TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 148, March 1970

AN ASYMPTOTIC PROPERTY OF GAUSSIAN PROCESSES. I[1]

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1. **Introduction.** The study of the asymptotic properties of stochastic processes has a long history. Such researches were developed from those of the asymptotic behavior of the first n of a sequence of independent random variables, namely, the so-called law of the iterated logarithm. A. Kolmogoroff proposed the final form of the very theory which is stated, without proof, in P. Lévy's book [4]. W. Feller [3] gave its complete proof. The final result on the Brownian motion corresponding to Feller's for the case of the partial sums of independent random variables were led by T. Sirao and T. Nisida [9]. Their result is stated below. We introduce the following notation. Let $M_a^+ =\{\phi;\ \phi$ is a positive, nondecreasing, real function on $[a, \infty)\}$. Let $\{B(t); 0<t<\infty\}$ a Brownian motion on a probability measure space (Ω, B, P). We put $v(t)=E(E(B(t)^2))^{1/2}=t^{1/2}$.

THEOREM A.

P(there is a $t_0(\omega)$ such that $|B(t)| \leqq v(t)\phi(t)$ for all $t > t_0(\omega)) = 1$ or 0,

according as, for some $a>0$,

$$\int_a^\infty (1/t)\phi(t) \exp\left(-\tfrac{1}{2}\phi^2(t)\right) dt, \qquad \phi \in M_a^+,$$

converges or diverges, respectively.

It will be possible to generalize in several ways Theorem A which is true for the Brownian motion. In this paper, by use of the method of T. Sirao [8], we will give some results on a generalization of Theorem A.

2. **Results.** Let $\{x(t),\ -\infty < t < \infty\}$ be a real, separable, measurable Gaussian process defined on a probability measure space $(\Omega, \mathscr{B}, P)$. Without loss of generality, we may assume that $E(x(t))=0$. We put $r(t, s)=E(x(t)x(s))$ and $E(x^2(t))=v^2(t)$. In the following, we will assume that $r(t, s)$ is continuous with respect to t and s and $v(t)$ is positive. And we put $\rho(t, s)=r(t, s)/(v(t)v(s))$.

In the following, we will obtain some results on the asymptotic behavior of the process $x(t)$ as t tends to infinity. To state the results, we introduce the following conditions.

Received by the editors April 1, 1969.

[1] Some results of this paper were previously announced in H. Watanabe [11], and an error found later in them which is corrected in this paper.

(A.1) There are positive constants δ_1, C_1 and T_1 such that there holds

$$\rho(t, t+h) \geqq 1-C_1h^{\alpha},$$

for all h in $(0, \delta_1)$ and all $t>T_1$ for some α with $0<\alpha\leqq 2$.

(A.1′) There are positive constants C_2, T_2 and η $(0<\eta<1)$ such that

$$\rho(t, t+h) \leqq \max(1-C_2h^{\alpha}, \eta)$$

for all $t>T_2$ for some α with $0<\alpha\leqq 2$.

(A.2) The limit

$$\lim_{s\to\infty} \rho(t, t+s)\cdot s = 0,$$

converges uniformly with respect to t.

THEOREM 1. *Suppose that condition* (A.1) *is satisfied. If* $\phi(t) \in M_a^+$ *and*

$$\int_a^{\infty} \phi(t)^{2/\alpha-1} \exp(-\tfrac{1}{2}\phi^2(t))\, dt < \infty,$$

for some $a>0$, *then we have*

$$P(\exists t_0(\omega), x(t) \leqq v(t)\phi(t) \textit{ for all } t \geqq t_0(\omega)) = 1.$$

THEOREM 2. *Suppose that condition* (A.1′) *and* (A.2) *are satisfied. If* $\phi(t) \in M_a^+$ *and*

$$\int_a^{\infty} \phi(t)^{2/\alpha-1} \exp(-\tfrac{1}{2}\phi^2(t))\, dt = \infty$$

for some $a>0$, *then we have*

$$P(x(t) > v(t)\phi(t) \textit{ i.o.}) = 1,$$

where i.o. means infinitely often.

Combining Theorems 1 and 2, we can easily show the following theorem.

THEOREM 3. *Assume that conditions* (A.1), (A.1′) *and* (A.2) *are satisfied at the same time for some* α. *Let* $\phi(t) \in M_a^+$ *for some* a. *Then*

$$P(\exists t_0(\omega), x(t) \leqq v(t)\phi(t) \textit{ for all } t \geqq t_0(\omega)) = 1 \textit{ or } 0,$$

according as the integral,

$$\int_a^{\infty} \phi(t)^{2/\alpha-1} \exp(-\tfrac{1}{2}\phi^2(t))\, dt$$

converges or diverges.

COROLLARY 1. *Under the same conditions in Theorem* 3, *we have, for every* $\varepsilon>0$, $P(\exists t_0(\omega), x(t)\leqq v(t)(2\log t+(2/\alpha+1+\varepsilon)\log\log t)^{1/2}$ *for all* $t>t_0(\omega))=1$. *Moreover, we have for any* $\varepsilon\geqq 0$,

$$P(x(t) > v(t)(2\log t+(2/\alpha+1-\varepsilon)\log\log t)^{1/2} \textit{ i.o.}) = 1.$$

From Corollary 1, assuming condition (A.1), it follows that, for every $\varepsilon > 0$,

$$P(\exists t_0(\omega),\, x(t) < v(t)((2\log t)^{1/2} + (1/\alpha + 1/2 + \varepsilon)\log\log t/(2\log t)^{1/2}) \quad \text{for all } t > t_0(\omega)) = 1.$$

Under the condition (A.1), the path functions $x(t)$ are almost certainly everywhere continuous. Then, for every fixed $t > 0$, the quantity

$$\eta(t) = \max_{0 \leqq u \leqq t} \frac{x(u)}{v(u)}$$

will have a definite meaning. Given any positive, monotone, nondecreasing continuous function $\phi(t)$ for large t, there is a $t_0(\omega)$ such that $x(t) < v(t)\phi(t)$ for all $t \geqq t_0$, if and only if there is a $t_0(\omega)$ such that $\eta(t) \leqq \phi(t)$ for all $t \geqq t_0$. Hence, assuming condition (A.1), for every $\varepsilon > 0$,

$$P(\exists t_0(\omega),\, \eta(t) \leqq (2\log t)^{1/2} + (1/\alpha + 1/2 + \varepsilon)\log\log t/(2\log t)^{1/2} \quad \text{for all } t \geqq t_0(\omega)) = 1. \tag{2.1}$$

H. Cramér [2] and M. G. Šur [7] have obtained the results corresponding to (2.1) in the case of $\alpha = 2$ when $x(t)$ is stationary.

