COUNTABLE PARACOMPACTNESS
AND WEAK NORMALITY PROPERTIES

BY

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In [4], Dowker proved that a normal space $X$ is countably paracompact if and only if its product with the closed unit interval is normal. In this paper, we prove an analogue of Dowker's theorem. Specifically, we define the term $\delta$-normal and then prove the following:

**Theorem 1.** A topological space is countably paracompact if and only if its product with the closed unit interval is $\delta$-normal.

After proving this theorem, we obtain similar results for the topological spaces studied in [7] and [11]. Also, cogent examples are given and the relation this note bears to the work of others is discussed.

We shall follow the terminology of [5] except we shall assume separation properties for a space only when these assumptions are explicitly stated.

For an infinite cardinal $m$, a set $A$ in a topological space will be called a $G_m$-set (respectively, a regular $G_m$-set) provided it is the intersection of at most $m$ open sets (respectively, at most $m$ closed sets whose interiors contain $A$). If $m = \aleph_0$, we shall use the familiar terms $G_\delta$-set and regular $G_\delta$-set.

It is clear that the zero-set of any continuous real valued function is a regular $G_\delta$-set and that the intersection of no more than $m$ such zero-sets is a regular $G_m$-set. In the remaining part of this paper, we shall use these facts without explicitly mentioning them.

**Definition.** For an infinite cardinal $m$, a topological space is $m$-normal if each pair of disjoint closed sets, one of which is a regular $G_m$-set, have disjoint neighborhoods. For $m = \aleph_0$, we shall use the more suggestive term $\delta$-normal.

Note that a normal space is $m$-normal and that a regular space is normal if and only if it is $m$-normal for every infinite cardinal $m$. On the other hand, a compact $T_1$-space that is not Hausdorff is $m$-normal for every infinite cardinal but yet it fails to be normal.

Recall that a space is $m$-paracompact if each open cover having cardinal less than or equal to $m$ has a locally finite open refinement. Characterizations of $m$-paracompact spaces may be found in [14] and [8].

**Theorem 2.** Each $m$-paracompact space is $m$-normal.

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Proof. Suppose \( X \) is \( m \)-paracompact and let \( A \) and \( B \) be disjoint closed sets such that \( B \) is a regular \( G_m \)-set. Then there is a family \( \mathscr{G} \), having cardinal less than or equal to \( m \), consisting of open neighborhoods of \( B \) such that \( B \) is the intersection of \( \{ G : G \in \mathscr{G} \} \). Let \( \Gamma \) be the set of all finite nonempty subfamilies of \( \mathscr{G} \). For \( \alpha \in \Gamma \), define \( G_\alpha \) to be the intersection of \( \{ G : G \in \alpha \} \). Then \( B \subseteq \bigcap G_\alpha \) and \( B = \bigcap G_\alpha \). The family of sets \( X \setminus A \cap G_\alpha \) is a directed open cover of \( X \). By Theorem 5 in [8], there exists a locally finite open cover \( \{ V_\alpha \} \) such that \( V_\alpha \cap A \cap G_\alpha = \emptyset \) for all \( \alpha \in \Gamma \). Now let \( U \) be the union of all sets \( V_\alpha \setminus G_\alpha \) and \( V \) be the union of the sets \( G_\alpha \setminus \{ V_\beta : \beta \not\subseteq \alpha \} \). Clearly, \( U \) is open, contains \( A \) and is disjoint from \( V \). Since \( \{ V_\alpha \} \) is locally finite it follows that \( V \) is open. For \( x \in B \), let \( y = \bigcup \{ \beta : x \in V_\beta \} \). Since the cover \( \{ V_\alpha \} \) is locally finite, \( y \in \Gamma \) and \( x \notin \bigcup \{ V_\beta : \beta \not\subseteq y \} \); whence \( x \in V \). Therefore \( B \subseteq V \) and the proof is complete.

For the important special case where \( m = \aleph_0 \), we have:

**Theorem 3.** Each countably paracompact space is \( \delta \)-normal.

The local weight of a topological space is the least cardinal \( m \) such that each point has a neighborhood base consisting of at most \( m \) elements.

