1. Introduction. If $M$ is a locally flat two-sided PL $m$-manifold in a PL $(m+1)$-manifold $N$ then clearly [2] $M$ can be approximated pointwise by locally flat embeddings from either side. Using a powerful result of Edwards and Kirby [7] we show conversely that $M$ has a collar on one side if $M$ can be approximated by locally flat embeddings from that side. As an application it follows that $M$ is locally flat (even if $M$ is one-sided in $N$) if $N \setminus M$ is 1-LC at each point of $M$, $M$ can be approximated by locally flat embeddings, and $m \geq 4$.

Let $I$ denote the interval $[0, 1]$ and $\text{Id}$ the identity mapping. Throughout we assume that $M$ is a closed PL $m$-manifold, $N$ is a PL $n$-manifold, $n = m+1$, and $M$ is topologically embedded in $N^\circ$ with two sides. We choose a metric denoted by $d$ on $N$ and on $M \times [-1, 1]$ we choose the product metric $\rho$. In case $A$, $B$ are subsets of $N$ and $h$ is a homeomorphism of $N$ we say that $h$ is an $\varepsilon$-push of $(N, A)$ keeping $B$ fixed if there is an isotopy $h_t$ of $N$ such that $h_0 = \text{Id}$, $h_1 = h$, and for each $t \in I$ $h_t$ is the identity on $B$ and outside the $\varepsilon$-neighborhood of $A$ and $d(h_t, \text{Id}) < \varepsilon$.

2. Preliminary results. Our proof depends heavily upon the following result of Edwards and Kirby [7].

**Lemma 1.** For each $\varepsilon > 0$ there is a $\delta > 0$ such that if $h: M \times [-\frac{1}{2}, \frac{1}{2}] \rightarrow M \times [-1, 1]$ is an embedding within $\delta$ of $\text{Id}|M \times [-\frac{1}{2}, \frac{1}{2}]$ then there is an isotopy $g_t: M \times [-\frac{1}{2}, \frac{1}{2}] \rightarrow M \times [-1, 1]$ such that $g_0 = h$, $g_1|M \times 0 = h|M \times 0$, and for each $t \in I$ $g_t|M \times \{-\frac{1}{2}, \frac{1}{2}\} = h|M \times \{-\frac{1}{2}, \frac{1}{2}\}$ and $\rho(g_t, \text{Id}) < \varepsilon$.

Thus we may define an isotopy $f_t$ on $M \times [-1, 1]$ by $f_t = g_t h^{-1}$ on $h(M \times [-\frac{1}{2}, \frac{1}{2}])$ and $f_t = \text{Id}$ elsewhere. Clearly $f_0 = \text{Id}$, $f_1 h = \text{Id}|M \times 0$, and $\rho(f_t, \text{Id}) < 2\varepsilon$. Now let $k$ be a homeomorphism of $M \times [-1, 1]$ commuting with the projection onto $M$, taking $M \times \frac{1}{2}$ onto $M \times t_0$ for some small $t_0 > 0$, and equal the identity on $M \times [-1, 0]$. Then, given $\varepsilon' > 0$, if $\varepsilon$ is small enough and $h(M \times 0) \subseteq M \times [-1, 0]$ then $H = k f_t k^{-1}|M \times [0, t_0]$ is an embedding satisfying $H|M \times t_0 = \text{Id}|M \times t_0$, $H|M \times 0 = h|M \times 0$, and $\text{diam } H(x \times [0, t_0]) < \varepsilon'$ for each $x \in M$.

Where two disjoint, close embeddings $h_0$, $h_1$ of $M$ into $N$ bound a common complementary domain, let $[h_0, h_1]$ denote the one which is near $M$. Thus the
remarks above can be interpreted as saying that \([h| M \times 0, \text{Id}| M \times 0]\) is an \(e\)-product if \(h\) is sufficiently close to \(\text{Id}| M \times [-\frac{1}{2}, \frac{1}{2}]\). Using radial engulfing we shall show that \(h\) can be chosen close to \(\text{Id}| M \times [-\frac{1}{2}, \frac{1}{2}]\) if \(h| M \times 0\) is close enough to \(\text{Id}| M \times 0\). First we need the codimension three version of Bing’s Engulfing Theorem A of [1]. Thus we adopt the following terminology of Bing. Let \(\{A_a\}\) be a collection of sets in \(N^o\), \(O\) an open subset of \(N\), and \(U\) a neighborhood of \(\overline{O}\). We say that finite \(r\)-complexes of \(U\) can be pulled into \(O\) along \(\{A_a\}\) if for each \(k\)-dimensional polyhedron \(P \subset U\) and each subpolyhedron \(Q \subset O\) such that \(R = \overline{P \setminus Q}\) is compact and \(R \subset N^o\), there is a homotopy \(h: R \times I \to N^o\) such that \(h_0 = \text{Id}, h_t(R) \subset O\), for each \(t \in I\) \(h_t|Q \cap R = \text{Id}| Q \cap R\), and for each \(x \in R\) \(h(x \times I)\) lies in an element of \(\{A_a\}\).