In this case, assuming conditions (A.1), (A.1′) and (A.2) and denoting $E(x(t)x(0))/(v(0))^2$ by $\rho(t)$, it follows that

$$\exists\alpha\ 2 \geqq \alpha > 0,\ \limsup_{t\to\infty} t^{-\alpha}(1-\rho(t)) < \infty, \tag{B.1}$$

$$\lim_{t\to\infty} \rho(t)t = 0. \tag{B.2}$$

Therefore the assumptions of Theorem 5.4 of J. Pickands III [6] are satisfied. Hence

$$P(\eta(t) - (2\log t)^{1/2} \to 0\ (t \to \infty)) = 1.$$

They are interested in the function ϕ_1 and ϕ_2 such that it holds

$$P(\exists t_0(\omega),\, \phi_1(t) \leqq \eta(t) \leqq \phi_2(t), \text{ for all } t \geqq t_0(\omega)) = 1.$$

We have only proposed some criteria concerning as $\phi_2(\cdot)$. Also, we remark that M. Nisio [5] proved under the weaker condition than (A.1), (A.1′) and (A.2) that

$$P\left(\lim_{t\to\infty} \frac{\eta(t)}{(2\log t)^{1/2}} = 1\right) = 1.$$

And also, we can easily show that the stationary Gaussian process with $\rho(t) = E(x(t)x(0)) = \exp(-|t|^\alpha)$ $(0 < \alpha \leqq 2)$ satisfies the conditions (A.1), (A.1′) and (A.2).

The proof of Theorems 1 and 2 will be carried out in the similar way to [8] and [10] and given in §§3 and 4, respectively.

Next, we will deal with another class of Gaussian processes including the Brownian motion. We introduce, here, the following conditions.

(C.1) There are such positive constants δ_2, C_3 and T_3 as rendering the following valid

$$\rho(t, t+h) \geqq 1-C_3(h/t)^{\alpha},$$

for all t and h such that $0<(h/t)<\delta_2$ and $t>T_3$ for some α with $0<\alpha\leqq 2$.

(C.1′) There are positive constants δ_3, C_4 and T_4 such that

$$\rho(t, t+h) \leqq 1-C_4(h/t)^{\alpha},$$

for all t and h such that $0<(h/t)<\delta_3$ and $t>T_4$, and for all t and s such that $(h/t)>\delta_3$ and $t>T_4$, $\rho(t, t+h)<1-C_4\delta_3^{\alpha}$, for some α with $0<\alpha\leqq 2$.

(C.2) The limit

$$\lim_{s\to\infty} \rho(t, ts)\log s = 0,$$

converges uniformly with respect to t.

THEOREM 4. *Suppose that condition* (C.1) *is satisfied. If* $\psi(t)\in M_a^+$ *and*

$$\int_a^\infty \frac{1}{t}\psi(t)^{2/\alpha-1}\exp\left(-\tfrac{1}{2}\psi^2(t)\right)dt < \infty,$$

for some $a>0$, *then we have*

$$P(\exists t_0(\omega), x(t) \leqq v(t)\psi(t), \textit{for all } t > t_0(\omega)) = 1.$$

THEOREM 5. *Suppose that* (C.1′) *and* (C.2) *are satisfied. If* $\psi(t)\in M_a^+$ *and*

$$\int_a^\infty \frac{1}{t}\psi(t)^{2/\alpha-1}\exp\left(-\tfrac{1}{2}\psi^2(t)\right)dt = \infty,$$

for some $a>0$, *then we have*

$$P(x(t) > v(t)\psi(t) \textit{ i.o.}) = 1.$$

Combination of Theorems 4 and 5 will lead to the following theorem and corollaries.

THEOREM 6. *Assume that conditions* (C.1), (C.1′) *and* (C.2) *are satisfied at the same time for some* α. *Let* $\psi(t)\in M_a^+$ *for some* a. *Then*

$$P(\exists t_0(\omega), x(t) \leqq v(t)\psi(t) \textit{ for all } t \geqq t_0(\omega)) = 1 \textit{ or } 0,$$

according as the integral

$$\int_a^\infty \frac{1}{t}x(t)^{2/\alpha-1}\exp\left(-\tfrac{1}{2}\psi^2(t)\right)dt$$

converges or diverges.

COROLLARY 1. *Under the same conditions as in Theorem* 6, *for every* $\varepsilon>0$, *we have*

$$P(\exists t_0(\omega), x(t) \leqq v(t)(2\log_{(2)}t+(2/\alpha+1+\varepsilon)\log_{(3)}t)^{1/2} \quad \textit{for all } t \geqq t_0(\omega)) = 1.$$

Moreover, for any $\varepsilon\geqq 0$, *we have*

$$P(x(t) > v(t)(2\log_{(2)}t+(2/\alpha+1-\varepsilon)\log_{(3)}t)^{1/2} \textit{ i.o.}) = 1.$$

The proof of Theorems 4 and 5 will be carried out directly by the minor change of the proof of Theorems 1 and 2, namely, by putting $t_{p,k}=2^p+2^pk/[\log p]^{1/\alpha}$ in $E(p;k)=\{y(t_{p,k})>\phi(t_{p,k})\}$ which is used in the proof of Theorems 1 and 2. Here, we make an important remark: the conditions (C.1)$\sim$(C.2) can be deduced by the time change $s=e^t$ from the conditions (A.1)$\sim$(A.2) and conversely. Therefore, by the time change, Theorems 4, 5 and 6 can be deduced from Theorems 1, 2 and 3 and conversely.

Some applications of Theorem 6 will be given in the following. Let $B(t)$ be a one dimensional Brownian motion on $(-\infty,\infty)$. We can easily show that it satisfies the conditions (C.1)$\sim$(C.2). Therefore, Theorem 6 contains as a special case Theorem A. Furthermore, we consider the process $X=\{x(t)=\int_0^t B(u)\,du, -\infty<t<\infty\}$. Then the process X is a Gaussian process with $E(x(t))=0$. Suppose that $0<s<t$, then, we have $r(t,s)=E(x(t)x(s))=s^2t/2-s^3/6$. Furthermore, if $0<t<t+h$,

$$\rho(t,t+h)=\frac{1+\frac{3}{2}(h/t)}{(1+h/t)^{3/2}}\sim 1-\frac{3}{8}\left(\frac{h}{t}\right)^2 \qquad \left(\frac{h}{t}\sim 0\right). \tag{2.2}$$

Hence, there is positive constant δ such that there holds

$$1-\tfrac{1}{2}(h/t)^2<\rho(t,t+h)<1-\tfrac{1}{4}(h/t)^2,$$

for all t and h such that $0<h/t<\delta$. Therefore, from this and (2.2), we can show that the conditions (C.1) and (C.1′) hold for the process X. Next,

$$\rho(t,ts)\log s=\frac{1+\frac{3}{2}(s-1)}{(s^3)^{1/2}}\log s\to 0 \quad (s\to\infty),$$

which justifies the condition (C.2). Accordingly, Theorem 6 is applicable to the present case.

