THE BLASCHKE CONDITION FOR
BOUND HOLOMORPHIC FUNCTIONS

BY
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1. Introduction. Let $U$ be the unit disc in the complex plane $C$ and $H^\infty(U)$ the space of all bounded holomorphic functions in $U$. Let $f \in H^\infty(U)$, $f \neq 0$, and let $\alpha_1, \alpha_2, \ldots$ be the zeros of $f$, listed according to multiplicities. For $0 < r < 1$, let $n(r)$ be the number of $\alpha$'s with $|\alpha| \leq r$. Then it is well known that the Blaschke condition

\begin{equation}
\int_0^1 n(r) \, dr < \infty
\end{equation}

is satisfied. This is a consequence of Jensen's formula in the form:

\begin{equation}
\int_0^r \frac{n(x)}{x} \, dx = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta - \log |f(0)|
\end{equation}

(see e.g. [11, §3.61]).

The aim of the present paper is to study the generalization of (1) to several variables. We show that the $N$-dimensional volume (given in terms of the Hausdorff measure) of the zero-set $Z(f)$ of a bounded holomorphic function $f$ in the unit polydisc $U^{N+1}$ or the unit ball $B_{N+1}$ in $C^{N+1}$ satisfies the generalized condition (Theorem 6.3).

In one variable, the condition (1) is also sufficient for the set $\{\alpha_n\}$ to be the zero-set of a bounded holomorphic function in $U$ (see [7, Theorem 15.21]). For more than one variable, this is no longer so, and we give two examples in §7. In this direction, Professor Rudin was the first to obtain a global condition sufficient for a subvariety $E$ in $U^N$ to be the zero-set of a bounded holomorphic function, namely the condition $\text{dist} (E, T^N) > 0$, where $T^N$ is the $N$-dimensional torus (see [6, Theorem 4.8.3]). Recently, Stout [10] has given a different set of sufficient conditions. No sufficient condition seems to be known in $B_N$.

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2. Hausdorff measures. A lemma on matrices. For our later applications, we summarize here the definition and some elementary properties of the Hausdorff measure.

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Let $A$ be a subset of a metric space $X$. Let $\delta(A)$ denote the diameter of $A$. Write $\delta^p(A) = [\delta(A)]^p$ for $p > 0$; $\delta^0(A) = 1$ if $A \neq \emptyset$, and $\delta^0(\emptyset) = 0$. For $p \geq 0$, $\epsilon > 0$, define

$$H_p(A; \epsilon) = \inf \left\{ \sum_{n=1}^{\infty} \delta^p(A_n) : A \subseteq \bigcup_{n=1}^{\infty} A_n \text{ and } \delta(A_n) < \epsilon \right\},$$

$$H_p(A) = \lim_{\epsilon \to 0^+} C_p H_p(A; \epsilon),$$

where $C_p = \pi^{p/2}(2^p \Gamma(p/2 + 1))$.

If $N$ is an integer, then $C_N$ is the volume of the ball $\{x \in \mathbb{R}^N : \sum_{i=1}^{N} x_i^2 \leq \frac{1}{4}\}$. $H_p(A)$ is called the $p$-dimensional Hausdorff measure or the Hausdorff $p$-measure of $A$. For any $A \subseteq X$, $H_p(A)$ equals the number of points in $A$.

For any $p \geq 0$, $H_p$ is a regular metric outer measure and hence the Borel sets are $H_p$-measurable (see [5, §12]). The Hausdorff measures have the following important elementary properties:

(i) If $H_p(A) < \infty$ and $r > p$, then $H_r(A) = 0$; hence if $A$ is $H_p$-$\sigma$-finite, then $H_r(A) = 0$.

(ii) Let $Y$ be a metric space and $f : X \to Y$ be a Lipschitz map with Lipschitz constant $\lambda$. Then for any $A \subseteq X$ and any $p \geq 0$, $H_p(f(A)) \leq \lambda^p H_p(A)$.

(iii) If $M$ is a $k$-dimensional $C^1$ submanifold of $\mathbb{R}^N$, then volume of $M = H_k(M)$.

The first two properties follow directly from definitions; for (iii), see Stolzenberg [9].

It follows from (i) that the singular locus $S$ of a pure $k$-dimensional analytic subvariety in $\mathbb{C}^N$ has Hausdorff $2k$-measure zero, since $S$ is the countable union of manifolds of real dimension at most $2(k - 1)$. By (iii) we see that the Lebesgue measure on $\mathbb{R}^N$ is equal to the Hausdorff $N$-measure on $\mathbb{R}^N$.

We insert here a lemma on matrices which must be well known. We include a proof for lack of a suitable reference. First, some definitions.

Let $A = (a_{mj})$ be any complex matrix with $n$ rows and $k$ columns (an $n \times k$ matrix). Let $a_{mj} = b_{mj} + ic_{mj}$. Then $\bar{A}$ will denote the real $2n \times 2k$ matrix obtained from $A$ by replacing each $a_{mj}$ by the $2 \times 2$ matrix

$$\begin{pmatrix} b_{mj} & c_{mj} \\ -c_{mj} & b_{mj} \end{pmatrix}.$$ 

It is easy to check that

(i) $(A + B)^\sim = \bar{A} + \bar{B}$,

(ii) $(\bar{A})^\sim = \bar{A}$, where $\bar{A} = (\bar{a}_{mj})$,

(iii) $(AB)^\sim = \bar{A}\bar{B}$, whenever the product $AB$ is defined,

(iv) $U$ is unitary implies $\bar{U}$ is orthogonal.

The first three statements can be verified by writing out both sides. The fourth follows from (ii) and (iii).
Let $k \leq n$, and let $A$ be any complex $n \times k$ matrix. Define

(5) $\Delta(A) = \sum |\text{det } A_x|^2,$

(6) $\tilde{\Delta}(\tilde{A}) = \left\{ \sum |\text{det } \tilde{A}_y|^2 \right\}^{1/2},$

where $A_x$ runs over all $k \times k$ submatrices of $A$ and $\tilde{A}_y$ runs over all $2k \times 2k$ submatrices of $\tilde{A}$.