**Theorem 4.** (i) A Hausdorff \( m \)-normal space having local weight \( \leq m \), is regular.

(ii) A Hausdorff \( m \)-normal space having cardinal \( \leq m \), is regular.

**Proof.** Under the hypotheses in each case, a singleton is a regular \( G_m \)-set. For emphasis, we state the following special case:

**Theorem 5.** If a \( \delta \)-normal Hausdorff space is either countable or satisfies the first axiom of countability, then it is regular.

**Corollary 6 (Aull [1]).** Each countably paracompact, first countable, Hausdorff space is regular.

**Example.** For each infinite cardinal \( m \), there is an \( m \)-normal, Hausdorff space which is not regular. Given \( m \), let \( w_a \) be the least ordinal having cardinal greater than \( m \). Denote by \( W^* \) the set of ordinals less than or equal to \( w_a \) and by \( W \), the set \( W^* \setminus \{ w_a \} \). In \( W^* \times W^* \setminus \{ (w_a, w_a) \} \), identify all points of \( W \times \{ w_a \} \). The quotient space \( X \) is Hausdorff but not regular (the images in \( X \) of the upper edge and the diagonal are not separated by disjoint open sets). Nonetheless, \( X \) is \( m \)-compact; hence it is \( m \)-paracompact and \( m \)-normal.

Throughout this paper \( I \) will denote the closed unit interval. In the proof of Theorem 1, we shall use the following lemma.

**Lemma 7.** For any topological space \( X \), the following are equivalent:

(a) \( X \) is countably paracompact.

(b) If \( g \) is a strictly positive lower semicontinuous function on \( X \), then there exist real valued functions \( l \) and \( u \) with \( l \) lower semicontinuous and \( u \) upper semicontinuous such that \( 0 \leq l(x) \leq u(x) \leq g(x) \) for all \( x \in X \).
(c) If $A$ is a closed subset of $X \times I$ and $K$ is closed in $I$ such that $A$ and $X \times K$ are disjoint, then $A$ and $X \times K$ have disjoint neighborhoods.

**Proof.** The equivalence of (a) and (c) is due to Tamano (Theorem 3.9 in [16]) while the equivalence of (a) and (b) is an easy consequence of Theorem 10 in [7].

**Proof of Theorem 1.** If $X$ is countably paracompact, then $X \times I$ is countably paracompact (Theorem 1 in [4]). By Theorem 3 above $X \times I$ is $\delta$-normal. Conversely, suppose $K$ is closed in $I$. Since $I$ is metrizable, $K$ is a regular $G_\sigma$-set. Therefore $X \times K$ is a regular $G_\sigma$-set in $X \times I$. In view of Lemma 7, the $\delta$-normality of $X \times I$ will imply that $X$ is countably paracompact.

**Theorem 8.** A closed continuous image of an $m$-normal space is $m$-normal.

**Proof.** Observe that a continuous inverse image of a regular $G_m$-set is regular $G_m$. Using this fact, the standard proof that a closed continuous image of a normal space is normal, becomes applicable here.

**Remark.** In general, it is not true that preimages of $m$-normal spaces are $m$-normal even for perfect maps (i.e., continuous closed maps for which the preimages of compact sets are compact). Note that in view of Theorem 1 and the fact that a space $X$ is always the perfect image of $X \times I$, it follows that if every perfect preimage of $X$ is $\delta$-normal, then $X$ is countably paracompact.

Many of the standard examples of nonnormal spaces, also, fail to be $\delta$-normal. Here we give a partial list of such examples. (i) The space $S \times S$ where $S$ is the reals with the half-open interval topology [15]. (ii) The space $X \times Y$ constructed by Michael [12]. (iii) The space $R^R$ where $R$ is the space of reals. (iv) The spaces constructed in problems 3K, 5I, 6P, 6Q of [5]. We shall use the space $S \times S$ to illustrate a technique that can be used to verify that these spaces are not $\delta$-normal.