THEOREM 7. *Let $B(t)$ be the Brownian motion. Let $\psi(t)\in M_a^+$, for some $a>0$. Then,*

$$P\left(\exists T_0(\omega),\int_0^T B(u)\,du\leqq\left(\frac{T^3}{3}\right)^{1/2}\psi(T)\text{ for all } T>T_0(\omega)\right)=1 \text{ or } 0,$$

according as the integral

$$\int_a^\infty \frac{1}{t}\exp\left(-\tfrac{1}{2}\psi^2(t)\right)dt$$

converges or diverges.

COROLLARY 1. *For every $\varepsilon>0$,*

$$P\left(\exists T_0(\omega),\int_0^T B(u)\,du\leqq\left(\frac{T^3}{3}\right)^{1/2}(2\log_{(2)}T+2\log_{(3)}T+\cdots(2+\varepsilon)\log_{(n)}T)^{1/2}\right.$$
$$\left.\text{for all } T>T_0(\omega)\right)=1.$$

While for any $\varepsilon \geqq 0$,

$$P\left(\int_0^T B(u)\,du > \left(\frac{T^3}{3}\right)^{1/2}(2\log_{(2)} T + 2\log_{(3)} T + \cdots (2-\varepsilon)\log_{(n)} T)^{1/2}\ i.o.\right) = 1.$$

COROLLARY 2.

$$P\left(\limsup_{T\to\infty}\left[\int_0^T B(u)\,du/(\tfrac{2}{3}T^3 \log_{(2)} T)^{1/2}\right] = 1\right) = 1.$$

3. Proof of Theorem 1.

LEMMA 3.1. *If Theorem* 1 *is true under the additional condition that for large* t,

$$(\log t)^{1/2} \leqq \phi(t) \leqq (3\log t)^{1/2},$$

it is true without the additional condition.

Proof. The proof of Lemma 3.1 is similar to Lemma 1 in T. Sirao [8]. Therefore, we will give only its sketch.

Assume that the statement of Theorem 1 is true for arbitrary ϕ such that

$$\int_a^\infty \phi(t)^{2/\alpha-1}\exp(-\tfrac{1}{2}\phi^2(t))\,dt < \infty \quad \text{for some } a > 0, \tag{3.1}$$

and

$$(\log t)^{1/2} \leqq \phi(t) \leqq (3\log t)^{1/2} \tag{3.2}$$

for large t.

Given any $\phi(t)$ satisfying only (3.1), the set $\{t;\ \phi(t) < (\log t)^{1/2}\}$ is bounded. For, if it is not, there are an infinite set $\{t_n\}$ such that $\phi(t_n) < (\log t_n)^{1/2}$. Then we can easily show that

$$\int_{t_1}^\infty \phi(t)^{2/\alpha-1}\exp(-\tfrac{1}{2}\phi^2(t))\,dt \geqq \left(\frac{(t_n-t_1)^2}{t_n}\right)^{1/2}(\log t_n)^{1/\alpha-1/2} \to \infty \qquad (n\to\infty),$$

which contradicts to (3.1).

Now put $\hat{\phi}(t) = \min(\max(\phi(t), (\log t)^{1/2}), (3\log t)^{1/2})$. Then,

$$\hat{\phi}(t) = \min(\phi(t), (3\log t)^{1/2})$$

for large t. Hence

$$\int^\infty \hat{\phi}(t)^{2/\alpha-1}\exp(-\tfrac{1}{2}\hat{\phi}^2(t))\,dt < \infty.$$

By definition, $(\log t)^{1/2} \leqq \hat{\phi}(t) \leqq (3\log t)^{1/2}$. Thus, Theorem 1 is valid for $\hat{\phi}(t)$, that is to say,

$$P(\exists t_0(\omega),\ x(t) \leqq v(t)\hat{\phi}(t), \text{ for all } t \geqq t_0(\omega)) = 1.$$

Recalling that $\hat{\phi}(t) \leqq \phi(t)$ for large t, we have the conclusion of Lemma 3.1.

By Lemma 3.1, it is sufficient to show Theorem 1, under the hypothesis (3.1) and (3.2). In the following, we put $y(t)=x(t)/v(t)$.

LEMMA 3.2. *Let* $\phi(t)$ *satisfy* (3.1) *and let* $E(p;k)=\{y(t_{p,k})>\phi(t_{p,k})\}$, where $t_{p,k}=p+k/[(\log p)^{1/\alpha}]$, $(p=1,\ldots, k=0, 1, 2, \ldots, [(\log p)^{1/\alpha}])$. *Then,*

$$\sum_{p=1}^{\infty}\sum_{k=0}^{[(\log p)^{1/\alpha}]} P(E(p;k)) < \infty. \tag{3.3}$$

Proof. Since

$$\int_x^{\infty} e^{-u^2/2}\,du \leqq \frac{1}{x}e^{-x^2/2} \quad \text{for any } x > 0,$$

we have for any p_0,

$$\sum_{p=p_0}^{\infty}\sum_{k=0}^{[(\log p)^{1/\alpha}]} P(E(p;k)) \leqq \frac{1}{(2\pi)^{1/2}}\sum_{p=p_0}^{\infty}\sum_{k=0}^{[(\log p)^{1/\alpha}]}\frac{1}{\phi(t_{p,k})}\exp\left(-\tfrac{1}{2}\phi^2(t_{p,k})\right) \equiv A.$$

By use of the monotonicity of $\phi(\cdot)$, we have

$$A \leqq (2\pi)^{-1/2}\sum_{p=p_0}^{\infty}(\log p)^{1/\alpha}\frac{1}{\phi(p)}\exp\left(-\tfrac{1}{2}\phi^2(p)\right) \equiv B.$$

Since $(\log p)^{1/2} \leqq \phi(p)$ for large p, if we take sufficiently large p_0, we obtain

$$\begin{aligned} B &\leqq (2\pi)^{-1/2}\sum_{p=p_0}^{\infty}\phi(p)^{2/\alpha-1}\exp\left(-\tfrac{1}{2}\phi^2(p)\right) \\ &\leqq (2\pi)^{-1/2}\int_{p_0}^{\infty}\phi(t)^{2/\alpha-1}\exp\left(-\tfrac{1}{2}\phi^2(t)\right)dt < \infty, \end{aligned}$$

which concludes the proof.

Let L be any positive constant. Let c be a real number which makes e^{qc} an integer for any positive integer q. Let $t_{p,k,q,m}=t_{p,k}+m/(e^{qc}[(\log p)^{1/\alpha}])$ and let

$$F_{q,m}(p;k) = y(t_{p,k,q,m}) \geqq \phi(t_{p,k})+\frac{2L}{\phi(t_{p,k})}\sum_{i=0}^{q-1}2^{-\alpha i},$$

where the integers m range from 0 to e^{qc}.