**Lemma 2.** Suppose \(r \leq n - 3\) and \(\{A_a\}\) is a collection of sets in \(N^n\) such that finite \(r\)-complexes in \(U\) can be pulled into \(O\) along \(\{A_a\}\). Then for each compact \(k\)-dimensional polyhedron \(P \subset U\), each \(q\)-dimensional polyhedron \(Q \subset O\) such that \(R = P \setminus Q\) is \(k\), \(q \leq r\), and each \(\epsilon > 0\) there is a homotopy \(h_t\) of \(N^n\) such that \(h_0 = \text{Id}, h_t(O) \supset P\), for each \(t \in I\) \(h_t|Q = \text{Id}| Q\), and for each \(x \in N^n\) either \(h(x \times I)\) is a point or else lies in the \(\epsilon\)-neighborhood of the sum of some \(k + 1\) elements of \(\{A_a\}\) if \(k \leq n - 4\) and some \(k + 3\) elements of \(\{A_a\}\) if \(k = n - 3\).

A proof can be constructed using piping (see Lemma 48 of [11]) and following the proof of Lemma 2.7 of [5].

**Proposition 3.** For each \(\epsilon > 0\) there is a \(\delta > 0\) such that for each pair of disjoint embeddings \(h_0, h_1: M \to N\) within \(\delta\) of \(\text{Id}| M\), there are strong \(\epsilon\)-deformation retractions of \([h_0, h_1]\) onto \(h_0(M)\) and \(h_1(M)\).

**Proof.** It is easy to show using the local contractibility of \(M\) that there is a neighborhood \(U\) of \(M\) and a \(\delta > 0\) such that for any \(\delta\)-embedding \(h: M \to N\), \(U/2\)-retracts onto \(h(M)\) (see Proposition 2 of [3]). Since \(M\) is two-sided \(\delta\) can be chosen so small that \([h_0, h_1]\) is defined for disjoint \(\delta\)-embeddings of \(M\) and there is an \(\epsilon/2\) retraction \(r\) or \(U\) onto \([h_0, h_1]\). Now if \(h_0\) and \(h_1\) are sufficiently close to \(\text{Id}| M\) then there is a homotopy \(r_t\) of \([h_0, h_1]\) in \(U\) with \(r_0 = \text{Id}|[h_0, h_1]\), for each \(t \in I\) \(r_t|h_0(M) = \text{Id}|h_0(M)\), \(d(r_t, \text{Id}| M) < \epsilon/2\), and \(r_t|h_0, h_1) = h_1(M)\). Thus \(rr_t\) is a strong \(\epsilon\)-deformation retraction of \([h_0, h_1]\) onto \(h_1(M)\).

**Proposition 4.** For each \(\epsilon > 0\) there is a \(\delta > 0\) such that for each pair \(h_0, h_1: M \to N\), \(n \geq 5\), of disjoint embeddings within \(\delta\) of \(\text{Id}| M\) each neighborhood \(U\) of \([h_0, h_1]\), and each open set \(O \supset h_0(M)\) there is an \(\epsilon\)-push \(H\) of \((N, M)\) fixed on \(h_0(M)\) and outside \(U\) such that \(H[h_0, h_1) = O\).

**Proof.** The proof is standard using Proposition 3 to construct the sets \(\{A_a\}\) of Lemma 2. Then using the standard dual skeleton argument \(H\) is constructed by pushing \(O\) out over the \((n - 3)\)-skeleton across to the dual 2-skeleton and then out over the rest of \([h_0, h_1]\). The first and last pushes are made using Lemma 2 and the
middle push (see Lemma 8.1 of [9]) preserves simplexes of some triangulation of $N$; thus $H$ can be made an $\varepsilon$-push of $(N, M)$ fixed outside $U$.

