APPROXIMATION BY POLYNOMIALS SUBORDINATE TO A UNIVALENT FUNCTION(¹)

BY

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This paper is concerned with approximating a function \( f(z) \) analytic and univalent in the unit disk \( E = \{ z : |z| < 1 \} \) by polynomials which are also univalent in \( E \). We are interested in such approximations where exactly one polynomial of each degree \( n, n \geq 1 \), is used and such that the polynomials are monotonically subordinate to each other.

Recall that \( f(z) \) is called subordinate to \( g(z) \) in \( E \) if both functions are analytic in \( E \) and if there exists a function \( \varphi(z) \) analytic in \( E \) and satisfying 

\[
|\varphi(z)| < 1, \quad \varphi(0) = 0
\]

such that \( f(z) = g(\varphi(z)) \). The existence of such a function \( \varphi(z) \) is implied by the conditions that \( g(z) \) is univalent in \( E \), \( f(0) = g(0) \) and \( f(E) \subseteq g(E) \). If \( f(z) \) is subordinate to \( g(z) \) in \( E \) we will write \( f(z) \preceq g(z) \) in \( E \).

Theorem 1 asserts that if \( f(z) \) is analytic and univalent in \( E \) then there exists a sequence \( \{ p_n(z) \} \), \( n = 1, 2, \ldots \), such that \( p_n(z) \) is a polynomial of degree \( n \) which is univalent in \( E \),

\[
p_1(z) \preceq p_2(z) \preceq p_3(z) \preceq \cdots
\]

and \( p_n(z) \to f(z) \) as \( n \to \infty \). This convergence is uniform in each compact subset of \( E \). The idea of proving this is to appropriately relate \( f(z) \) to the partial sums \( s_n(z) \) of the power series for \( f(z) \). Although \( s_n(z) \) are not, in general, univalent in \( E \) (they are in \( |z| < \frac{1}{2} \) [12]) they are univalent in \( |z| < r, 0 < r < 1 \), for all large \( n \) once \( r \) is given. This eventually leads to a chain of univalent polynomials

\[
p_{n_1}(z) \preceq p_{n_2}(z) \preceq p_{n_3}(z) \preceq \cdots
\]

such that \( p_{n_j}(z) \) has degree \( n_j, n_1 < n_2 < n_3 < \cdots \), and \( p_n(z) \to f(z) \) as \( j \to \infty \). These polynomials are so related that it is still possible to find polynomials of the remaining degrees which fill in the chain and such that \( p_n(z) \to f(z) \).

This method is equally adaptable when \( f(z) \) maps \( E \) one-to-one onto a convex domain in order to produce a similar chain of convex, univalent polynomials. In fact, our argument may be applied to any one of a number of classes of univalent functions.

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The possibility of such an approximation to convex mappings was implicitly raised by G. Pólya and I. J. Schoenberg in [10]. They considered analytic functions

\[ f(z) = \sum_{k=0}^{\infty} a_k z^k \]

which map \( E \) one-to-one onto a convex domain. For each integer \( n, n \geq 1 \), the de la Vallée Poussin means of \( f(z) \) are defined by

\[ V_n(z) = \frac{n}{n+1} \sum_{k=0}^{n} \frac{n(n-1)}{(n+1)(n+2)} a_{k+1} z^k + \cdots + \frac{n(n-1) \cdots 1}{(n+1)(n+2) \cdots (2n)} a_n z^n. \]

They showed that \( V_n(z) \) is a convex, univalent function, \( V_n(z) \leq f(z) \), and \( V_n(z) \to f(z) \) in \( E \), and conjectured that

\[ V_1(z) \leq V_2(z) \leq V_3(z) \leq \cdots. \]

This conjecture remains open and its validity would yield a somewhat more explicit proof of the existence of such a chain except that we also require \( p_n(z) \) to have the exact degree \( n \).