If $A$ and $B$ are two $n \times n$ matrices, then $\text{det } (BA) = \text{det } B \cdot \text{det } A$. This has the following generalization.

**Cauchy-Binet Theorem.** Let $k \leq n$. Let $A$ be an $n \times k$ matrix and $B$ a $k \times n$ matrix. Then $\text{det } (BA)$ is equal to the sum of all the $\binom{n}{k}$ products which can be made by taking a minor of order $k$ from certain $k$ columns of $B$ and a minor of order $k$ from the corresponding rows of $A$. (See Aitken [1, §36].)

**Lemma 2.1.** Let $A$ be any complex $n \times k$ matrix, $k \leq n$. Then $\Delta(A) = \tilde{\Delta}(\tilde{A})$.

**Proof.** Let $A'$ denote the transpose of $A$. Then by the Cauchy-Binet Theorem,

$$\Delta(A) = \text{det } (A'A), \quad \tilde{\Delta}(\tilde{A}) = \left( \text{det } (\tilde{A}'\tilde{A}) \right)^{1/2}.$$ 

Therefore if $U$ is a unitary $n \times n$ matrix, and $\tilde{U}$ the associated orthogonal matrix, then

$$\Delta(A) = \Delta(UA), \quad \tilde{\Delta}(\tilde{A}) = \tilde{\Delta}(\tilde{U}\tilde{A}).$$

Given $A$, we choose a unitary $n \times n$ matrix $U$ such that the bottom $(n-k)$ rows of $U$ are orthogonal to each of the $k$ columns of $A$. Then the bottom $(n-k)$ rows of $UA$ are zero. Let $B$ be the $k \times k$ submatrix of $UA$ formed by the first $k$ rows. Then

$$\Delta(A) = \Delta(UA) = |\text{det } B|^2.$$

Now $\tilde{U}\tilde{A} = (UA)'$ has the bottom $2(n-k)$ rows equal to zero and the $2k \times 2k$ submatrix formed by the first $2k$ rows of $\tilde{U}\tilde{A}$ is $\tilde{B}$. Therefore

$$\tilde{\Delta}(\tilde{A}) = \tilde{\Delta}(\tilde{U}\tilde{A}) = |\text{det } \tilde{B}|.$$ 

It remains to show that $|\text{det } \tilde{B}| = |\text{det } B|^2$.

Assume first that $B$ has $k$ distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ and that $\lambda_i \neq \lambda_j$ for all $i, j$. To each $\lambda_i$ corresponds an eigenvector $(z_1, \ldots, z_k)$ of $B$. It is easy to verify that then $(z_1, iz_1, \ldots, z_k, iz_k)$ is an eigenvector of $\tilde{B}$ with the same eigenvalue $\lambda_i$. Since $\tilde{B}$ is real $\tilde{\lambda}_i$ is also an eigenvalue of $\tilde{B}$. Thus $\lambda_1, \tilde{\lambda}_1, \ldots, \lambda_k, \tilde{\lambda}_k$ are $2k$ distinct eigenvalues of $\tilde{B}$, so there are no others. Hence

$$\text{det } \tilde{B} = \lambda_1 \tilde{\lambda}_1 \cdots \lambda_k \tilde{\lambda}_k = |\lambda_1 \cdots \lambda_k|^2 = |\text{det } B|^2.$$ 

The general case follows by continuity.
3. **The volume of an analytic variety.** A pure \( k \)-dimensional analytic subvariety in \( \mathbb{C}^N \) is the closure of a complex \( k \)-dimensional manifold, viz., the set of its regular points. A complex analytic manifold is also a real analytic manifold. To get a formula for the volume of an analytic subvariety, we shall use the following facts concerning the volume of manifolds in \( \mathbb{R}^N \). For an account of these, see Schwartz [8, Chapter IV, §10].

Let \( M \) be an open subset of a \( C^1 \) submanifold of dimension \( k \) in \( \mathbb{R}^N \). Suppose \( M \) is homeomorphic to an open subset \( \Omega \) in \( \mathbb{R}^k \) under the map \( \Phi: \Omega \to M \), where \( \Phi \) and its inverse \( \Phi^{-1} \) are both of class \( C^1 \). Let \( L \) be the Jacobian matrix of \( \Phi \). Define \( J\Phi = (\Delta(L))^{1/2} \) where \( \Delta(L) \) is given by formula (5). Then by Theorem 107 of [8, p. 688], the \( k \)-dimensional volume of \( M \) is given by

\[
H_k(M) = \int_{\Omega} J\Phi(x) \, dH_k(x).
\]

**Theorem 3.1.** Let \( \Omega \) be an open subset of \( \mathbb{R}^k \) and let \( \Phi: \Omega \to \mathbb{R}^N \) be a \( C^1 \) map from \( \Omega \) into \( \mathbb{R}^N \). Then for any Lebesgue measurable subset \( A \) of \( \Omega \),

\[
\int_{\Phi(A)} H_0(A \cap \Phi^{-1}(y)) \, dH_k(y) = \int_A J\Phi(x) \, dH_k(x).
\]

This is proved in Federer [3, Theorem 4.5], for Lipschitz maps and the measure \( \mathcal{L}^k \). The same proof works for \( C^1 \) maps and Hausdorff measures.