Let $\bigcup_{m=0}^{e^{qc}} F_{q,m}(p;k)=F_q(p;k)$ and let $\bigcup_{q=0}^{\infty} F_q(p;k)=F(p;k)$. Then, we have, for almost all $y(\cdot,\omega)$,

$$F(p;k) = \left\{y(t_{p,k}+s/[(\log p)^{1/\alpha}]) \geqq \phi(t_{p,k})+\frac{2L}{\phi(t_{p,k})}\sum_{i=0}^{\infty}2^{-\alpha i} \text{ for some } 0 \leqq s \leqq 1\right\},$$

since almost all path functions $y(t,\omega)$ are continuous.

LEMMA 3.3.

$$\sum_{p=1}^{\infty}\sum_{k=0}^{[(\log p)^{1/\alpha}]} P(F(p;k)) < +\infty.$$

Before proving this lemma, we will use it to carry out the proof of Theorem 1. By use of the Borel-Cantelli theorem, it follows that

$$P\Big(\exists p_0, \forall p \geqq p_0, 0 \leqq \forall s \leqq 1, 1 \leqq \forall k \leqq [(\log p)^{1/\alpha}], y(t_{p,k}+s/[(\log p)^{1/\alpha}]) \leqq \phi(t_{p,k})+\frac{2L}{\phi(t_{p,k})}\sum_{i=0}^{\infty} 2^{-\alpha i}\Big) = 1.$$

By the monotonicity of ϕ, we find

$$P\Big(\exists T_0(\omega), y(T) \leqq \phi(T)+\frac{2L}{\phi(T)}\sum_{i=0}^{\infty} 2^{-\alpha i}, \text{ for all } T \geqq T_0(\omega)\Big) = 1.$$

Let $\tilde{\phi}(t)=\phi(t)-3L'/\phi$, where $L'=L\sum_{i=0}^{\infty} 2^{-\alpha i}$. Then, for sufficiently large t, it holds that

$$(a \log t)^{1/2} \leqq \tilde{\phi}(t) \leqq (3 \log t)^{1/2}$$

with some $0<a<1$, and

$$\int^{\infty} \tilde{\phi}(t)^{2/\alpha-1} \exp(-\tfrac{1}{2}\tilde{\phi}^2(t))\, dt < \infty.$$

Since we can repeat the similar discussion for this $\tilde{\phi}$, we can obtain that

$$P(\exists T_0(\omega), y(T) \leqq \tilde{\phi}(T)+2L'/\tilde{\phi}(T) \text{ for all } T > T_0(\omega)) = 1.$$

Since $\tilde{\phi}(T)+2L'/\tilde{\phi}(T)<\phi(T)$ for sufficiently large T, we have the conclusion of Theorem 1.

Proof of Lemma 3.3. In the following, we will sometimes denote simply $F_{q,m}$ F_q and F in place of $F_{q,m}(p; k)$, $F_q(p; k)$ and $F(p; k)$ respectively. Since the relation $F_{q-1} \cap F_q = F_{q-1}$, we have

$$\begin{aligned} P(F_q) &= P(F_{q-1})+P(F'_{q-1} \cap F_q) \\ &\leqq P(F_{q-1})+\sum_{m=0}^{e^{qc}} P(F'_{q-1} \cap F_{q,m}) \\ &\leqq P(F_{q-1})+\sum_{m=0}^{e^{qc}} P(F'_{q-1,m_1} \cap F_{q,m}), \end{aligned}$$

for any integer m_1 with $0 \leqq m_1 \leqq e^{(q-1)c}$, where F' means the complementary set of F.

LEMMA 3.4. *If we take m_1 satisfying the relation $|me^{-qc}-m_1e^{-(q-1)c}|<e^{-(q-1)c}$ there exists a positive constant C_8 independent of p, k and q such that*

$$P(F'_{q-1,m_1} \cap F_{q,m}) \leqq C_8 e^{-2qc} P(E(p; k)).$$

Proof of Lemma 2.4. From the definition of set $F_{q,m}$, we can write

$$Q \equiv P(F'_{q-1,m_1} \cap F_{q,m})$$
$$= P\left(y(t_{p,k,q-1,m}) \leqq \phi(t_{p,k}) + \frac{2L}{\phi(t_{p,k})} \sum_{i=0}^{q-2} 2^{-\alpha i}, \right.$$
$$\left. y(t_{p,k,q,m}) > \phi(t_{p,k}) - \frac{2L}{\phi(t_{p,k})} \sum_{i=0}^{q-1} 2^{-\alpha i} \right).$$

Now, let X and Y be two independent random variables obeying the Gaussian distribution with the mean zero and variance 1, and let $\rho_{m_1,m}$ be the correlation coefficient between $y(t_{p,k,q-1,m_1})$ and $y(t_{p,k,q,m})$. Then

$$Q \equiv P\left((1-\rho_{m_1,m}^2)^{1/2} Y + \rho_{m_1,m} X \leqq \left(\phi + (2L/\phi) \sum_{i=0}^{q-2} 2^{-\alpha i}\right), X \geqq \left(\phi + (2L/\phi) \sum_{i=0}^{q-1} 2^{-\alpha i}\right)\right)$$
$$\leqq P\left(X \geqq \left(\phi + (2L/\phi) \sum_{i=0}^{q-1} 2^{-\alpha i}\right), (1-\rho_{m_1,m}^2)^{1/2} Y \right.$$
$$\left. \leqq \left(\phi + (2L/\phi) \sum_{i=0}^{q-2} 2^{-\alpha i}\right) - \rho_{m_1,m}\left(\phi + (2L/\phi) \sum_{i=0}^{q-1} 2^{-\alpha i}\right)\right)$$
$$= P\left(X \geqq \left(\phi + (2L/\phi) \sum_{i=0}^{q-1} 2^{-\alpha i}\right)\right)$$
$$\times P\left(Y > -(1-\rho_{m_1,m}^2)^{-1/2}(1-\rho_{m_1,m})\left(\phi + (2L/\phi) \sum_{i=0}^{q-2} 2^{-\alpha i}\right) \right.$$
$$\left. + (1-\rho_{m_1,m}^2)^{-1/2} \rho_{m_1,m} (2L/\phi) 2^{-\alpha(q-1)}\right),$$

where $\phi = \phi(t_{p,k})$. By use of assumption (A.1), for large p,

$$\rho_{m_1,m} \geqq 1 - C_1 \left(\frac{|m_1/e^{(q-1)c} - m/e^{qc}|}{[(\log p)^{1/\alpha}]} \right)^{\alpha}$$
$$\geqq 1 - C_5/((\log p) e^{(q-1)\alpha}),$$

where C_5 is an absolute positive constant. Taking account of this estimate and (3.2),

$$-(1-\rho_{m_1,m}^2)^{-1/2}(1-\rho_{m_1,m})\left(\phi + (2L/\phi) \sum_{i=0}^{q-1} 2^{-\alpha i}\right) \geqq -C_6,$$

where C_6 is a positive constant independent of p and q. Also, for a convenient positive constant C_7,

$$\frac{\rho_{m_1,m}}{(1-\rho_{m_1,m}^2)^{1/2}} \frac{2L}{\phi} \frac{1}{2^{\alpha(q-1)}} > C_7 \left(\frac{e^c}{4}\right)^{\alpha(q-1)}.$$