Disregarding the demand that \( p_n(z) \) be a polynomial, it is easy to find a chain

\[ f_1(z) \leq f_2(z) \leq f_3(z) \leq \cdots \]

such that \( f_n(z) \to f(z) \) and we do not have \( f_n(z) = f(z) \) for all \( n \). One simply sets \( f_n(z) = f(r_n z) \) where \( \{r_n\} \) is an increasing sequence of real numbers in the interval \( 0 < r < 1 \) such that \( r_n \to 1 \). A less trivial situation in the case \( f(z) \) is univalent and convex in \( E \) is obtained by considering the functions

\[ f(z, \alpha) = \frac{1}{\alpha} \int_{0}^{\alpha} f(ze^{i\theta}) \, d\theta. \]

If \( \{\alpha_n\} \) is any sequence of real numbers in the interval \( 0 < \alpha < 2\pi \) which monotonically converges to zero then \( f(z, \alpha_n) \) is univalent and convex, \( f(z, \alpha_1) \leq f(z, \alpha_2) \leq f(z, \alpha_3) \leq \cdots \) and \( f(z, \alpha_n) \to f(z) \) in \( E \) [6].

Interest in (continuous as well as discrete) chains of subordinate functions has occurred elsewhere and we would like to point out the recent papers by C. Pommerenke [7], [8], [9] as well as [1] by A. Bielecki and Z. Lewandowski.

The earliest consideration of this kind presumably is due to C. Carathéodory in his idea of kernels.

**Lemma 1.** Let \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) be analytic in \( E \) and set \( s_n(z) = \sum_{k=0}^{n} a_k z^k \).

Let \( \{n_j\} \) be a strictly increasing sequence of positive integers and let \( \{r_j\} \) be a sequence of real numbers such that \( 0 < r_j < 1 \) and \( r_j \to 1 \) as \( j \to \infty \). Then \( s_{n_j}(r_j z) \to f(z) \) uniformly in each disk \( |z| \leq r, \ 0 < r < 1 \).
Proof. Suppose that $|z| \leq r$ and $0 < r < 1$.

$$|f(z) - s_n(r, z)| = \left| \sum_{k=0}^{n} a_k(1 - r^2)^k z^k + \sum_{k=n+1}^{\infty} a_k z^k \right|$$

$$\leq (1 - r) \sum_{k=0}^{n} k|a_k|r^k + \sum_{k=n+1}^{\infty} |a_k|r^k$$

$$\leq (1 - r) \sum_{k=0}^{\infty} k|a_k|r^k + \sum_{k=n+1}^{\infty} |a_k|r^k.$$

Given a positive number $e$ there is an integer $N$ such that $\sum_{k=n+1}^{\infty} |a_k|r^k < e/2$ if $n \geq N$. Since $n_j \geq j$ this implies that for $j \geq N$, $\sum_{k=n_j+1}^{\infty} |a_k|r^k < e/2$. As $r_j \to 1$ there is an integer $N'$ such that if $j \geq N'$ then

$$(1 - r_j) \sum_{k=0}^{n} k|a_k|r^k < \frac{e}{2}.$$

Choosing $N'' = \max(N, N')$ we see that $|f(z) - s_n(r, z)| < e$ for $j \geq N''$ and this proves the lemma.

Theorem 1. Let $f(z)$ be analytic and univalent in $E$. There exists a sequence of polynomials $\{p_n(z)\}, n = 1, 2, 3, \ldots$, where $p_n(z)$ has degree $n$ and is univalent in $E$, such that

$$p_1(z) \subset p_2(z) \subset p_3(z) \subset \cdots$$

in $E$ and $p_n(z) \to f(z)$ in $E$. The convergence is uniform in compact subsets of $E$.

Proof. Let the power series for $f(z)$ be $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and let $s_n(z) = \sum_{k=0}^{n} a_k z^k$.

Then $s_n(z) \to f(z)$ and $s_n(z) \to f'(z)$ as $n \to \infty$ uniformly in each disk $|z| \leq r$, where $0 < r < 1$.

Let us introduce the notation:

$$\Delta_g(u, v) = (g(u) - g(v))/(u - v), \quad \text{if } u \neq v,$$

$$= g'(u), \quad \text{if } u = v,$$

for each function $g(z)$ analytic in $E$. If $u \neq v$, $|u| \leq r$, $|v| \leq r$, $0 < r < 1$, we may write

$$g(u) - g(v) = \int_v^u g'(z) \, dz,$$

where the path of integration is the line segment from $v$ to $u$. Since the points on this line segment satisfy $|z| \leq r$, it is clear that

$$\max_{|u| \leq r, |v| \leq r} |\Delta_g(u, v)| = \max_{|u| \leq r} |g'(z)|.$$

Applying this to $g_n(z) = f(z) - s_n(z)$ we conclude that $\Delta_{g_n}(u, v)$ is uniformly small for large $n$ if $|u| \leq r$, $|v| \leq r$, since $g_n(z)$ is uniformly small for large $n$ in $|z| \leq r$.