Now let \( V \) be a pure \( k \)-dimensional analytic subvariety in a domain \( \Omega \) in \( \mathbb{C}^N \). Let \( S \) be the singular locus of \( V \). Then as we have seen \( H_{2k}(S) = 0 \) and so \( H_{2k}(V) = H_{2k}(V-S) \). We shall establish the following integral geometric formula for the volume of an analytic variety:

**Theorem 3.2.** Let \( V \) be a pure (complex) \( k \)-dimensional analytic subvariety in a domain \( \Omega \) in \( \mathbb{C}^N \). Let the \( k \)-dimensional coordinate subspaces of \( \mathbb{C}^N \) be enumerated in some order. Let \( \pi_j \) be the projection from \( \mathbb{C}^N \) onto the \( j \)th subspace and write \( \hat{z}_j = \pi_j(z) \). Then

\[
H_{2k}(V) = \sum_{j=1}^{m} \int_{\pi_\Omega} H_0(V \cap \pi_j^{-1}(\hat{z}_j)) \, dH_{2k}(\hat{z}_j),
\]

where \( m = \binom{N}{k} \).

**Proof.** Let \( R = V-S \) be the set of regular points of \( V \). We shall show that (9) holds for \( R \). Since both sides of (9) are regular Borel measures, it is sufficient to show that it holds for \( R \cap K \), \( K \) any compact subset of \( \Omega \).

For each \( z \in R \), we can find a neighborhood \( B \) of arbitrarily small diameter which is holomorphically homeomorphic to a closed polydisc \( A \) in \( \mathbb{C}^k \). The sets \( B \cap K \) form a covering of \( R \cap K \) in the sense of Vitali. By the classical covering theorem of Vitali, there exists a countable disjoint family \( \{ B_j \cap K \}_{j=1}^{\infty} \) such that

\[
H_{2k}(R \cap K - \bigcup_{j=1}^{\infty} B_j \cap K) = 0.
\]

Hence it suffices to prove the formula (9) for each set \( B_j \cap K \).
Thus let $B$ be a subset of $R$ which is holomorphically homeomorphic to a closed polydisc $A$ in $C^k$, under the map

$$F: A \rightarrow B, \quad F(x) = (f_1(x), \ldots, f_k(x)).$$

Let $L$ be the complex Jacobian matrix of $F$ and $\bar{L}$ the associated real Jacobian matrix. Let $JF = \Delta(L)$ and $\bar{JF} = \Delta(\bar{L})$ as given by (5) and (6). Then by (7),

$$H_{2k}(B) = \int_A JF(x) \, dH_{2k}(x).$$

This holds since the boundary of $A$ has measure zero and so does its image, the boundary of $B$, on account of property (ii) of the Hausdorff measures. By Lemma 2.1, $JF = \bar{JF}$ and so

$$H_{2k}(B) = \int_A JF(x) \, dH_{2k}(x) = \sum_{j=1}^m \int_A |\text{det } L_j(x)|^2 \, dH_{2k}(x),$$

where the $L_j$'s are the $k \times k$ submatrices of $L$. With $\pi_j$, as defined in the statement of the theorem, each $L_j$ can be regarded as the Jacobian matrix of the map $F_j = \pi_j \circ F: A \rightarrow C^k$ mapping $A$ into the $j$th $k$-dimensional coordinate subspace of $C^k$. $F$ being a homeomorphism implies

$$H_0(A \cap F_j^{-1}(\bar{z})) = H_0(B \cap \pi_j^{-1}(\bar{z})).$$

So since $JF_j = |\text{det } L_j|^2$, Theorem 3.1, formula (8) gives

$$\int_A |\text{det } L_j(x)|^2 \, dH_{2k}(x) = \int_{\pi_j B} H_0(B \cap \pi_j^{-1}(\bar{z})) \, dH_{2k}(\bar{z}).$$

Substituting in (10) we get

$$H_{2k}(B) = \sum_{j=1}^m \int_{\pi_j B} H_0(B \cap \pi_j^{-1}(\bar{z})) \, dH_{2k}(\bar{z}).$$

Noting that $H_0(B \cap \pi_j^{-1}(\bar{z})) = 0$ if $\bar{z}, \notin \pi_j B$, we may replace the domain of integration $\pi_j B$ by $\pi_j \Omega$ and so complete the proof of (9) for the set $B$.

For later application, we give here a generalization of (9):

**Theorem 3.3.** With the notation as in Theorem 3.2, let $f$ be a nonnegative Borel function which vanishes outside $V$. Then

$$\int_{\Omega} f(z) \, dH_{2k}(z) = \sum_{j=1}^m \int_{\pi_j \Omega} dH_{2k}(\bar{z}) \int_{\pi_j^{-1}(\bar{z})} f(z) \, dH_0(z).$$

**Proof.** The proof of Theorem 3.2 shows that (9) holds for any open subset of $V$. Hence since both sides of (9) are regular Borel measures, it holds for any Borel subset $A$ of $V$. Thus (11) holds if $f = \chi_A$, the characteristic function of $A$; hence it holds if $f$ is any nonnegative simple Borel function vanishing outside $V$. If $f$ is any nonnegative Borel function, then there is an increasing sequence of nonnegative
simple Borel functions \( s_n \) such that \( \lim s_n = f \). Since (11) holds for each \( s_n \), the monotone convergence theorem shows that it holds for \( f \).

4. The multiplicity function. Let \( f \) be a holomorphic function in a domain \( \Omega \) in \( \mathbb{C}^N \). For each \( a \in \Omega \), we define the zero-multiplicity \( \mu(a) = \mu_f(a) \) of \( f \) at \( a \) as follows: If \( f \equiv 0 \), then \( \mu(a) = \infty \). If \( f \neq 0 \), then \( f \) has an expansion of the form

\[
f(z) = f_n(z-a) + f_{n+1}(z-a) + \cdots
\]

in a neighborhood of \( a \), where \( f_j \) is a homogeneous polynomial of degree \( j \) and \( f_n \neq 0 \). Define \( \mu(a) = m \).

The following observations can be made:

(i) If \( f = gh \), then \( \mu_f = \mu_g + \mu_h \).

(ii) \( \mu(a) \) does not depend on the choice of coordinates at \( a \).

(iii) If \( 0 \in \Omega \) and \( \mu(0) = m > 0 \), then by the Weierstrass preparation theorem, there is a coordinate system \( z_1, \ldots, z_N \) such that

\[
f = uW
\]

in a neighborhood of 0, where \( u \) has no zeros in that neighborhood and \( W \) is a Weierstrass polynomial of degree \( m \) in \( z_N \).

