_j(A) = (Ay_j, y_j) = Q_j \sum_k \phi_k(A_k A A_k),
\]

we obtain a positive functional in \( N \) of norm not exceeding \((\varepsilon + \|x\|)^2\). There is a unique \( y_j \) (respectively \( \theta_j \)) in \( H \) (respectively in \( N \)) such that \( Q_j y_j = y_j \) (respectively \( Q_j \theta_j = \phi_j \)) for each \( Q_j \) and \((1 - P_j) y_j = 0 \) (respectively, \((1 - P_j) \theta_j = 0 \)). We have that \( \theta_j(A) = (Ay_j, y_j) \) for each \( A \) in \( \mathcal{A} \) and that \( \|\theta_j\| \leq (\varepsilon + \|x\|)^2 \). Then we have that

\[
P_j|(Ax, x)| \leq \|A\| |P_j(x - y_j)| |P_j x|
+ \|A\| |P_j(x - y_j)| |P_j y_j| + |\theta_j(A)| \leq 2\varepsilon \|A\| \|x + 2\|x\|\| |P_j + |\theta_j(A)|.
\]

Since \( \theta_j(1) \leq (\varepsilon + \|x\|)^2 P_j \), there is a unique \( \theta \) in \( N \) such that \( P_j \theta = \theta_j \) for each \( P_j \). This means that

\[
|(Ax, x)| \leq 2\varepsilon \|A\| \|x + 2\|x\|\| + |\theta(A)|
\]

for every \( A \) in \( \mathcal{A} \). Now it becomes obvious that the \( \sigma(\mathcal{A}, N) \)-topology is finer than the weak topology on the unit sphere of \( \mathcal{A} \).

Conversely, let \( \phi \) be a functional in \( N \) and let \( U \) be a partial isometry of \( \mathcal{A} \) such that \( U \cdot \phi \) is positive and \( U U^* \cdot \phi = \phi \). There is a sequence \( \{P_n\} \) of mutually orthogonal projections in \( \mathcal{Z} \) such that \( P_n \phi(U) = C_n \) is invertible with inverse \( D_n \) in \( \mathcal{Z} P_n \) and such that \((1 - \sum P_n) \phi(U) = 0 \). Then \( \sum D_n U \cdot \phi = \psi \) is in \( N \). Since \( \phi(1) = \sum P_n \), the functional \( \psi \) is in \( S \). We then have that \( \phi(A) = (Ax, y) \) where \( x = \sum C_n (1 - L_n) \) and \( y = U(1 - L_n) \). Thus a net \( \{A_n\} \) in the unit sphere of \( \mathcal{A} \) converges to \( A \) in the \( \sigma(\mathcal{A}, N) \)-topology whenever \( \{A_n\} \) converges to \( A \) in the \( \sigma(\mathcal{A}, \mathcal{A}) \)-topology. Q.E.D.

Remark. In the notation of Theorem 7 we have that the closure of \( N \) in the uniform topology of \( \mathcal{A}^* \) is equal to the closure of \( \mathcal{A} \) in \( \mathcal{A}^* \) and that \( \mathcal{A} \) is the dual of the closure of the module \( N \).
Bibliography


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