Since $f(z)$ is univalent in $E$, $f'(z)$ does not vanish, and, therefore,

$$m_j(r) = \min_{|u| \leq r, |v| \leq r} |\Delta_j(u, v)| > 0,$$
where $0 < r < 1$. Because $\Delta_{n_1}(u, v) = \Delta_1(u, v) - \Delta_{n_1}(u, v)$ there exists an integer $N$ such that if $n \geq N$ then $m_{n_1}(r) \geq m_1(r)/2$. In particular, this implies that $s_n(z)$ is univalent in $|z| \leq r$ for all $n \geq N$.

Let $D(r)$ denote the image of $|z| < r$ under $f(z)$ and let $D_s(r)$ denote the image of $|z| < r$ under $s_n(z)$, where $0 < r \leq 1$. Also, let $d(A, B)$ denote the distance between the sets $A$ and $B$ and let $\delta A$ denote the boundary of the set $A$. If we set $d(r) = d(D(r), \delta D(1))$, where $0 < r < 1$, then $d(r) \rightarrow \infty$ since $D(r)$ is bounded and $D(1)$ is not the whole plane. Also, $d(r) > 0$ as $f(z)$ is an open mapping. There is an integer $N'$ such that

$$|f(z) - s_n(z)| < \frac{d(r)}{2}$$

for $|z| \leq r$, $0 < r < 1$, if $n \geq N'$. Consequently, if $n \geq N'$ then $d_n(r) = d(D_n(r), \delta D(1)) > d(r)/2$.

Let $r_1$ be any fixed number such that $0 < r_1 < 1$ and choose an integer $n_1 \geq \max(N, N')$, where $N = N(r_1)$, $N' = N'(r_1)$ was obtained by the two previous arguments. This implies that $m_{n_1}(r_1) \geq m_1(r_1)/2$ and $d_{n_1}(r_1) > d(r_1)/2$.

We continue our argument with the assumption that $f(z)$ is not a polynomial. Then the selection of $n_1$ can be made so that $s_{n_1}(z)$ has exactly the degree $n_1$. If $E_n(r) = f^{-1}(D_n(r))$ then there is a number $\rho_1$, $0 < \rho_1 < 1$, such that $E_{n_1}(r_1)$ is contained in $|z| \leq \rho_1$, as, otherwise, $d_n(r_1) = 0$. Choose the number $r_2$ such that $r_1 < r_2 < 1$ and $r_2 > (1 + r_1)/2$. Then $d(D_{n_1}(r_1), \delta D(r_2)) > 0$ and, as always, $d(r_2) > 0$.

Arguing as before we conclude that there is an integer $N = N(r_2)$ such that if $n \geq N$ then $m_{n_2}(r_2) \geq m_2(r_2)/2$. Also there is an integer $N' = N'(r_2)$ such that if $n \geq N'$ then $|f(z) - s_n(z)| < d^*/2$ for $|z| \leq r_2$, where

$$d^* = \min(d(r_2), d(D_{n_1}(r_1), \delta D(r_2))).$$

This implies that $d_{n_2}(r_2) > d(r_2)/2 > 0$ and

$$d(D_{n_1}(r_1), \delta D_{n_2}(r_2)) > \frac{1}{2}d(D_{n_1}(r_1), \delta D(r_2)) > 0$$

for $n \geq N'$. The last condition and the facts that $D_{n_1}(r_1)$ is connected and $D_{n_2}(r_2)$ have a common point, namely $a_0$, implies that $D_{n_1}(r_1) = D_{n_2}(r_2)$.