(iv) The converse of (iii) is also true, viz., if (12) holds, then \( \mu(0) = m \), the degree of \( W \) in \( z_N \).

The first two observations follow easily from definition (see [6, 1.1.6]); (iv) is a consequence of the uniqueness of the Weierstrass polynomial for \( f \).

Proposition 4.1. Let \( f \) be a holomorphic function in a domain \( \Omega \) in \( \mathbb{C}^N \), \( N > 1 \). Then \( \mu \) is constant on each (connectivity) component of the set of regular points of \( V = Z(f) \).

Proof. Without loss of generality, let 0 be a regular point of \( V \). Let \( \mu(0) = m \). If \( m = \infty \), then \( f \equiv 0 \) and the proposition is trivial. Suppose \( m < \infty \). At a regular point, \( V \) is an \((N-1)\)-dimensional manifold. By the definition of a manifold (see [4, I.B. 8]), there exist a neighborhood \( U_0 \) of 0 and a mapping \( F: U_0 \to \mathbb{C} \) which is nonsingular at 0 such that \( V \cap U_0 = Z(F) \). \( F \) being nonsingular at 0 implies that

\[
\frac{\partial F(0)}{\partial z_j} \neq 0 \text{ for some } j, 1 \leq j \leq N.
\]

By renaming the coordinates, we may assume that \( j = N \) and \( \frac{\partial F(0)}{\partial z_N} \neq 0 \). By the implicit function theorem, there is a disc \( D \) containing \( 0 \in \mathbb{C} \) and there is a function \( \varphi(z') \) holomorphic in \( D^{N-1} \) such that \( (z', z_N) \in D^{N-1} \times D \) and \( f(z', z_N) = 0 \) if and only if \( z_N = \varphi(z') \). Define \( p(z) = z_N - \varphi(z') \), \( z \in D^N \). Then in a possibly smaller neighborhood of \( 0 \in \mathbb{C}^N \) (again denoted by \( U_0 \)) we have

\[
Z(f) \cap U_0 = Z(F) \cap U_0 = Z(p) \cap U_0.
\]

Thus in \( U_0, f(z) = 0 \) if and only if \( p(z) = 0 \). Since \( p(0', z_N) \neq 0, f(0', z_N) \neq 0 \). Therefore by the Weierstrass preparation theorem, \( f = uW \) in a neighborhood \( U_1 \) of 0, where \( u(z) \neq 0 \) for all \( z \in U_1 \) and \( W \) is a Weierstrass polynomial of degree \( m \) in \( z_N \). Let
Let $W = \prod_{i=1}^{s} p_i^{n_i}$ be the factorization of $W$ into irreducible factors. By taking $U_1$ small enough, we may assume that each $p_i$ is holomorphic in $U_1$ and $U_1 \subseteq U_0$. Then

$$Z(p) \cap U_1 = Z(f) \cap U_1 = Z(W) \cap U_1 = \bigcup_{i=1}^{s} Z(p_i).$$

Since $Z(p)$ is an irreducible variety at 0, the uniqueness of the irreducible decomposition of analytic varieties shows that $s = 1$ and $p_1 = p$. Thus $W = p^{n_1}$. Since $W$ is of degree $m$ and $p$ is of degree 1 in $z_N$, we must have $n_1 = m$. So $f = up^m$ in $U_1$.

Now let $b \in U_1 \cap V$. We claim that $\mu(b) = m$. Let $w = z - b$, $z \in U_1$. Then $f(z) = u(z)[p(z)]^m = \bar{u}(w)[\bar{p}(w)]^m$, where

$$\bar{p}(w) = p(w + b) = w_n + b_n - \varphi(w' + b') = w_n + \varphi(w').$$

Since $b \in V$, $\varphi(0') = b_n - \varphi(b') = 0$. Therefore $\bar{p}$ is a Weierstrass polynomial at $b$, hence so is $\bar{p}^m$. Since $f = \bar{u}p^m$, the observation (iv) above shows that $\mu(b) = m$.

It follows then that $\mu$ is constant on each connected component of the set of regular points of $V$. Q.E.D.

This shows that the restriction of $\mu$ to the regular points of $V$ is a continuous function. In general we have

**Proposition 4.2.** Let $f$ be a holomorphic function in a domain $\Omega$ in $\mathbb{C}^N$. Then its multiplicity function $\mu$ is upper semicontinuous in $\Omega$.

**Proof.** In the definition of $\mu$, the homogeneous polynomials $f_i$ are obtained by rearrangement of the Taylor series of $f$ at $a$. Thus if $f$ has a partial derivative of total order $n$ which is different from zero at $a$, then $\mu(a) \leq n$. Now let $a \in \Omega$ and $\mu(a) = m$. Then $f$ has a partial derivative of total order $m$ different from zero at $a$. By the continuity of the partial derivative, it is different from zero in a neighborhood $U$ of $a$. So for any $b \in U$, $\mu(b) \leq m$.

Combining this with Theorem 3.3, noting that the zero-set of a nontrivial holomorphic function is a pure dimensional subvariety of codimension one, we get

**Corollary 4.3.** Let $f$ be a holomorphic function in a domain $\Omega$ in $\mathbb{C}^{N+1}$, $f \neq 0$. Let $\mu$ be its multiplicity function. Then

$$\int_{\Omega} \mu(z) \, dH_{2N}(z) = \sum_{j=1}^{N+1} \int_{\pi_j(\Omega)} \int_{\pi_j^{-1}(a)} \mu(z) \, dH_0(z_j)$$

where $\pi_j : \mathbb{C}^{N+1} \rightarrow \mathbb{C}^N$, $\pi_j(z) = (z_1, \ldots, z_{j-1}, 0, z_{j+1}, \ldots, z_{N+1}) \equiv \tilde{z}_j$.