Now, we select c such that $C_7(e^c/4)^{\alpha(q-1)} > 2C_6$ for all $q > 2$ and $(e^c/4) > 1$. Then we have

$$Q \leqq P(X \geqq \phi) P(Y \geqq (C_7/2)(e^c/4)^{(q-1)\alpha/2})$$
$$\leqq P(E(p;k))(2\pi)^{-1/2} \frac{2}{C_7(e^c/4)^{(q-1)\alpha/2}} \exp\left(-\tfrac{1}{2}\left(\frac{C_7^2}{4}\right)\left(\frac{e^c}{4}\right)^{(q-1)\alpha}\right).$$

Since

$$\lim_{q\to\infty} e^{2qc}\frac{1}{(e^c/4)^{(q-1)\alpha/2}}\exp\left(-\tfrac{1}{2}\left(\frac{C_7^2}{4}\right)\left(\frac{e^c}{4}\right)^{(q-1)\alpha}\right) = 0,$$

the above expression is smaller than the following quantity, $C_8e^{-2qc}P(E(p;k))$, where C_8 is an absolute constant independent of p and q. This concludes the proof.

From Lemma 3.4, it follows that

$$\begin{aligned} P(F_q) &\leqq P(F_{q-1})+C_8e^{-qc}P(E(p;k)) \\ &\leqq C_8\sum_{i=0}^{q-1} e^{-ic}P(E(p;k)). \end{aligned}$$

When $q\to\infty$,

$$P(F(p;k)) \leqq C_8\sum_{i=0}^{\infty} e^{-ic}P(E(p;k)) = \text{Constant}\times P(E(p;k)).$$

Hence, for sufficiently large $\tilde{p}$,

$$\sum_{p=\tilde{p}}^{\infty}\sum_{k=0}^{[(\log p)^{1/\alpha}]} P(F(p;k)) \leqq \text{Constant}\times \sum_{p=0}^{\infty}\sum_{k=0}^{[(\log p)^{1/\alpha}]} P(E(p;k)),$$

which is convergent by Lemma 3.2. This concludes the proof of Lemma 3.3.

4. Proof of Theorem 2.

LEMMA 4.1. *If Theorem 2 is true under the additional assumption that for large t,*

$$(\log t)^{1/2} \leqq \phi(t) \leqq (3\log t)^{1/2},$$

it is true without the additional assumption.

Proof. The proof of Lemma 4.1 is similar to Lemma 1 in T. Sirao [8]. We assume that Theorem 2 is true under the additional assumption. Let $\phi(t)$ be an arbitrary function with $\int^\infty \phi(t)^{2/\alpha-1}\exp(-\frac{1}{2}\phi^2(t))\,dt=\infty$. Let

$$\hat{\phi}(t) = \min(\max(\phi(t), (\log t))^{1/2}, (3\log t)^{1/2}).$$

Suppose that there exists an infinite sequence $\{t_n\}$ such that $\phi(t_n)<(\log t_n)^{1/2}$ and $t_n\to\infty$ $(n\to\infty)$. Then, $\int^\infty \hat{\phi}(t)^{2/\alpha-1}\exp(-\frac{1}{2}\hat{\phi}^2(t))\,dt=\infty$, because in this case $\hat{\phi}(t_n)=(\log t_n)^{1/2}$ and therefore,

$$\begin{aligned} \int_{t_1}^{\infty} \hat{\phi}(t)^{2/\alpha-1}\exp(-\tfrac{1}{2}\hat{\phi}^2(t))\,dt &\geqq \int_{t_1}^{t_n} \hat{\phi}(t)^{2/\alpha-1}\exp(-\tfrac{1}{2}\hat{\phi}^2(t))\,dt \\ &\geqq (t_n-t_1)\hat{\phi}(t_n)^{2/\alpha-1}\exp(-\tfrac{1}{2}\hat{\phi}^2(t_n)) \\ &= \frac{t_n-t_1}{(t_n)^{1/2}}(\log t_n)^{1/\alpha-1/2}. \end{aligned}$$

Next, suppose that $\phi(t)>(\log t)^{1/2}$ for large t. In this case,

$$\min(\phi(t), (3\log t)^{1/2}) = \hat{\phi}(t).$$

Therefore, for all large t,

$$\phi(t)^{2/\alpha-1}\exp(-\tfrac{1}{2}\phi^2(t)) \leqq \hat{\phi}(t)^{2/\alpha-1}\exp(-\tfrac{1}{2}\hat{\phi}^2(t)).$$

Hence,

$$\int^{\infty} \hat{\phi}(t)^{2/\alpha-1}\exp(-\tfrac{1}{2}\hat{\phi}^2(t))\,dt = \infty.$$

Furthermore,

$$(\log t)^{1/2} \leqq \hat{\phi}(t) \leqq (3\log t)^{1/2}.$$

Consequently, by assumption of Lemma 4.1,

$$P(\exists\{T_n(\omega)\},\, T_1 < T_2 < \cdots < T_n \to \infty\ (n\to\infty),\ y(T_n) > \hat{\phi}(T_n)) = 1.$$

But, by Theorem 1, clearly,

$$P((3\log T_n(\omega)^{1/2}) > y(T_n(\omega))\forall n > \exists n_0(\omega)) = 1.$$

Therefore,

$$P((3\log T_n)^{1/2} > \hat{\phi}(T_n), \forall n > \exists n_0(\omega)) = 1.$$

Hence,

$$P(\hat{\phi}(T_n) = \max(\phi(T_n), (\log T_n)^{1/2}) > \phi(T_n), \forall n > \exists n_0(\omega)) = 1.$$

Thus

$$P(y(T_n) > \phi(T_n), \forall n > \exists n_0(\omega)) = 1,$$

which concludes the proof.

As in §2, we consider the family of events $E(p;k)$, where p ranges over the positive integers and integer k ranges over $0 \leqq k \leqq [(\log p)^{1/\alpha}]$. We enumerate events $E(p;k)$ such that $n<n'$ if and only if $p+k/[(\log p)^{1/\alpha}] < p'+k'/[(\log p')^{1/\alpha}]$ if we put $E_n = E(p;k)$ and $E_{n'} = E(p';k')$. Then, the proof of Theorem 2 is carried out by proving the following sequence of lemmas. For, then, by Chung-Erdös [1] we can see that $P(E_n \text{ i.o.}) = 1$.

LEMMA 4.2. $\sum_{n=1}^{\infty} P(E_n) = +\infty$.