If we set $N^* = \max(N, N')$ then for $n \geq N^*$ $m_{n_2}(r_2) \geq (m_2(r_2)/2)$, $d(D_{n_1}(r_1), \delta D_{n_2}(r_2)) > 0$ and $d_{n_2}(r_2) > 0$. The first two conditions and $s_{n_2}(0) = s_{n_1}(0) = a_0$ show that $s_n(2z)$ is univalent in $|z| \leq 1$ and $s_{n_2}(r_1z) = s_{n_2}(r_2z)$ in $E$. Since $f(z)$ is not a polynomial there is such an integer $n = n_2$ so that $n_2 \geq N^*$, $n_2 > n$, and $s_{n_2}(z)$ has degree $n_2$. Moreover, since $d_{n_2}(r_2) > 0$ this whole argument may be repeated.

Namely, we first consider the set $E_{n_2}(r_2)$ to show that there is a number $r_3$ such that $(1 + r_2)/2 < r_3 < 1$ and $d(D_{n_2}(r_2), \delta D(r_3)) > 0$.

We then find an integer $N$ associated with the condition $m_{n_2}(r_3) \geq m_1(r_3)/2$ and an integer $N'$ associated with the inequality $|f(z) - s_n(z)| < d^*/2$ for $|z| \leq r_3$, where

$$d^* = \min(d(r_3), d(D_{n_2}(r_2), \delta D(r_3))).$$
This leads to an integer $n_3$ so that $n_3 > n_2$, $s_{n_3}(z)$ has degree $n_3$, $m_{n_3}(r_3) \geq m_f(r_3)/2$, $d(D_{n_3}(r_3), \delta D_{n_3}(r_3)) > 0$, and $d_{n_3}(r_3) > 0$. In particular, $s_{n_3}(r_3z)$ is univalent in $|z| \leq 1$ and $s_n(r_3z) < s_n(r_3z)$ in $E$.

This argument, therefore, can be continued indefinitely and it yields the following conclusion. There is a sequence of real numbers $\{r_j\}$, a strictly increasing sequence of integers $\{n_j\}$ and a sequence of polynomials $\{p_{n_j}(z)\}$ such that $0 < r_1 < 1$, $r_j \to 1$ and $p_{n_j}(z) = s_{n_j}(r_jz)$ are univalent in $|z| \leq 1$. Moreover, $p_{n_j}(z)$ has degree $n_j$ and

$$p_{n_1}(z) < p_{n_2}(z) < p_{n_3}(z) < \ldots$$

in $E$. In addition,

$$m_{p_{n_1}}(1) \geq m_f(r_1)/2$$

and

$$d(D_{n_1}(r_1), \delta D(r_{j+1})) > 0.$$ 

The fact that $r_j \to 1$ follows from the demand that

$$(1+r_j)/2 < r_{j+1} < 1.$$ 

We also note that because of Lemma 1 $p_{n_j}(z) \to f(z)$ uniformly in every disk $|z| < r$, $0 < r < 1$.

The remaining part of our argument consists of interjecting appropriate polynomials between successive pairs in the chain

$$p_{n_1}(z) < p_{n_2}(z) < p_{n_3}(z) < \ldots.$$ 

Suppose that $p(z)$ and $q(z)$ are such a pair and have degrees $n$ and $k$, respectively, where $n < k$. Then $m_{p}(1) > 0$ and therefore $p(z)$ is univalent in $|z| < R$ for some $R > 1$. If $1 < \rho < R$ and $P(z) = p(\rho z)$ then $m_P(1) > 0$. Let

$$Q(z) = P(z) + b z^{n+1}$$ 

so that

$$|\Delta_Q(u, v)| \geq m_P(1) - (n+1)|b|$$

in $|z| \leq 1$. This implies that for all sufficiently small values of $b$

$$m_q(1) \geq \frac{1}{2} m_P(1)$$

so that, in particular, $Q(z)$ is univalent in $E$. If $A = p(E)$ and $B = q(E)$ then we have the condition $d(A, \delta B) > 0$. Because of the kind of arguments made earlier in this proof it follows that for $p > 1$ and sufficiently near 1 we can conclude that $p(E) \subset P(E) \subset q(E)$ from the relations $d(p(E), \delta P(E)) > 0$, $d(P(E), \delta q(E)) > 0$, and $p(0) = P(0) = q(0)$.