5. The mean value of a plurisubharmonic function. We recall the definition of a plurisubharmonic function: Let $\Omega$ be a domain in $\mathbb{C}^N$. A function $u : \Omega \rightarrow (-\infty, \infty)$ is called plurisubharmonic if

(i) $u$ is upper semicontinuous,

(ii) for any $z$ and $w \in \mathbb{C}^N$, the function $\lambda \rightarrow u(z + \lambda w)$ is subharmonic where it is defined.
For subharmonic functions, the following is well known.

**Theorem 5.1.** Let \( u \) be a subharmonic function in the unit disc \( U \), \( u \not\equiv -\infty \). Let \( m_1 \) be the Lebesgue measure on \( T \) normalized so that \( m_1(T) = 1 \). Let

\[
M_1(r) = \int_T u(r\lambda) \, dm_1(\lambda), \quad 0 \leq r < 1.
\]

Then (i) \( M_1(r) > -\infty \) if \( r > 0 \),

(ii) \( M_1(r) \leq M_1(s) \) if \( r \leq s \),

(iii) \( M_1(r) \) is a convex function of \( \log r \) in the interval \( (0, 1) \), i.e.

\[
M_1(r) \leq \alpha M_1(r_1) + (1 - \alpha) M_1(r_2), \quad \text{whenever } 0 \leq \alpha \leq 1, \quad 0 < r_1 \leq r_2 < 1.
\]

See [7, Chapter 17] and [12, Chapter 2]. This has been generalized to plurisubharmonic functions (see [12]). We give here a further generalization.

Let \( \Omega \) be a domain in \( \mathbb{C}^n \). Then \( \Omega \) is called a complete circular domain if \( z = (z_1, \ldots, z_n) \in \Omega \) and \( |\lambda| \leq 1 \) imply \( Az = (\lambda z_1, \ldots, \lambda z_n) \in \Omega \). Thus \( U^N \) and \( B_N \) are complete circular domains. A measure \( m \) on \( \Omega \) is said to be circularly invariant if \( m(E) = m(E) \) for any measurable subset \( E \) of \( \Omega \) and any \( \lambda \in T \), where \( E_\lambda = \{Az : z \in E\} \). The Lebesgue measure is circularly invariant. Such measures were first considered by Bochner [2]. His method can be applied to prove the following.

**Theorem 5.2.** Let \( \Omega \) be a bounded complete circular domain in \( \mathbb{C}^n \). Let \( m \) be a positive circularly invariant measure on \( \Omega \) normalized so that \( m(\Omega) = 1 \). Let \( u \) be a plurisubharmonic function in \( \Omega \), \( u \not\equiv -\infty \), and let

\[
M(r) = \int_\Omega u(rz) \, dm(z), \quad 0 \leq r \leq 1.
\]

Then (i) \( M(r) > -\infty \) if \( r > 0 \) and \( m \) is the Lebesgue measure,

(ii) \( M(r) \leq M(s) \) if \( r \leq s \),

(iii) \( M(r) \) is a convex function of \( \log r \) in the interval \( (0, 1) \).

**Proof.** To prove (ii) and (iii), we define for each \( z \in \Omega \), \( u_\lambda(z) = u(\lambda z) \). Then \( u_\lambda \) is a subharmonic function in a neighborhood of \( \bar{U} \). So by Theorem 5.1, we have for all \( z \in \Omega \) and \( 0 \leq r \leq s \leq 1 \),

\[
\int_T u(r\lambda z) \, dm_1(\lambda) \leq \int_T u(s\lambda z) \, dm_1(\lambda).
\]

Since \( m \) is circularly invariant,

\[
\int_\Omega u(rz) \, dm(z) = \int_\Omega u(\lambda z) \, dm(z) \quad \text{for all } \lambda \in T.
\]

Hence integrating over \( T \), we get

\[
\int_\Omega u(rz) \, dm(z) = \int_\Omega dm_1(\lambda) \int_T u(\lambda z) \, dm(z) = \int_\Omega dm(z) \int_T u(r\lambda z) \, dm_1(\lambda)
\]

by Fubini's theorem.
From (14), we get by integrating over \( \Omega \),

\[
\int_\Omega dm(z) \int_\tau u(r\lambda z) \, dm_1(\lambda) \leq \int_\Omega dm(z) \int_\tau u(x\lambda z) \, dm_1(\lambda).
\]

Hence substitution of (15) gives

\[
\int_\Omega u(rz) \, dm(z) \leq \int_\Omega u(sz) \, dm(z), \quad 0 \leq r \leq s \leq 1,
\]

which is (ii).

Let \( M_z(r) = \int_\tau u(r\lambda z) \, dm_1(\lambda), z \in \Omega \). Then

\[
M_z(r\alpha) + (1-\alpha)M_z(r_2)
\]

whenever \( 0 \leq \alpha \leq 1, 0 < r_1 \leq r_2 < 1 \). Integrating over \( \Omega \) and noting that by (15),

\[
M(r) = \int_\Omega M_z(r) \, dm(z),
\]

we get

\[
M(r\alpha) + (1-\alpha)M(r_2).
\]

This proves (iii).

To prove (i), let \( m \) be the Lebesgue measure on \( \Omega \). Then since the Jacobian of the transformation \( z \rightarrow rz \) is \( r^{2N} \),

\[
M(r) = \frac{1}{r^{2N}} \int_{r\Omega} u(z) \, dm(z).
\]

We note further that the proof of (ii) gives the following: If the ball \( B = B(a, r) \) of center \( a \) and radius \( r \) is contained in \( \Omega \), then

\[
u(a) \leq \frac{1}{m(B)} \int_B u(z) \, dm(z).
\]

Now suppose \( M(r_0) = -\infty \) for some \( r_0 > 0 \). Then there exist \( a \in r_0\Omega \) and a number \( r_1 > 0 \) such that \( B(a, 3r_1) \subseteq \Omega \) and

\[
\int_{B(a, r_1)} u(z) \, dm(z) = -\infty.
\]

By (16), \( u(a) = -\infty \). Let \( z' \in B(a, r_1) \). Then

\[
B(a, r_1) \subseteq B(z', 2r_1) \subseteq B(a, 3r_1).
\]

Therefore

\[
\int_{B(a, r_1)} u(z) \, dm(z) = -\infty,
\]

which implies as before \( u(z') = -\infty \). So \( \Omega_0 \), the interior of the set \( \{ z \in \Omega : u(z) = -\infty \} \), is a nonempty open set. If \( z \) is a limit point of \( \Omega_0 \) in \( \Omega \), then by the above argument, we also have \( u(z) = -\infty \) and \( z \in \Omega_0 \); hence \( \Omega_0 \) is also closed in \( \Omega \). Since \( \Omega \) is connected, \( \Omega_0 = \Omega \) and \( u \equiv -\infty \), a contradiction. Q.E.D.