In $S \times S$ let $A$ be the set of points $(x, -x)$ where $x$ is rational and $B$ be the set of such points for irrational $x$. Then $A$ is closed and $B$ is a regular $G_\sigma$-set while these sets do not have disjoint neighborhoods. To show this, one can exploit the fact that the irrationals are not an $F_\sigma$-set in the reals (cf. [12]).

**Definition.** A space will be called $\delta$-normally separated if each closed set and each zero set disjoint from it are completely separated. A space will be termed weakly $\delta$-normally separated if each regular closed set (i.e., the closure of an open set) and zero-set disjoint from it are completely separated.

**Remark.** The properties of being $\delta$-normal and $\delta$-normally separated are, unfortunately, not comparable for arbitrary topological spaces. In a space where every regular $G_\sigma$-set is a zero-set, $\delta$-normal separation implies $\delta$-normality, but not conversely (see the example at the end of this paper). On the other hand, Hewitt's example [6] of an infinite regular Hausdorff space on which each continuous real-valued function is constant, is a $\delta$-normally separated space which is not $\delta$-normal (cf. Remark following Theorem 13). The author does not know whether among completely regular spaces, $\delta$-normal separation implies $\delta$-normality.
Clearly each normal space is $\delta$-normally separated. Likewise, $\delta$-normal separation implies weak $\delta$-normal separation and the converse is true for $\delta$-normal spaces.

The concept of $\delta$-normal separation is not a new one. P. Zenor introduced this idea in [17] and used the term Property Z.

The $\delta$-normal separation of a space $X$ can be characterized in terms of properties of the ring $C(X)$ of a real-valued continuous function on $X$.

**Theorem 9.** A topological space $X$ is $\delta$-normally separated if and only if for each $f \in C(X)$ and each closed set $A$ on which $f$ is strictly positive, there exists a unit $u$ of the ring $C(X)$ such that $fu$ is identically one on $A$.

**Proof.** Assume $X$ is $\delta$-normally separated and that $f$ and $A$ have the given properties. Then there exists a nonnegative element $h$ of $C(X)$ which vanishes on $A$ and assumes the value 1 everywhere on the zero-set of $f$. Then the ring inverse of $\frac{1}{f} + h$ is the desired unit. The converse is obvious.

We shall now proceed to state and prove the analogue of Theorem 1 for the $\delta$-normal separation and weak $\delta$-normal separation properties. To achieve this, we need to recall the definitions of cb-spaces and weak cb-spaces. A space $X$ is a cb-space (respectively, weak cb-space) provided every locally bounded real valued function on $X$ (respectively, every locally bounded lower semicontinuous function on $X$) is bounded above by a continuous function. Information concerning cb-spaces and weak cb-spaces may be found in [7] and [11], respectively. In comparing Theorem 1 with Theorem 11 below, it is useful to remember that a space is cb if and only if it is weak cb and countably paracompact.

**Lemma 10.** (a) Each cb-space is $\delta$-normally separated.

(b) Each weak cb-space is weakly $\delta$-normally separated.

**Proof.** We shall prove (a) and make parenthetical comments to indicate the proof of (b). Let $A$ be closed and $Z$ be a zero-set disjoint from $A$. Given a nonnegative function $h$ in $C(X)$ such that $Z$ is the zero-set of $h$, define $g(x) = 1 + h(x)$ for $x$ not in $A$ and $g(x) = h(x)$ for $x$ belonging to $A$. Then $g$ is lower semicontinuous (normal lower semicontinuous if $A$ is regular closed). Clearly $g$ is strictly positive. By Theorem 1 in [7] (Theorem 3.1 in [11] for (b)) there is a strictly positive real valued continuous function $f$ such that $f \leq g$. Then the function $hf$ completely separates $A$ and $Z$.

**Theorem 11.** Let $X$ be a topological space. Then

(a) $X$ is a cb-space if and only if $X \times I$ is $\delta$-normally separated.

(b) $X$ is a weak cb-space if and only if $X \times I$ is weakly $\delta$-normally separated.