As $Q(z) \to P(z)$ as $b \to 0$ uniformly in $|z| \leq 1$ it is also possible to deduce that

$$d(p(E), \delta Q(E)) > 0 \quad \text{and} \quad d(Q(E), \delta q(E)) > 0$$

for sufficiently small $b$. Choosing $b \neq 0$ and sufficiently small so that these several
conditions hold we see that \( Q(z) \) is univalent in \( E \), has degree \( n+1 \) and \( p(z) \subset Q(z) \) in \( E \). Moreover, \( m\sigma(1) > 0 \) and \( d(Q(E), \delta q(E)) > 0 \) so that the argument that just was applied to \( p(z) \) and \( q(z) \) can again be applied to \( Q(z) \) and \( q(z) \). Continuing in this manner we obtain the chain of univalent polynomials

\[
p(z) \subset p_{n+1}(z) \subset p_{n+2}(z) \subset \cdots \subset p_{k-1}(z) \subset q(z),
\]

where \( p_j(z) \) has degree \( j \). In fact, there is an infinite such chain between \( p(z) \) and \( q(z) \), but for our purposes we end with \( p_{k-1}(z) \).

Now we return to the earlier chain

\[
p_{n_1}(z) \subset p_{n_2}(z) \subset p_{n_3}(z) \subset \cdots .
\]

If \( n_1 \neq 1 \) set \( p_1(z) = a_0 + b_1z \) where \( b_1 \neq 0 \) and is sufficiently small so the closed disk \( |w - a_0| \leq |b_1| \) is contained in \( p_{n_1}(E) \). Between successive pairs of polynomials in this new chain interject appropriate polynomials, as just described. This produces the chain

\[
p_1(z) \subset p_2(z) \subset p_3(z) \subset \cdots ,
\]

where \( p_a(z) \) is univalent in \( E \) and is a polynomial of degree \( n \). Moreover, a subsequence of \( \{p_n(z)\} \), namely \( \{p_n(z)\} \), converges to \( f(z) \) uniformly in compact subsets of \( E \). The proof will be complete if we can show that if \( \{\epsilon_j\} \) is any sequence of positive numbers such that \( \epsilon_j \to 0 \) then the interjected polynomials can be so chosen to fill out the chain

\[
p_{n_1}(z) \subset p_{n_2}(z) \subset p_{n_3}(z) \subset \cdots
\]

so that

\[
|p_a(z) - p_n(z)| < \epsilon_j
\]

for \( z \) in \( E \) and for \( n < n < n_{j+1}, j = 1, 2, \ldots \).

In order to do this consider the previous argument where \( p(z) \) and \( q(z) \) are successive pairs in the sequence \( \{p_n(z)\} \) and have degrees \( n \) and \( k \), respectively, with \( n < k \). That argument yielded real numbers \( R_\mu \) and \( \delta_\mu, \mu = n+1, \ldots, k-1 \), such that \( R_\mu > 1, \delta_\mu > 0 \) and successively defining \( p_\mu(z) \) by

\[
p_{\mu+1}(z) = p_\mu(p_\mu z) + b_\mu z^{\mu+1}
\]

the chain between \( p(z) \) and \( q(z) \) results, where \( p_\mu \) and \( b_\mu \) are restricted only by the conditions \( 1 < p_\mu < R_\mu \) and \( 0 < |b_\mu| < \delta_\mu \). The recursive definitions lead to the formula

\[
p_{\mu}(z) = p_{\mu}(p_{\mu} p_{\mu+1} \cdots p_{\mu-1} z) + b_{\mu+1}(p_{\mu} + p_{\mu+2} \cdots p_{\mu-1}) z^{\mu+1} + b_{\mu+2}(p_{\mu+2} p_{\mu+3} \cdots p_{\mu-1}) z^{\mu+1} + \cdots + b_{\mu+1} p_{\mu+1}^{\mu-1} p_{\mu+2}^{\mu-1} + b_{\mu} z^{\mu} .
\]

Let \( \epsilon \) be any given positive number. Since \( p_\mu(z) \) is uniformly continuous in \( E \) there exist numbers \( \rho_\mu \) such that \( 1 < \rho_\mu < R_\mu \) and the product \( \rho_{n+1} \rho_{n+2} \cdots \rho_{k-1} \) is so close to 1 that

\[
|p_a(z) - p_\mu(p_{\mu} p_{\mu+1} \cdots p_{\mu-1} z)| < \epsilon / 2
\]