**Lemma 5.** For each $\varepsilon > 0$ there is a $\delta > 0$ such that if $h: M \times 0 \rightarrow M \times [-1, 1]$ is a locally flat embedding within $\delta$- of $\text{Id}|M \times 0$ such that $h(M \times 0) \cap M \times 0 = \emptyset$ then there is a homeomorphism $H: M \times I \rightarrow [M \times 0, h]$ such that $H_0 = \text{Id}|M$, $H_1 = h$, and $\text{diam } H(x \times I) < \varepsilon$ for each $x \in M$.

**Outline of Proof.** The proof is implicit in Wright's proof in [10]. From the remarks following Lemma 1 it is sufficient to show that $h$ can be extended to $M \times [-\frac{1}{2}, \frac{1}{2}]$ so that $\rho(h, \text{Id}|M \times [-\frac{1}{2}, \frac{1}{2}]) < \delta$ where $\delta$ is given by Lemma 1 with $\varepsilon$ replaced by some positive number depending on $\varepsilon$. Thus we need only to produce a $\delta > 0$ such that for each locally flat $\delta$-approximation $h$ of $\text{Id}|M \times 0$, each extension of $h$ to $M \times [-\frac{1}{2}, \frac{1}{2}]$ so that for each $t \in [-\frac{1}{2}, \frac{1}{2}]\rho(h(x, t), (x, 0)) < \delta$, and each number $t_0 \in (0, 1)$ there is an $\varepsilon$-push $H$ of $(M \times [-1, 1], M \times 0)$ fixed on $h(M \times [-1, t_0])$ such that $h(M \times t_0)$ is separated from $h(M \times 1)$ by $M \times t'$ for some $t' \in (-\varepsilon, \varepsilon)$. However for $\delta$ chosen by Proposition 4, an embedding $h: M \times [-\frac{1}{2}, \frac{1}{2}] \rightarrow M \times [-1, 1]$ such that $\rho(h(x, t), (x, 0)) < \delta$ for all $t \in [-\frac{1}{2}, \frac{1}{2}]$, and $0 < t_0 < t_1 < \frac{1}{2}$ we apply Proposition 4 with $h_0(x) = (x, \lambda)$ for each $x \in M$ ($\lambda = \delta$ if $h(M \times 0)$ is separated from $M \times \delta$ by $h(M \times \frac{1}{2})$ and $\lambda = -\delta$ in the other case), $h_1(x) = h(x, t_1)$ for all $x \in M$, $U =$ the component of $M \times [-1, 1]h(M \times t_0)$ containing $[h_0, h_1]$, and $0 =$ the component of $M \times [-1, 1]\setminus h(M \times t_0)$ which does not contain $M \times 0$ where $0 < \eta < \lambda$ and $[M \times \eta, h_0] \subseteq [h_0, h_1]$. Thus there is an $\varepsilon$-push $G$ of $(M \times [-1, 1], M \times 0)$ fixed outside $U$ (and hence on $h(M \times t_0)$) such that $G[h_0, h_1] \subseteq O$ and therefore $Gh(M \times (t_0, t_1))$ contains $M \times \eta$. The embedding $H$ is obtained now by choosing a fine partition $0 = t_0 < t_1 < \cdots < t_k = \frac{1}{2}$ and an embedding $H': M \times [-\frac{1}{2}, \frac{1}{2}] \rightarrow M \times [-\delta, \delta]$ extending $h|M \times 0$ such that for each $i, H'(M \times (t_i, t_{i+1})) \supseteq M \times \eta_i$ for some $\eta_i \in (-\delta, \delta)$ and diam $H'(M \times (t_i, t_{i+1}))$ is small. Finally move $\eta_i$ to $\pm i/2k$ by a homeomorphism $F$ of $M \times [-1, 1]$ leaving $h(M \times 0)$ fixed. We define $H =FH'$.

This completes the outline of the proof.

3. **The main results.**

**Theorem 6.** Suppose $M$ is a closed PL $m$-manifold, $m \geq 4$, $N$ is a PL $n$-manifold, and $n = m + 1$. If $M$ is topologically embedded in the interior of $N$ as a two-sided subset then $M$ has a collar on one side if $M$ can be pointwise approximated by locally flat embeddings on that side.