If \( f \) is a holomorphic function in \( \Omega \), then \( \log |f| \) is a plurisubharmonic function in \( \Omega \). Thus we have the following corollary which will be used in the proof of the Blaschke condition.
Corollary 5.3. Let $\Omega = U^N$ or $B_N$ and let $m$ be the Lebesgue measure on $\Omega$ normalized so that $m(\Omega) = 1$. Let $f \in H(\Omega), f \neq 0$. Then

$$\int_{\Omega} \log |f(rz)| \, dm(z) > -\infty \quad \text{if } 0 < r \leq 1.$$  

If $f(0) \neq 0$, then

$$\int_{\Omega} \log |f(z)| \, dm(z) \geq \log |f(0)|.$$  

If $f \in H^a(\Omega)$, then $\log |f| \in L^1(m)$. 

6. The Blaschke condition. We now show that the generalized Blaschke condition holds for bounded holomorphic functions in several complex variables. We begin with a (well-known) lemma.

Lemma 6.1. Let $X$ be a Lebesgue measurable subset of $\mathbb{R}^k$, $I$ an interval in $\mathbb{R}$. For each positive integer $k$, let $m_k$ be the Lebesgue measure on $\mathbb{R}^k$. Let $f: I \times X \to [-\infty, \infty]$ be a function satisfying the conditions

(i) for each $t \in I$, $x \mapsto f(t, x)$ is Lebesgue measurable,

(ii) for each $x \in X$, $t \mapsto f(t, x)$ is increasing.

Then $f$ is a Lebesgue measurable function in $I \times X$.

Proof. In what follows, measurable will mean Lebesgue measurable. It is sufficient to prove that

$$A = \{(t, x) \in I \times X : f(t, x) > \alpha\}$$  

is measurable for every real number $\alpha$. Since $I$ is $\sigma$-compact and $X$ is the union of an $F_\sigma$ and a set of measure zero, we may assume that they are compact.

Let $\varepsilon > 0$ be given. Choose points $t_0, t_1, \ldots, t_n \in I$ such that $t_0 < t_1 < \cdots < t_n$, $[t_0, t_n] = I$ and $m_i(I_i) \leq \varepsilon$ where $I_i = [t_i, t_{i+1}]$, $0 \leq i \leq n-1$. Let $A_i = \{x : f(t_i, x) > \alpha\}$. By condition (i), each $A_i$ is measurable. By (ii), $A_i \subseteq A_j$ if $i \leq j$. Let $B = \bigcup_{i=0}^{n-1} (I_i \times A_i)$, $C = \bigcup_{i=0}^{n-1} (I_i \times A_{i+1})$. Then $B$ and $C$ are measurable subsets of $I \times X$ and by the condition (ii), it is easy to check that $B \subseteq A \subseteq C$. Now

$$C - B = \bigcup_{i=0}^{n-1} (I_i \times (A_{i+1} - A_i)).$$  

Since $(A_{i+1} - A_i) \cap (A_i - A_{i-1}) = \emptyset$ for all $i$, we have

$$m_{N+1}(C - B) = \sum_{i=0}^{n-1} m_1(I_i)m_N(A_{i+1} - A_i) \leq \varepsilon \sum_{i=0}^{n-1} m_N(A_{i+1} - A_i) = \varepsilon m_N\left(\bigcup_{i=0}^{n-1} (A_{i+1} - A_i)\right) \leq \varepsilon m_N(X).$$
Let $\varepsilon = 1, \frac{1}{2}, \frac{1}{3}, \ldots$. Then we see that there is an increasing sequence $B_k$ and a decreasing sequence $C_k$ of measurable sets such that with $E = \bigcup_{k=1}^{\infty} B_k$, $F = \bigcup_{k=1}^{\infty} C_k$, we have $E \subseteq A \subseteq F$ and $m_{N+1}(F - E) = 0$. Hence $A$ is Lebesgue measurable.

For our application, we note that the Lebesgue measure in $\mathbb{R}^N$ coincides with the Hausdorff measure $H_N$ in $\mathbb{R}^N$. For what follows, we shall use the following notation: $N$ will denote a positive integer. For $j = 1, 2, \ldots, N+1$, $\pi_j$ will denote the projection on $\mathbb{C}^{N+1}$ defined by

$$\pi_j(z) = (z_1, \ldots, z_{j-1}, 0, z_{j+1}, \ldots, z_{N+1})$$

and we write $\tilde{z}_j = \pi_j(z)$. $\Omega$ will denote $\Omega^{N+1}$ or $B_{N+1}$ and for $0 < r < 1$, $\Omega(r)$ is the corresponding domain of radius $r$.

**Lemma 6.2.** Let $f \in H(\Omega)$, $f \neq 0$ and $\mu$ its multiplicity function. Let $V_r = Z(f) \cap \Omega(r)$. Then for each $j$, $1 \leq j \leq N+1$, the function

$$F(r, \tilde{z}_j) = \int_{V_r \cap \pi_j^{-1}(\tilde{z}_j)} \mu(z) \, dH_0(z)$$

is Lebesgue measurable in $(0, 1) \times \pi_j \Omega$.