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for \( z \in E, \mu = n + 1, \ldots, k - 1 \). Let such a choice of \( \rho_n \) be made. From (1) it follows that if \( z \in E \) then

\[
|p_n(z) - p_n(\rho_n \rho_{n+1} \cdots \rho_{n+\mu} - z)| \leq |b_{n+1}|(\rho_{n+1} \cdots \rho_{n+\mu - 1})^{n+1} + |b_{n+2}|(\rho_{n+2} \cdots \rho_{n+\mu - 1})^{n+2} + \cdots + |b_{\mu} - 1|^{n+1} + |b_{\mu}|
\]

\[
\leq (\rho_{n+1} \rho_{n+2} \cdots \rho_{k-1})^{n+1}(|b_{n+1}| + |b_{n+2}| + \cdots + |b_{\mu}|)
\]

\[
\leq (\rho_{n+1} \rho_{n+2} \cdots \rho_{k-1})^{n+1}(|b_{n+1}| + |b_{n+2}| + \cdots + |b_{k-1}|).
\]

Now choose the numbers \( b_{\mu}, \mu = n + 1, \ldots, k - 1 \), so that \( 0 < |b_{\mu}| < \delta_{\mu} \) and the last expression in the previous line does not exceed \( \epsilon/2 \). This produces an appropriate chain

\[ p(z) \subset p_{n+1}(z) \subset \cdots \subset p_{k-1}(z) \subset q(z) \]

satisfying the additional condition

\[ |p_n(z) - p_n(z)| < \epsilon \]

for \( z \in E, \mu = n + 1, \ldots, k - 1 \). This argument applied to \( p_n(z) \) and \( p_{n+1}(z) \), with \( \epsilon = \epsilon_j \), yields the desired conclusion that \( p_n(z) \to f(z) \) as \( n \to \infty \) uniformly in compact subsets of \( E \). This completes the proof in the case \( f(z) \) is not a polynomial.

The case when \( f(z) \) is a univalent polynomial can be treated in an even simpler way. Suppose that \( f(z) = \sum_{n=0}^N a_n z^n, \ \ \ a_N \neq 0 \), is univalent in \( E \). Let \( \{r_j\} \) be a sequence of real numbers such that \( 0 < r_1 < r_2 < r_3 < \cdots \) and \( r_n \to 1 \) and set \( f_1(z) = f(r_n z) \). Choose \( b_1 \neq 0 \) so small that \( f_1(z) = a_0 + b_1 z \) is subordinate to \( f(z) \) say by demanding the disk \( |w - a_0| \leq |b_1| \) is covered by \( f_1(E) \). The relation between \( p_1(z) \) and \( p_N(z) \) yields the existence of polynomials \( p_\mu(z), \mu = 2, \ldots, N-1 \) univalent in \( E \) such that \( p_n(z) \) has degree \( n \) and

\[ p_1(z) \subset p_2(z) \subset \cdots \subset p_{N-1}(z) \subset p_N(z), \]

as was already shown in an earlier argument in this proof.

Next choose \( b_{N+1} \neq 0 \) so small that

\[ p_{N+1}(z) = f_2(z) + b_{N+1} z^{N+1} \]

is univalent in \( E \) and such that \( d(p_{N+1}(E), \delta f_{N+1}(E)) > 0 \) and \( d(p_{N+1}(E), \delta f_{N+1}(E)) > 0 \).

This is possible since \( f_2(z) \) is univalent in a disk \( |z| < R \) with \( R > 1 \), \( d(f_1(E), \delta f_2(E)) > 0 \) and \( d(f_2(E), \delta f_2(E)) > 0 \).

At the next step we select \( b_{N+2} \neq 0 \) so small that \( p_{N+2}(z) = f_3(z) + b_{N+2} z^{N+1} \) is univalent in \( E \) and such that \( d(p_{N+2}(E), \delta f_{N+2}(E)) > 0 \) and \( d(p_{N+2}(E), \delta f_{N+2}(E)) > 0 \).