**Proof.** Clearly it is sufficient to show that for any $\varepsilon > 0$ there is a $\delta > 0$ such that for any pair of disjoint locally flat embeddings $h_0, h_1: M \rightarrow N$ within $\delta$ of $\text{Id}|M$, there is a homeomorphism $H: M \times I \rightarrow [h_0, h_1]$ such that $H_0 = h_0$, $H_1 = h_1$, and $\text{diam } H(x \times I) < \varepsilon$ for each $x \in M$. $H$ is constructed roughly as follows: Using a sequence of small engulfings move collars on $h_0(M)$ and $h_1(M)$ so that the whole collars are very close. Then apply local contractibility [7] to make the collars close. 

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agree on a little stretch in the middle. Thus $[h_0, h_1]$ is homeomorphic to $M \times I$ by the standard push-pull technique. We now give a rigorous argument.

First select $\delta_1 > 0$ and $\eta_1 > 0$ so that $d(x, y) < \eta_1$, $x, y \in M \Rightarrow d(h(x), h(y)) < \varepsilon/3$ for any $\delta_1$-embedding of $M$ into $N$. Next pick $\eta_2$ using Lemma 5 with $\varepsilon$ replaced by $\eta_1$. Now there are $\delta_2 > 0$ and $\delta_3 > 0$ such that for any $\delta_2$-approximation $h$ of $\text{Id}/M$ and any pair $(x, y)$ of points in $h(M)$ with $d(x, y) < 3\delta_3$, $d(h^{-1}(x), h^{-1}(y)) < \eta_2$.

Finally we choose $\delta_4 > 0$ using Proposition 4 with $\varepsilon$ replaced by $\min \{\varepsilon/3, \delta_3\}$. Let $\delta = \min \{\delta_1, \delta_2, \delta_3, \delta_4\}$.

Now let $h_0, h_1 : M \to N$ be disjoint locally flat embeddings within $\delta$ of $\text{Id}|M$. We can now assume that $\delta$ is so small that $h_0(M)$ is two-sided; then identifying $M$ with $\mathbb{R}/\partial M$ extend $h_0$ to an embedding (still denoted $h_0$) of $M \times [0, 1]$ into $N$ such that

$$x, y \in M \times [0, 1], \rho(x, y) < \eta_1 \Rightarrow d(h_0(x), h_0(y)) < \varepsilon/3.$$ 

Let $O \subseteq h_0(M \times [0, 1])$ be a neighborhood of $h_0(M)$ so small that $x, y \in O$, $d(x, y) < 3\delta_3 \Rightarrow \rho(h_0^{-1}(x), h_0^{-1}(y)) < \eta_2$. Now apply Proposition 4 to obtain a min $\{\varepsilon/3, \delta_3\}$-push $F$ of $(N, M)$, fixed on $h_0(M)$, such that $F[h_0, h_1] \subseteq 0$. Then $d(h_0(x), Fh_1(x)) \leq d(h_0(x), h_1(x)) + d(h_1(x), Fh_1(x)) < 3\delta_3$. Thus $\rho(x, h_0^{-1}Fh_1(x)) < \eta_2$.

Therefore we can apply Lemma 5 and obtain an embedding $G : M \times I \to M \times [-1, 1]$ such that $G_0 = \text{Id}|M$, $G_1 = h_0^{-1}Fh_1$, and $\text{diam} G(x \times I) < \eta_1$. Thus $\text{diam} h_0 G(x \times I) < \varepsilon/3$. Now define $H$ to be $F^{-1}h_0 G$. Then $H_0 = F^{-1}h_0 G_0 = F^{-1}h_0 = h_0$, $H_1 = F^{-1}h_0 G_1 = F^{-1}h_0 h_0^{-1}Fh_1 = h_1$, and for each $x \in M$, $H(x \times I) = F^{-1}(h_0 G(x \times I))$ has diameter < $\varepsilon$. This completes the proof of Theorem 6.

For the proof of the next theorem we need one more preliminary result in order to apply Theorem 2.

**Proposition 7.** Suppose that the closed m-manifold $M$ embedded in the interior of $N^n$, $n = m + 1$, separates $N$ into two components $U$ and $V$ and $U$ is 1-ULC. For each $\varepsilon > 0$ there is a neighborhood $O$ of $U \cup M$ such that for each closed set $C \subseteq U$ there is a closed set $B$, $B \subseteq U$, and a homotopy $h_t$ of $O$ in $N$ such that:

1. $h_0 = \text{Id}$,
2. $h_1(O) \subseteq U$,
3. $h_t|B = \text{Id}|B$ for each $t \in I$,
4. $h_t(O \setminus B) \cap C = \emptyset$ for each $t \in I$, and
5. $\text{diam } h_t(\text{Id}) < \varepsilon$ for each $t \in I$.