**Proof.** Fix $j$, $1 \leq j \leq N+1$. Clearly $F(r, \tilde{z}_j)$ is an increasing function of $r$ for each $\tilde{z}_j$. Thus in view of Lemma 6.1, we need only show that $F_r : \tilde{z}_j \to F(r, \tilde{z}_j)$ is Lebesgue measurable for each $r$.

Fix $r \in (0, 1)$. Let $S$ be the singular locus of $Z(f)$. Then $H_{2N}(S) = 0$; hence by property (ii) of the Hausdorff measures, $H_{2N}(\pi_j S) = 0$. We shall show that $F_r$ is a Borel function on $\mathbb{R} = \pi_j \Omega - \pi_j S$. This will imply that $F_r$ is Lebesgue measurable on $\pi_j \Omega$.

The value $n = F(r, \tilde{z}_j)$ is a nonnegative integer or $\infty$. Suppose $n \neq 0$ or $\infty$. Then $V_r \cap \pi_j^{-1}(\tilde{z}_j)$ consists of only a finite number of points. If $\tilde{z}_j \in R$, then each point of $V_r \cap \pi_j^{-1}(\tilde{z}_j)$ is a regular point of $Z(f)$. By Proposition 4.1, $\mu(z)$ is constant in a neighborhood of each such point. So $F_r$ is constant in a neighborhood of $\tilde{z}_j$. If $n = 0$, then $f$ has no zeros on $V_r \cap \pi_j^{-1}(\tilde{z}_j)$. By the continuity of $f$, it has no zeros in a neighborhood of $V_r \cap \pi_j^{-1}(\tilde{z}_j)$. Thus for each integer $n$, $0 \leq n < \infty$, the set $A_n = \{\tilde{z}_j \in R : F(r, \tilde{z}_j) = n\}$ is an open set in $\mathbb{R}$. Since $A_\infty = \{\tilde{z}_j \in R : F(r, \tilde{z}_j) = \infty\} = R - \bigcup_{n=1}^{\infty} A_n$, we see that $A_\infty$ is a closed set of $R$. This shows that $F_r$ is a Borel function on $\mathbb{R}$ and the proof is complete.

**Theorem 6.3.** Let $f \in H^\infty(\Omega)$, $f \neq 0$ and $|f| \leq 1$. Let $\mu$ be its multiplicity function. Then

$$\int_0^1 dr \int_{\Omega(r)} \mu(z) \, dH_{2N}(z) < \infty.$$  

If $f(0) \neq 0$, then

$$\int_0^1 dr \int_{\Omega(r)} \mu(z) \, dH_{2N}(z) \leq c(\Omega) \log \frac{1}{|f(0)|}$$

where $c(U^{N+1}) = (N+1)!^N$, $c(B_{N+1}) = (N+1)!^N/N!$. 

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Proof. Assume first that \( f(\hat{z}_j) \neq 0 \) for all \( j, 1 \leq j \leq N + 1 \). Let \( f_j(z_j) = f(z) \) and let \( \mu_j(z_j) \) be the zero-multiplicity of \( f_j \) at \( z_j \). It is easily seen that \( \mu_j(z_j) \geq \mu(z) \). Let \( n_j(r) \) be the number of zeros of \( f_j \) in \( \pi_j^{-1}(\hat{z}_j) \cap \Omega(r) \), counting multiplicities. Let \( V_r = Z(f) \cap \Omega(r) \). By Corollary 4.3,

\[
\int_{\Omega(r)} \mu(z) \, dH_{2N}(z) = \sum_{j=1}^{N+1} \int_{\pi_j^{-1}(\hat{z}_j) \cap \Omega(r)} \mu(z) \, dH_0(z_j).
\]

Since \( \hat{z}_j \notin \pi_j \Omega(r) \) implies \( V_r \cap \pi_j^{-1}(\hat{z}_j) = \emptyset \), this can be written

\[
\int_{\Omega(r)} \mu(z) \, dH_{2N}(z) = \sum_{j=1}^{N+1} \int_{\pi_j^{-1}(\hat{z}_j)} \mu(z) \, dH_0(z_j) = \sum_{j=1}^{N+1} \int_{\pi_j^{-1}(\hat{z}_j)} F(r, \hat{z}_j) \, dH_{2N}(\hat{z}_j)
\]

where

\[
F(r, \hat{z}_j) = \int_{\pi_j^{-1}(\hat{z}_j)} \mu(z) \, dH_0(z_j).
\]

By Lemma 6.2, \( F \) is Lebesgue measurable in \((0, 1) \times \pi_j \Omega \) and Fubini's theorem applies to give

\[
\int_0^1 dr \int_{\pi_j \Omega} F(r, \hat{z}_j) \, dH_{2N}(\hat{z}_j) = \int_{\pi_j \Omega} dH_{2N}(\hat{z}_j) \int_0^1 F(r, \hat{z}_j) \, dr.
\]

Since \( \mu(z) \leq \mu_{\hat{z}_j}(z_j) \), we have

\[
F(r, \hat{z}_j) \leq \int_{\pi_j^{-1}(\hat{z}_j)} \mu_{\hat{z}_j}(z_j) \, dH_0(z_j) = n_j(\rho)
\]

where \( \rho = r \) if \( \Omega = U^{N+1} \) and \( \rho = (r^2 - \|\hat{z}_j\|^2)^{1/2} \) if \( \Omega = B_{N+1} \) (\( \|\hat{z}_j\| \) is Euclidean norm of \( \hat{z}_j \)). Noting that \( dr/d\rho = r/\rho \leq 1 \), we get by Jensen's formula (2),

\[
\int_0^1 n_j(\rho) \, dr = \int_0^1 n_j(r) \, dr \leq \log \frac{1}{|f(\hat{z}_j)|} \text{ if } \Omega = U^{N+1};
\]

and

\[
\int_0^1 n_j(\rho) \, dr = \int_0^\alpha n_j(\rho) \, d\rho \leq \int_0^\alpha n_j(\rho) \, d\rho; \quad \left( a = (1 - \|\hat{z}_j\|^2)^{1/2} \right)
\]

\[
\leq \log \frac{1}{|f(\hat{z}_j)|} \text{ if } \Omega = B_{N+1}.
\]