This process can be continued to yield the chain

\[ p_1(z) \subset p_2(z) \subset p_3(z) \subset \cdots, \]

where each \( p_n(z) \) is a polynomial of degree \( n \) univalent in \( E \). Moreover, given any sequence of positive real numbers \( \{\epsilon_j\} \) such that \( \epsilon_j \to 0 \) we can choose the numbers \( b_{N+j} \), such that \( |b_{N+j}| < \epsilon_j \). This ensures that \( p_n(z) \to f(z) \) uniformly in \( E \).
THEOREM 2. Suppose that $f(z)$ is analytic and univalent in $E$ and maps $E$ onto a convex domain. There exists a sequence $\{p_n(z)\}$, $n=1, 2, \ldots$, such that $p_n(z)$ is a polynomial of degree $n$ that maps $E$ one-to-one onto a convex domain,

$$p_1(z) \subseteq p_2(z) \subseteq p_3(z) \subseteq \cdots$$

in $E$, and $p_n(z) \to f(z)$ uniformly in each compact subset of $E$.

**Proof.** A function $g(z)$ analytic in $E$ and satisfying $g'(0) \neq 0$ maps $E$ one-to-one onto a convex domain if and only if

$$\text{Re} \left\{ \frac{z g''(z)}{g'(z)} + 1 \right\} > 0$$

for $z$ in $E$. Thus, if $0 < r < 1$ then such a function satisfies

$$m^*_n(r) = \min_{|z| \leq r} \text{Re} \left\{ \frac{z g''(z)}{g'(z)} + 1 \right\} > 0.$$

The proof of this theorem is essentially the same as that given for Theorem 1 where the role of $m_g(r)$ in that theorem is now replaced by $m_g^*(r)$. Again letting $s_n(z)$ be the $n$th partial sum of the power series for $f(z)$, we note that $s_n(z) \to f(z)$ and $s'_n(z) \to f'(z)$ uniformly in $|z| \leq r$, $0 < r < 1$. Consequently, there exists an integer $N=N(r)$ such that if $n = N$ then

$$m^*_n(r) \geq m^*_r(r)/2.$$

In particular, such functions $s_n(z)$ are univalent and convex in $|z| \leq r$. Once this can be done it is clear that the other parts of the argument in Theorem 1 can be combined with this to yield (in the case $f(z)$ is not a polynomial) the polynomials $p_n(z) = s_n(rz)$ such that $0 < r_1 < 1$, $r_j \to 1$, $p_{n_j}(z)$ is univalent and convex in $E$ and has degree $n_j$, and $n_1 < n_2 < n_3 < \cdots$. Also,

$$p_{n_1}(z) \subseteq p_{n_2}(z) \subseteq p_{n_3}(z) \subseteq \cdots$$

in $E$,

$$m^*_{p_{n_j}}(1) \geq m^*_r(r)/2$$

and

$$d(D_{n_j}(r_j), \delta D(r_{j+1})) > 0.$$

In order to interject convex univalent polynomials between two polynomials $p(z)$ and $q(z)$ which are consecutive in the sequence $\{p_n(z)\}$ first notice that $m^*_p(1) > 0$ and therefore $p(z)$ is univalent and convex in $|z| < R$ for some $R > 1$. Setting $P(z) = p(\rho z)$ for $1 < \rho < R$ we see that $m^*_p(1) > 0$, and, consequently, if

$$Q(z) = P(z) + b z^{n+1}$$

then

$$m^*_Q(1) \geq \frac{1}{2} m^*_p(1) > 0$$

for all sufficiently small values of $b$. This implies that $Q(z)$ is univalent and convex.
in $E$. All the other arguments of Theorem 1 are applicable. In particular, at this point the relations between $p(z)$ and $q(z)$ now hold between $Q(z)$ and $q(z)$ so that the argument may be repeated. This produces the appropriate chain

$$p_1(z) \leq p_2(z) \leq p_3(z) \leq \cdots,$$

and, as before, a more careful construction of the interjected polynomials shows that such polynomials can be chosen which also satisfy $p_n(z) \to f(z)$ uniformly in compact subsets of $E$.

This outlines the proof when $f(z)$ is not a polynomial, and an argument like that given at the end of Theorem 1 takes care of the case when $f(z)$ is a polynomial.