**Proof.** The necessity follows from Lemma 10 above; the sufficiency from Corollary 12 and Theorem 13 in [9].

**Remark,** In both Theorems 1 and 11, $I$ may be replaced by any infinite compact
metric space. Also, note that Theorem 10 in [9] implies that a variation of (a) in the above theorem is valid when \( I \) is replaced by an infinite product of intervals.

**Corollary 12.** (a) Each countably compact space is both \( \delta \)-normal and \( \delta \)-normally separated.

(b) A completely regular, pseudocompact space is weakly \( \delta \)-normally separated.

**Proof.** Since countably compact spaces are \( cb \) (Corollary 3 in [7]) and completely regular pseudocompact spaces are weak \( cb \) (Corollary 3.8 in [11]), this theorem follows immediately from Lemma 10.

It is well known that a normal pseudocompact Hausdorff space is countably compact. In [17], Zenor shows that normality may be replaced by \( \delta \)-normal separation. Here we show the condition can be further weakened to \( \delta \)-normality.

**Theorem 13.** A completely regular space is countably compact if and only if it is \( \delta \)-normal and pseudocompact.

**Proof.** By Corollary 12 above, a pseudocompact \( \delta \)-normal space is also \( \delta \)-normally separated. This theorem now follows from Zenor’s result (Theorem 3 in [17]).

**Remark.** In Theorem 13, it is essential that the space be completely regular; for there exist regular, countably paracompact, Hausdorff spaces, that are not countably compact, on which every real valued function is constant [10]. Such a space can be obtained by altering slightly the construction used by Hewitt in [6].

For a completely regular space \( X \), let \( vX \) denote the Hewitt realcompactification. In [5, p. 120], it is noted that the normality of \( X \) and of \( vX \) are independent of each other. The same is true for \( \delta \)-normality and \( \delta \)-normal separation. To see this, first, let \( X \) be a completely regular pseudocompact space that is not countably compact (the Tychonoff plank will do nicely). Then \( vX \) is compact and hence is both \( \delta \)-normal and \( \delta \)-normally separated, but \( X \) has neither of these properties.

On the other hand, let \( P \) be the product \( R^c \) of \( c \) (\( c = \text{card } R \)) copies of the reals \( R \) and let \( X \) be associated \( \Sigma \)-product. Then \( X \) is normal, and \( vX = P \) (see [3]) but \( P \) is not countably paracompact. Whence it follows from Theorem 1 that \( P \) is not \( \delta \)-normal and from Theorem 11 that \( P \) is not \( \delta \)-normally separated.

The situation for weak \( \delta \)-normal separation is entirely different as Theorems 14 and 17 below will show.

**Theorem 14.** If a completely regular Hausdorff space \( X \) is weakly \( \delta \)-normally separated then \( vX \) is as well.

**Proof.** If \( A \) is regular closed in \( vX \) and \( Z \) is a zero-set in \( vX \), then \( A \cap X \) is regular closed in \( X \) and \( Z \cap X \) is a zero-set in \( X \). Moreover, \( A \) and \( Z \) are the closures in \( vX \) of \( A \cap X \) and \( Z \cap X \) respectively (for the latter see 8.8(b) in [5]). If \( f \) is a continuous real valued function on \( X \) which completely separates \( A \cap X \) and \( Z \cap X \), then its extension to \( vX \) clearly separates \( A \) from \( Z \).
Corollary 15. Any product of complete separable metric spaces is weakly $\delta$-normally separated.

Proof. In [3], it is proved that any such product is $vX$ for some normal space $X$.

In order to obtain a partial converse of Theorem 14, we prove the following lemma which seems to be of independent interest. A point $x$ of a space $X$ is a $q$-point [13] if it has a sequence $\{U_n\}$ of neighborhoods such that if $\{x_n\}$ is a sequence of distinct points with $x_n \in U_n$, then this sequence has an accumulation point.

Lemma 16. If every point of $vX \setminus X$ is a $q$-point of $vX$, then every pair of disjoint sets $A, Z$ where $A$ is regular closed in $X$ and $Z$ is a zero-set in $X$, have disjoint closures in $vX$.