**Proof.** The proof is in two steps.

**Step 1.** There is a neighborhood $P$ of $N \setminus U$ such that for any closed set $C \subseteq U$ there is a closed neighborhood $D$ of $N \setminus U$, $C \cap D = \emptyset$, and an $\varepsilon/2$-map $r$ of $P$ into $N \setminus C$ such that $r[D] = \text{Id}|D$ and $r(P \cap U) \subseteq U \setminus C$.

**Step 2.** There is a neighborhood $O'$ of $U \cup M$ such that for any closed set $B \subseteq U$ there is a closed set $E$, $B \subseteq E \subseteq U$, and a strong $\varepsilon/2$-retraction $r_t$ of $O'$ onto $E$ such that $r_t(N \setminus E) \subseteq P$ for each $t \in I$. 

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Now define \( h_t(x) = x \) for \( x \in B \) and \( t \in I \) and \( h_t(x) = r(x) \) for \( x \in O \cap D \). Since \( r(x) \in P \) for \( x \in O \cap D \), \( h_t(x) \) is defined for all \( t \in I \) and \( x \in O \cap D \). Since \( r: B \cap D = \partial D \cap B \), \( h_t \) is continuous for all \( t \in I \). It is clear that \( h_t \) satisfies the conclusion of the proposition. Next we prove Step 1. Since \( M \) is an ANR and separates \( N \) there is a polyhedral neighborhood \( P \) of \( N \setminus U \) and an \( \varepsilon/8 \)-retraction \( f \) of \( P \) onto \( N \setminus U \). Now let \( \varepsilon = \min \{ \varepsilon/8, d(M, C)/2 \} \). Since \( U \) is 1-ULC, \( U \) is ULC\(^{-1} \). Thus there is a sequence \( 0 < \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_n = \varepsilon \) such that each map \( f: s^i \to U \) with \( \operatorname{diam} f(s^i) < 2\varepsilon_i \) can be extended to a map of the \((i+1)\)-ball \( B^i \) into \( U \) such that \( \operatorname{diam} f(B^{i+1}) < \varepsilon_{i+1} \). Now take a triangulation \( T' \) of \( P \) with mesh less than \( 2\varepsilon_1/3 \) and so small that for each \( \sigma \in T' \) \( \operatorname{diam} f(\sigma) < 2\varepsilon_1/3 \). Since \( f|\partial M = Id|\partial M \) there is a neighborhood \( D' \) of \( N \setminus U \) such that \( d(f|D', Id|D') < 2\varepsilon_1/3 \). Choose a refinement \( T \) of \( T' \) so fine that the simplicial neighborhood \( D \) of \( N \setminus U \) in \( T' \) is contained in \( D' \). Define \( r|D = Id|D \) and extend \( r \) skeletonwise to \( T \) as follows. For each vertex \( a^0 \in \partial D \) pick \( r(a^0) \in U \) satisfying \( d(r(a^0), f(a^0)) < \varepsilon_1/3 \). Then for each 1-simplex \( a^1 \in \partial D \), \( \operatorname{diam} r(a^1) < 2\varepsilon_1 \). Suppose \( r \) has been extended to \( r: D \cup |T^k| \to N \) so that for each \( \sigma^{k+1} \in T \) \( \operatorname{diam} r(\sigma^{k+1}) < 2\varepsilon_{k+1} \). Thus for each \( \sigma^{k+1} \in T \) we can extend \( r|\sigma^{k+1} \) to \( \sigma^{k+1} \) so that \( \operatorname{diam} r(\sigma^{k+1}) < 2\varepsilon_{k+2} \). By induction we have \( r \) defined on all of \( P \) so that \( \operatorname{diam} r(\sigma) < \eta \) for all simplices \( \sigma \in T \). However, \( d(r(\sigma), N \setminus U) < 3\varepsilon_1/3 < \eta \) and \( \operatorname{diam} r(\sigma) < \eta \) imply that \( r(\sigma) \subset N \setminus C \) for each \( \sigma \in T \). Moreover for each point \( p \in P \) there is a vertex \( v \) of \( T \) such that
\[
d(p, r(P)) \leq d(p, v) + d(v, r(v)) + d(r(v), r(p)) < \eta + (\varepsilon/8 + \eta) + \eta < \varepsilon/2.
\]
This completes the proof of Step 1. Step 2 can be proved similarly.