**Remarks.** 1. The method used to prove Theorems 1 and 2 also applies if $f(z)$ maps $E$ one-to-one onto a domain starlike with respect to the point $f(0)$. This yields univalent polynomials $\{p_n(z)\}$ which are starlike with respect to $f(0)$. One simply needs to note that a function $g(z)$ analytic in $E$ and satisfying $g'(0) \neq 0$ is univalent and starlike (with respect to $g(0)$) in $E$

$$\Re \left\{ \frac{zg'(z)}{g(z) - g(0)} \right\} > 0.$$ 

The role of $m_0(r)$ in Theorem 1 is now replaced by

$$\min_{z \in \mathbb{D}} \Re \left\{ \frac{zg'(z)}{g(z) - g(0)} \right\}.$$

One can readily state suitable general conditions on a given function $f(z)$ analytic and univalent in $E$ in order that the ideas of Theorem 1 yield a proof of the existence of such polynomials which additionally have the given property of $f(z)$. One such general statement is the following.

Let $w = F(u_1, u_2, \ldots, u_n)$ be defined for certain complex $n$-tuples $(u_1, u_2, \ldots, u_n)$. Then $f(z)$ has a certain "given property" if

$$w = G(z) = F(z, f(z), f'(z), \ldots, f^{(n-1)}(z))$$

is a nonconstant analytic function in $E$ and if the values of $G(z)$ are assumed to lie in a given open set. This includes as special cases starlike and convex (univalent) functions; namely, the value region for $G(z)$ in each case is $\Re w > 0$ and the functions $w = F(u_1, \ldots, u_n)$ are

$$w = u_1u_2/(u_2 - f(0)) \quad \text{and} \quad w = u_1u_2/u_3 + 1,$$

respectively. This idea also can be expressed if $G(z)$ is replaced by a function of several variables. In particular, this would include the case of all univalent functions (Theorem 1). There the value region is $w \neq 0$ for the function $\Delta_2(u, v)$.

Finally, we mention one more example of this type of approximation, namely for the case $f(z)$ is close-to-convex. Such functions are defined by the conditions that they be analytic in $E$ and that there exist an analytic, univalent and convex
function \( g(z) \) such that \( \text{Re} \left\{ \frac{f'(z)}{g'(z)} \right\} > 0 \) in \( E \). Such functions \( f(z) \) are univalent in \( E \) [4]. If \( 0 < r < 1 \) then \( |g'(z)| \geq m_0(r) \) and, consequently,

\[
\left| \frac{f'(z)}{g'(z)} - \frac{s_n'(z)}{g'(z)} \right| \leq \frac{1}{m_0(r)} |f'(z) - s_n'(z)|
\]

for \( |z| \leq r \). This shows that

\[
s_n'(z)/g'(z) \rightarrow f'(z)/g'(z)
\]

uniformly in \( |z| \leq r \). Setting

\[
\tilde{m}_f(r) = \min_{|z| \leq r} \text{Re} \left\{ \frac{f'(z)}{g'(z)} \right\}
\]

we see that \( \tilde{m}_f(r) \) can play the role of \( m_f(r) \) in the argument of Theorem 1. This produces an appropriate chain of polynomials each of which is close-to-convex (relative to \( g(z) \)) in \( E \).

2. The proof of Theorem 1 is very constructive and when combined with precise inequalities for univalent functions gives specific information on how well \( p_n(z) \) approximates \( f(z) \). What is sought is an estimate on

\[
\max_{|z| \leq r} |f(z) - p_n(z)|
\]

in terms of \( n \) and \( r \). We may assume that \( f'(0) = 1 \), as otherwise the factor \( |f'(0)| \) will generally appear. It is known that \( s_n(z) \) are univalent in

\[
|z| < 1 - 6(\log n)/n
\]

for \( n \geq 17 \) [5], and, in fact, this can be slightly improved using the more recent result, \( m_f(r) \geq (1 - r^2)/r^2 \) (see [2, p. 120] or [3]). The quantity \( d(r) \) can also be estimated from below in a fairly simple fashion to get

\[
d(r) \geq \frac{1}{4}((1-r)/(1+r))^2.
\]

One only needs consider the interplay of these numbers \( 1 - 6(\log n)/n \) and \( \frac{1}{4}((1-r)/(1+r))^2 \), noticing that the interjected polynomials bear no significance as the sequence \( \{e_j\} \) in the proof is arbitrary. One finally applies these relations between \( r_j \) and \( n_j \) into the proof of Lemma 1 along with the estimate \( |a_k| < e_k \). A similar procedure is possible for other classes of univalent functions. In particular, use could be made of the results in [11, pp. 404-408].

REFERENCES


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