Proof. Suppose on the contrary that $p$ belongs to the closure of both $A$ and $Z$ and let $\{U_n\}$ be a sequence of open neighborhoods of $p$ given by the definition of $q$-points. Let $G$ denote the interior of $A$ in $X$ and $f \in C(X)$ be a function whose zero-set is $Z$. By our assumption $p$ belongs to the closure of $G \cap \{x : |f(x)| < 1/n\}$ (call this set $H_n$) for each positive integer $n$. Whence $U_n \cap H_n$ is nonempty for each $n$. Pick $x_n$ from this set. Clearly, we may assume that the $x_n$ are distinct. Since $A$ and $Z$ are disjoint, it follows that $\{H_n\}$ is locally finite. Thus the sequence $\{x_n\}$ has no accumulation point. But this is impossible since $p$ is a $q$-point.

Theorem 17. If $vX$ is locally compact (more generally, if each point in $vX \setminus X$ has a compact neighborhood in $vX$), then $X$ is weakly $\delta$-normally separated if and only if $vX$ has the same property.

A converse for Theorem 14 is not possible, without some sort of restriction on $vX$. This is shown in the example below.

It is natural to ask what relation the above results bear to the well known unanswered question [4, p. 221]: Must the product of a normal Hausdorff space with the closed unit interval be normal? In this regard, first, observe that if $X$ is normal, then $X \times I$ is normal provided it is $\delta$-normal. This fact suggests the following question: If $X$ is a regular, $\delta$-normal space, must $X \times I$ be $\delta$-normal? Except for noting that without the assumption that the space is regular, the answer to this question is negative (see p. 221 in [4]), the author has not obtained any significant clues concerning the answer to this question. On the other hand, the answer to the corresponding question for $\delta$-normal separation is negative. This is the substance of the following example.

Example. Let $X$ and $X^*$ be the spaces constructed on pp. 240, 241 of [11]. There it is pointed out that $X$ is locally compact, countably paracompact but not a $cb$-space while $X^*$ is $\sigma$-compact but not locally compact and that $X^* = vX$. It is a simple matter (using Theorem 9 and the special relation that $X$ bears to $X^*$) to show that $X$ is also $\delta$-normally separated. Since $X$ is not a $cb$-space, it follows from Theorem 11 that $X \times I$ is not $\delta$-normally separated (or even weakly $\delta$-normally separated). It is, however, $\delta$-normal.
Also, in view of Theorem 2.8 in [2], note that \( v(X \times I) = vX \times I \). Now since \( vX \times I = X^* \times I \) is Lindelöf and regular, it is normal. Nonetheless \( X \times I \) fails to be weakly \( \delta \)-normally separated. This shows that the restriction on \( vX \) in Theorem 17 cannot be entirely suppressed.

In [14], Morita obtained the following generalization of Dowker's theorem [4]: A space \( X \) is \( m \)-paracompact and normal if and only if \( X \times I^m \) is normal. In view of Morita's result, it is natural to ask: What condition on \( X \) is necessary and sufficient for \( X \times I^m \) to be \( m \)-normal? Theorem 2 implies that \( m \)-paracompactness of \( X \) is a sufficient condition; however the author has been unable to determine whether \( m \)-paracompactness is also necessary. The chief stumbling block is the lack of a characterization of \( m \)-paracompactness similar to that for countable paracompactness given by Tamano (Lemma 7 above).

In contrast to the obstacles encountered in attempting to obtain an analogue of Theorem 1 for uncountable cardinals, Theorem 11 (as pointed out in the Remark following that theorem) can be extended by merely giving an appropriate meaning to the term \( m \)-normal separation. Specifically, define a space to be \( m \)-normally separated provided the intersection of any family consisting of at most \( m \) zero-sets is completely separated from any closed set disjoint from it. Then for any space \( X \), \( X \times I^m \) is \( m \)-normally separated if and only if \( X \) is an \( H(m) \)-space in the sense of [9].

\[ \text{References} \]

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