**Lemma 8.** Suppose that \( M^n \) is a closed two-sided submanifold of the interior of \( N^n \), \( n = m + 1 \geq 5 \). If \( N \setminus M \) is 1-ULC then for each \( \varepsilon > 0 \) there is a PL \( \varepsilon \)-push \( H \) of \((N, M)\) such that \( H(M) \cap M = \emptyset \).

**Proof.** Suppose that \( W \) is a connected open neighborhood of \( M \) which is separated into \( U \) and \( V \) by \( M \). Then apply Proposition 7 with \( N \) replaced by \( W \) and \( \varepsilon \) by \( \varepsilon/3n \) to obtain polyhedral neighborhoods \( O_1 \) of \( W \setminus U \) and \( O_2 \) of \( W \setminus V \). Apply Proposition 7 for each closed subset of \( W \setminus M \) containing \( W \setminus O_1 \cap O_2 \) and let \( \{ A_n \} \) be the tracks of all points under all such homotopies of \( O_1 \cap O_2 \) into \( O_1 \cap U \) and \( O_2 \cap V \). Now take a triangulation \( T \) of \( O_1 \cap O_2 \) with mesh less that \( \varepsilon/3 \) and apply Lemma 2 with \( P \) replaced by \( |T^{n-3}| \), \( N \) by \( O_1 \), \( O \) by \( O_1 \cap U \) and \( U \) by \( O_1 \cap O_2 \). Thus there is an \( \varepsilon/3 \)-push \( H_2 \) of \((W, M)\) fixed on \( U \setminus O_1 \) such that \( H_2(U) \supset T^{n-3} \). Similarly there is an \( \varepsilon/3 \)-push \( H_3 \) of \((W, M)\) fixed on \( V \setminus O_2 \) such that \( H_3(V) \supset T^{n-3} \). Using Lemma 8.1 of [9] there is an \( \varepsilon/3 \)-push \( H_3 \) of \((W, M)\) such that \( H_3 H_2(U) \cup H_2(V) = W \). Thus \( H_2^{-1} H_3 H_2(U) \cup V = W \) and \( H = H_2^{-1} H_3 H_2(U) \cup V \). Let \( H = H_2^{-1} H_3 H_2(U) \cup V \). Then \( H \) is an \( \varepsilon \)-push of \((N, M)\) such that \( H(M) \subset U \).

Clearly \( H^{-1}(M) \subset U \) therefore if \( M \) can be approximated sufficiently close by locally flat embeddings then it can be approximated from both sides and so \( M \) is
bicollared. In fact, we can construct a double cover $\tilde{N}$ of $N$ in the case that $M$ is one-sided in $N$ and $\tilde{M}$ (the part of $\tilde{N}$ covering $M$) is two-sided in $\tilde{N}$. Moreover, if $M$ can be approximated then so can $\tilde{M}$. Therefore $\tilde{M}$ is bicollared and so $M$ has a closed normal 1-disk bundle neighborhood. Thus we have the following.

**Theorem 9.** Suppose that $M$ is a closed (possibly one-sided) $m$-manifold in the interior of the $n$-manifold $N$, $n = m + 1 \geq 5$. If $M$ can be pointwise approximated by locally flat embeddings and $N \setminus M$ is 1-ULC, then $M$ has a normal 1-disk bundle neighborhood.

We remark in conclusion that it follows from the annulus conjecture [8], Connell’s approximation theorem [6], and [7] that any locally flat $(n - 1)$-sphere in $S^n$, $n \geq 5$, is $\epsilon$-tame; thus for the case of spheres, Theorems 6 and 9 can be strengthened to $\epsilon$-taming results. In fact, since the locally flat side approximations become levels in the collar, there is an $\epsilon$-taming result if $M$ can be approximated by PL embeddings from both sides. Therefore it follows from the results here and in [4] that a closed PL $m$-manifold $M$ in the interior of an $n$-manifold $N$ is $\epsilon$-tame if its complement is 1-ULC, it can be approximated by PL embeddings, $n \geq 5$, and $m \neq n - 2$. Moreover, given one of these embeddings $h: M \to N$ and an $\epsilon > 0$ there is a $\delta > 0$ such that if $g$ is also one that is within $\delta$ of $h$ then there is an $\epsilon$-push $p$ of $(N, h(M))$ such that $h = pg$.

**References**


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