LEMMA 4.3. *For any pair of (n, h) with $n \geqq h$, there exist $c(h) > 0$ and $H(n, h) > n$ such that for any $m \geqq H(n, h)$*

$$P(E_m \mid E'_h \cap E'_{h+1} \cap E'_{h+2} \cap \cdots \cap E'_n) \geqq c(h)P(E_m).$$

LEMMA 4.4. *There exist two absolute constants K_1 and K_2 with the following property: to each E_n there corresponds a set of events $E_{n_1}, E_{n_2}, \ldots, E_{n_s}$ belonging to $\{E_n\}$ such that*

$$\sum_{i=1}^{s} P(E_n \cap E_{n_i}) < K_1 P(E_n)$$

and that for any other E_m than E_{n_i} $(1 \leqq i \leqq s)$ which stands after E_n in the sequence (viz. $m > n$) the inequality

$$P(E_n \cap E_m) \leqq K_2 P(E_n)P(E_m)$$

holds.

Proof of Lemma 4.2. Since $P(E_n)=P(E(p;k))$, it is sufficient to show that

$$\sum_{p=p_0}^{\infty}\sum_{k=0}^{[(\log p)^{1/\alpha}]} P(E(p;k)) = \infty,$$

for some p_0. The left hand side of the above equality is greater than

$$\sum_{p=p_0}^{\infty}\sum_{k=0}^{[(\log p)^{1/\alpha}]} 2^{-1}(2\pi)^{-1/2}\frac{1}{\phi(t_{p,k})}\exp\left(-\tfrac{1}{2}\phi^2(t_{p,k})\right) \equiv R,$$

because $\int_x^\infty e^{-u^2/2}\,du \geqq (1/2x)e^{-x^2/2}$ $(x\geqq 1)$. By the monotonicity of the function $1/\phi(x)\exp(-\frac{1}{2}\phi^2(x))$ of x for large x,

$$\begin{aligned} R &\geqq 2^{-1}(2\pi)^{-1/2}\sum_{p=p_0}^{\infty}[(\log p)]^{1/\alpha}\frac{1}{\phi(p+1)}\exp\left(-\tfrac{1}{2}\phi^2(p+1)\right) \\ &\geqq \left(\frac{\log p_0}{\log(p_0+1)}\right)^{1/\alpha}\frac{1}{3^{1/\alpha}}2^{-1}(2\pi)^{-1/2}\sum_{p=p_0}^{\infty}\phi(p+1)^{2/\alpha-1}\exp\left(-\tfrac{1}{2}\phi^2(p+1)\right) \\ &\geqq \text{Constant}\times\int_{p_0+1}^{\infty}\phi(t)^{2/\alpha-1}\exp\left(-\tfrac{1}{2}\phi^2(t)\right)dt, \end{aligned}$$

which concludes the proof.

Proof of Lemma 4.3. The proof is similar to the discussion in pp. 146–148 of T. Sirao [8]. Therefore, we will give only an outline of it. Let

$$F_n(a) = \{\omega;\, \phi(t_{p,k}) \leqq y(t_{p,k}) \leqq \phi(t_{p,k})+a\}.$$

Then, for large n,

$$P(E_n) \geqq P(F_n(a)) \geqq \tfrac{1}{2}P(E_n)(1-2\exp[-a\phi(t_{p,k})]).$$

Therefore, given a pair of (n, h), we can take $a>0$ such that

$$P(F_m(a)) \geqq \tfrac{1}{3}P(E_m) \qquad \text{for all } m \geqq n \tag{4.1}$$

and

$$P\left(\bigcap_{i=h}^{n}(E_i'\cap E_i(a))\right) \geqq \tfrac{1}{2}P\left(\bigcap_{i=h}^{n}E_i'\right),$$

where $E_n(a)=\{\omega;\, y(t_{p,k})+a\geqq 0\}$. Hence, we have

$$P(E_m \mid E_h'\cap E_{h+1}'\cap\cdots\cap E_n') \geqq \tfrac{1}{2}P\left(F_m(a)\,\Big|\,\bigcap_{i=h}^{n}(E_i'\cap E_i(a))\right). \tag{4.2}$$

Let

$$\rho_{i,m} = \rho(y(p_i+k_i/[(\log p_i)^{1/\alpha}]),\ \ y(p_m+k_m/[(\log p_m)^{1/\alpha}]))$$

and $\rho_m=\max\{|\rho_{i,m}|;\, 0\leqq i\leqq n-m\}$. Then by condition (A.2), we can see that for large m,

$$\rho_m^{-1/2} \geqq 4(\log p_m)^{1/2} \geqq \phi(p_m+k_m/[(\log p_m)^{1/\alpha}])$$

and $\lim_{m\to\infty}\rho_m=0$.

Now, let $X_i = y(p_i + k_i/[(\log p_i)^{1/\alpha}])$. Let $A_i = [-a, \phi(p_i + k_i/[\log p_i]^{1/\alpha}))$ for $i=0, 1, 2, \ldots, n-h$. Let $Y_m = y(p_m + k_m/[(\log p_m)^{1/\alpha}])$ and $B_m = [\phi(p_m + k_m/[(\log p_m)^{1/\alpha}]), \phi(p_m + k_m/[(\log p_m)^{1/\alpha}]) + a)$, where $m = n+1, n+2, \ldots$. Then, the following lemma is available (Lemma 4 in T. Sirao [8]).

LEMMA 4.5. *Let $\{X_1, X_2, \ldots, X_n, Y_m, m=1, 2, \ldots\}$ be a sequence of standard Gaussian random variables and $\rho_{i,m} = E(X_i Y_m)$, $0 \leqq i \leqq k$ and $m \geqq 1$. We assume that $\rho_m = \max\{|\rho_{i,m}|; 0 \leqq i \leqq k\} \to 0$ as $m \to \infty$ and A_i $(i=1, 2, \ldots, k)$ are bounded Borel sets. Then, for any sequence of Borel sets $B_m \subset [-\rho_m^{-c}, \rho_m^{-c}]$ with $c<1$, we have*

$$\frac{P(Y_m \in B_m \mid X_i \in A_i, i = 0, 1, 2, \ldots, k)}{P(Y_m \in B_m)} \to 1 \qquad (m \to \infty).$$

Proof. We will give only its sketch. Denoting by P_X the probability law of $\{X_1, X_2, \ldots, X_n\}$ and by $p_m(X_1, X_2, \ldots, X_n)$ the conditional expectation of Y_m for given values of $X_1, X_2, \ldots, X_n$, respectively, we have

$$\begin{aligned}
&P(X_i \in A_i, i = 1, 2, \ldots, Y_m \in B_m) \\
&= \int_{x_1 \in A_1} \cdots \int_{x_n \in A_n} \left\{ \int_{z \in B_m} \frac{1}{(2\pi(1-\gamma^2)^{1/2})} \exp\left(-\frac{1}{2(1-\gamma^2)} (z - p(x_1, x_2, \ldots, x_n))^2\right) dz \right\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot P_X(dx_1, \ldots, dx_n) \\
&= \int_{x_1 \in A_1} \cdots \int_{x_n \in A_n} \left\{ \int_{z \in B_m} \frac{1}{(2\pi(1-\gamma^2)^{1/2})} \exp\left(-\tfrac{1}{2}z^2 + \theta\right) dz \right\} P_X(dx_1, \ldots, dx_n),
\end{aligned}$$

where

$$\theta = -\{\gamma^2 z^2 - 2z p_m(x_1, \ldots, x_n) + p_m^2(x_1, \ldots, x_n)\}/(2(1-\gamma^2))$$

and $\gamma^2 = E(p_m^2(X_1, \ldots, X_n))$. From the assumptions of the lemma, we can easily infer that γ^2 and p_m tend to zero uniformly with respect to $x_1, \ldots, x_n$ and z as m tends to infinity, giving the conclusion of the lemma.

Thus we have

$$P\left(F_m(a) \,\middle|\, \bigcap_{i=h}^{n} (E_i' \cap E_i(a)) \geqq \tfrac{1}{2} P(F_n(a))\right). \tag{4.3}$$

By combining (4.1), (4.2), and (4.3), we obtain

$$P(E_m \mid E_h' \cap E_{h+1}' \cap \cdots \cap E_n') \geqq P(E_m)/12 \quad \text{for large } m,$$

which concludes the proof.

Proof of Lemma 4.4. In the following, if $E_n = E(p; k)$ and $E_{n'} = E(p'; k')$, we put $t_{p,k} = t_n$ and $t_{p',k'} = t_{n'}$. Let $A_n = \{n'; \rho(t_n, t_{n'}) < (\phi(t_n)\phi(t_{n'}))^{-1}\}$ for each fixed n. Let us take an arbitrary n' in A_n. Then

$$\begin{aligned}
P(E_n \cap E_{n'}) &= P(y(t_n) \geqq \phi(t_n), y(t_{n'}) \geqq \phi(t_{n'})) \\
&\leqq \text{Constant} \times P(E_n) P(E_{n'}).
\end{aligned}$$

In fact, if we denote, by X and Y two independent standard Gaussian random variables, we can write

$$P(E_n \cap E_{n'}) = P(X \geqq \phi(t_{n'}), \rho X + (1-\rho^2)^{1/2} Y \geqq \phi(t_n)),$$

where $\rho=\rho(t_n, t_{n'})$. In case of $\rho \leqq 0$, we have

$$\begin{aligned}P(E_n \cap E_{n'}) &\leqq P(X \geqq \phi(t_{n'}),\ Y \geqq \phi(t_n)/(1-\rho^2)^{1/2})\\&\leqq P(X \geqq \phi(t_{n'}))P(Y \geqq \phi(t_n))\\&= P(E_{n'})P(E_n).\end{aligned}$$

Next, we consider the case of $0<\rho \leqq 1/(\phi(t_n)\phi(t_{n'}))$. At first, we divide it into three parts as follows

$$\begin{aligned}P(E_n \cap E_{n'}) = P(\phi(t_{n'}) &\leqq y(t_{n'}) \leqq 2\phi(t_{n'}), 2\phi(t_{n'}) \geqq y(t_n) \geqq \phi(t_n))\\&+P(\phi(t_{n'}) \leqq y(t_n), 2\phi(t_{n'}) \leqq y(t_{n'}))\\&+P(2\phi(t_{n'}) \leqq y(t_{n'}), \phi(t_{n'}) \leqq y(t_n) \leqq 2\phi(t_{n'})).\end{aligned}$$

On the first term of the right hand side, we have

$$\begin{aligned}&P(\phi(t_{n'}) \leqq y(t_{n'}) \leqq 2\phi(t_{n'}), \phi(t_n) \leqq y(t_n) \leqq 2\phi(t_{n'}))\\&\quad= P(\phi(t_{n'}) \leqq X \leqq 2\phi(t_{n'}), \phi(t_n) \leqq \rho X+(1-\rho^2)^{1/2} Y \leqq 2\phi(t_{n'}))\\&\quad\leqq P(\phi(t_{n'}) \leqq X \leqq 2\phi(t_{n'}), (\phi(t_n)-2\rho\phi(t_{n'}))/(1-\rho^2)^{1/2} \leqq Y)\\&\quad\leqq P(\phi(t_{n'}) \leqq X)P((\phi(t_n)-2/\phi(t_{n'})) \leqq Y)\\&\quad\leqq \text{Constant} \times P(\phi(t_{n'}) \leqq X)P(\phi(t_n) \leqq Y),\end{aligned}$$

since

$$\lim_{n\to\infty} P(\phi(t_n)-2/\phi(t_{n'}) \leqq Y)/P(\phi(t_n) \leqq Y) = e^2.$$

On the second term (also on the third term), we can see that

$$\begin{aligned}P(y(t_n) &\geqq \phi(t_n), y(t_{n'}) \geqq 2\phi(t_{n'}))\\&\leqq P(y(t_{n'}) > 2\phi(t_{n'})) \leqq \text{Constant} \times 1/\phi(t_{n'}) \exp(-2\phi^2(t_{n'}))\\&\leqq \text{Constant} \times P(E_n)P(E_{n'}).\end{aligned}$$

Next, we consider the case of $n' \notin A_n$. Then, we have, for sufficiently large n,

$$t_{n'} < t_n+K \log t_n, \tag{4.4}$$

where K is an absolute constant. For, from the definition, for large n,

$$\rho(t_n, t_{n'}) \geqq \frac{1}{\phi(t_n)\phi(t_{n'})} \geqq \frac{C_9}{(\log t_n \log t_{n'})^{1/2}}, \tag{4.5}$$

where C_9 is an absolute positive constant. Taking account of the condition (A.2), we can write

$$\rho(t_n, t_{n'}) \leqq \varepsilon(t_n, t_{n'})/|t_{n'}-t_n|, \tag{4.6}$$

where $\varepsilon(t_n, t_{n'}) \to 0$ ($|t_n-t_{n'}| \to 0$). Otherwise, suppose that there is an infinite sequence $\{n_j\}$ such that $t_{n_i'} > t_{n_j}+K \log t_{n_j}$, we have, by (4.5) and (4.6),

$$C_9 \frac{K \log t_{n_j}}{\log t_{n_j} \log(t_{n_j}+K \log t_{n_j})} \leqq C_9 \frac{(t_{n_j'}-t_{n_j})}{(\log t_{n_j} \log t_{n_j'})^{1/2}} \leqq \varepsilon(t_{n_j}, t_{n_j'}),$$

from which as j tends to infinity we have the contradiction $0 < C_9 K \leqq 0$, which concludes (4.4).

Therefore, we can write such as $A'_n = \{E_{n_i}; i = 1, 2, \ldots, s\}$. If $n_i \in A'_n$,

$$\begin{aligned} P(E_n \cap E_{n_i}) &= P(y(t_n) \geqq \phi(t_n), y(t_{n_i}) \geqq \phi(t_{n_i})) \\ &\leqq P(y(t_n) \geqq \phi(t_n), y(t_{n_i}) \geqq \phi(t_n)) \qquad \text{for large } n, \end{aligned}$$

because $\phi(t_n) \leqq \phi(t_{n_i})$. The last quantity is expressed by the following integral

$$\frac{1}{2\pi(1-\rho_i^2)^{1/2}} \int_\phi^\infty \int_\phi^\infty \exp\left(-\frac{(x^2 - 2\rho_i xy + y^2)}{2(1-\rho_i^2)}\right) dy\, dx,$$

where $\phi = \phi(t_n)$ and $\rho_i = \rho(t_n, t_{n_i})$. Rotating the axes by $\pi/4$, we obtain

$$\begin{aligned} P(E_n \cap E_{n_i}) &\leqq \frac{1}{2\pi(1-\rho^2)^{1/2}} \int_{2^{1/2}\phi}^\infty \int_{-(x-2^{1/2}\phi)}^{(x+2^{1/2}\phi)} \exp\left(-\frac{(1-\rho_i)x^2 + (1+\rho_i)y^2}{2(1-\rho_i^2)}\right) dy\, dx \\ &\leqq (2\pi)^{-1/2} \int_{(2/(1+\rho_i))^{1/2}\phi}^\infty \exp\left(-\frac{x^2}{2}\right) dx \\ &\leqq (2\pi)^{-1/2}\phi^{-1}\left(\frac{1+\rho_i}{2}\right)^{1/2} \exp\left(-\frac{1-\rho_i^2}{2(1+\rho_i)^2}\phi^2\right) \exp\left(-\frac{\phi^2}{2}\right) \\ &\leqq 2P(y(t_n) \geqq \phi(t_n))\left(\frac{1+\rho_i}{2}\right)^{1/2} \exp\left(-\frac{1-\rho_i^2}{2(1+\rho_i)^2}\phi^2\right) \\ &\leqq 2P(E_n) \exp\left(-\tfrac{1}{8}(1-\rho_i^2)\phi^2(t_n)\right). \end{aligned}$$

Therefore,

$$\sum_{i=1}^s P(E_n \cap E_{n_i}) \leqq \text{Constant} \times \sum_{i=1}^s \exp\left(-d(1-\rho_i^2)\phi^2(t_n)\right),$$

where d is an absolute positive constant.

In order to prove Lemma 4.4, it is sufficient to show that $\sum_{i=1}^s \exp(-d(1-\rho_i^2)\phi^2)$ is bounded with respect to p. For this purpose, we shall divide it into two parts and write

$$\sum_{i=1}^s \exp(-d(1-\rho_i^2)\phi^2(t_n)) = \sum\nolimits^{(1)} \exp(-d(1-\rho_i^2)\phi^2(t_n)) + \sum\nolimits^{(2)} \exp(-d(1-\rho_i^2)\phi^2(t_n)),$$

where $\sum^{(1)}$ stands for the summation over i such that $\rho_i \geqq (1-(\log t_n)^{-1/2})^{1/2}$ and $\sum^{(2)}$ stands for the remainder of $\sum$.

By the definition of $\sum^{(2)}$ and (3.2), we have, for large n,

$$(1-\rho_i^2)\phi^2(t_n) \geqq (\log t_n)^{1/2}.$$

Since the number of n_i in $\sum^{(2)}$ does not exceed $2K(\log t_n)^{1/\alpha+1}$ for large p,

$$\begin{aligned} \sum\nolimits^{(2)} \exp(-d(1-\rho_i^2)\phi^2(t_n)) &\leqq \sum\nolimits^{(2)} \exp(-d(\log t_n)^{1/2}) \\ &\leqq 2K(\log t_n)^{1/\alpha+1} \exp(-d(\log t_n)^{1/2}) \end{aligned}$$

which is bounded with respect to p, giving the desired result.

Next, we shall prove that $\sum^{(1)} \exp(-d(1-\rho_i^2)\phi^2(t_n))$ is bounded with respect to n. For n_i, which is added to $\sum^{(1)}$, we have

$$1 - (\log t_n)^{-1/2} \leqq \rho^2(t_n, t_{n'}).$$

By assumption of Theorem 2, there is a $\delta > 0$ such that if $|t_{n'} - t_n| < \delta$, then $\rho(t_n, t_{n'}) \leqq \eta$. Therefore, $|t_{n'} - t_n| < \delta$ for large p. Hence, we find

$$1 - (\log t_n)^{-1/2} \leqq 1 - C_2 |t_{n'} - t_n|^\alpha,$$

from which follows that $p' \leqq p+1$ for large p. Thus, we can show that for an appropriate positive constant D

$$(1-\rho_i^2) \geqq D(k^\alpha/(\log(p+1))) \quad \text{or} \quad D(k^\alpha/(\log p)),$$

where different i corresponds to different k. Since $\phi(t_n) > (\log p)^{1/2}$ for large p, we have that

$$\sum^{(1)} \exp(-d(1-\rho_i^2)\phi^2(t_n)) < \sum_{k=0}^{\infty} \exp(-\text{Constant} \times k^\alpha) < +\infty,$$

which concludes the proof of Theorem 2.

References

1. K. L. Chung and P. Erdös, *On the application of the Borel-Cantelli lemma*, Trans. Amer. Math. Soc. **72** (1952), 179–186. MR **13**, 567.

2. H. Cramér, *On the maximum of a normal stationary stochastic process*, Bull. Amer. Math. Soc. **68** (1962), 512–516. MR **25** #3562.

3. W. Feller, *The general form of the so-called law of the iterated logarithm*, Trans. Amer. Math. Soc. **54** (1943), 373–402. MR **5**, 125.

4. P. Lévy, *Théorie de l'addition des variables aleatoires*, Gauthier-Villars, Paris, 1937.

5. M. Nisio, *On the extreme values of Gaussian processes*, Osaka J. Math. **4** (1967), 313–326. MR **37** #2309.

6. J. Pickands III, *Maxima of stationary Gaussian processes*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **7** (1967), 190–223. MR **36** #955.

7. M. G. Šur, *On the maximum of a Gaussian process*, Theor. Probability Appl. **10** (1965), 354–357.

8. T. Sirao, *On the continuity of Brownian motion with a multidimensional parameter*, Nagoya Math. J. **16** (1960), 135–156. MR **22** #8570.

9. T. Sirao and T. Nisida, *On some asymptotic properties concerning Brownian motion*, Nagoya Math. J. **4** (1952), 97–101. MR **13**, 761.

10. T. Sirao and H. Watanabe, *On the upper and lower class for stationary Gaussian processes*, Trans. Amer. Math. Soc. (to appear.)

11. H. Watanabe, *An asymptotic property of Gaussian stationary processes*, Proc. Japan Acad. **44** (1968), 895–896.

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