Thus

\[
\int_0^1 F(r, \hat{z}_j) \, dr \leq \log \frac{1}{|f(\hat{z}_j)|}.
\]

Integrating (19) with respect to \( r \) and substituting (20) and (21), we get

\[
\int_0^1 dr \int_{\Omega(r)} \mu(z) \, dH_{2N}(z) \leq \sum_{j=1}^{N+1} \int_{\pi_j \Omega} \log \frac{1}{|f(\hat{z}_j)|} \, dH_{2N}(\hat{z}_j).
\]

Since \( f(\hat{z}_j) \neq 0 \) for all \( j \), Corollary 5.3 shows that each integral on the right is finite and (17) is proved.
If \( f(0) \neq 0 \), Corollary 5.3 gives

\[
\int_{\mathbb{S}^N} \log \frac{1}{|f(\xi)|} \, dH_{2N}(\xi) \leq H_{2N}(\pi, \Omega) \log \frac{1}{|f(0)|}.
\]

Hence,

\[
\int_0^1 \int_{\mathbb{S}^N} \mu(z) \, dH_{2N}(z) = \sum_{j=1}^{N+1} H_{2N}(\pi, \Omega) \log \frac{1}{|f(0)|}.
\]

Putting \( c(\Omega) = \sum_{j=1}^{N+1} H_{2N}(\pi, \Omega) \), we get (18).

The case when \( f(z_j) = 0 \) for some \( j \) can be reduced to the first case as follows. We do this separately for \( U_{N+1} \) and \( B_{N+1} \).

For \( U_{N+1} \). If \( f(z_j) \equiv 0 \) for some \( j \), then there is a positive integer \( \alpha_j \) such that \( g_j(z) = f(z) / z_j^{\alpha_j} \) is holomorphic in \( U_{N+1} \) and \( g_j(z_j) \neq 0 \). Doing this for all \( j \), we get nonnegative integers \( \alpha_j \) such that

\[
f(z) = z_1^{\alpha_1} \cdots z_{N+1}^{\alpha_{N+1}} g(z)
\]

where \( g \) is holomorphic in \( U_{N+1} \) and \( g(z_j) \neq 0 \) for all \( j \). Since \( |f(z)| \leq 1 \) as \( z \) tends to \( T_{N+1} \), the same is true for \( g \), so that \( |g| \leq 1 \) in \( U_{N+1} \) by the maximum modulus theorem. Thus the first part of the proof applies to \( g \). An easy computation shows that each factor \( z_j^{\alpha_j} \) contributes \( \alpha_j \pi^{N}/(2N+1) \) to the integral in (17). Thus with \( \mu_g = \text{multiplicity function of } g \),

\[
\int_0^1 \int_{\mathbb{S}^N} \mu(z) \, dH_{2N}(z) = \int_0^1 \int_{\mathbb{S}^N} \mu_g(z) \, dH_{2N}(z) + \frac{\pi^{N}}{2N+1} \left( \sum_{j=1}^{N+1} \alpha_j \right) < \infty.
\]

For \( B_{N+1} \), we have the following lemma.

**Lemma 6.4.** Let \( f \in H(B_N), f \neq 0 \). Then there exists a coordinate system \( z_1, \ldots, z_N \) such that \( f(z_j) \neq 0 \) for all \( j \).

**Proof.** Without loss of generality, we may assume that \( f \in H(\overline{B}_N) \). Let \( e_1, \ldots, e_N \) be \( N \) points on \( S^{2N-1} \) which form an orthogonal basis for \( C^N \). We shall show that there exists a unitary transformation \( A \) such that \( f(Ae_1)f(Ae_2) \cdots f(Ae_N) \neq 0 \). Then \( A(\overline{B}_N) \) will give the required coordinate system.

If \( f(e_1) \neq 0 \), we take \( A = I \) the identity transformation. If \( f(e_1) = 0 \), then since \( f \) is not identically zero on \( S^{2N-1} \), there is an \( \xi_1 \in S^{2N-1} \) such that \( f(\xi_1) \neq 0 \). Since the unitary transformations are transitive on \( S^{2N-1} \), we can find a unitary transformation \( A_1 \) such that \( A_1e_1 = \xi_1 \). By the continuity of \( f \), there is a neighborhood \( W_1 \) of \( \xi_1 \) such that \( f(e) \neq 0 \) for all \( e \in W_1 \).

If \( f(A_1e_2) \neq 0 \), we take \( A_2 = I \). If \( f(A_1e_2) = 0 \), then since \( Z(f) \) is nowhere dense on \( S^{2N-1} \), there is an \( \xi_2 \) arbitrarily close to \( A_1e_2 \) such that \( f(\xi_2) \neq 0 \). Then \( \xi_2 = A_2A_1e_2 \) for some unitary transformation \( A_2 \). We can choose \( \xi_2 \) so close to \( A_1e_2 \) that \( A_2 \xi_1 = A_1e_1 \) is in \( W_1 \). Fix such an \( A_2 \). Then there exists a neighborhood \( W_2 \) of \( \xi_2 \) such that \( f(\xi) \neq 0 \) for all \( \xi \in W_2 \).

Continuing the process \( N \) times, we get unitary transformations \( A_1, \ldots, A_N \) such that if \( A = A_NA_{N-1} \cdots A_1 \), then \( f(Ae_j) \neq 0 \) for all \( j \). This completes the proof of the lemma and that of the theorem.
7. **Examples.** In contrast to the theorem in one variable, the Blaschke condition is not sufficient for an analytic subvariety to be the zero-set of a bounded holomorphic function in $U^2$ or $B_2$. In fact there are analytic subvarieties which satisfy the Blaschke condition and which are determining sets for bounded holomorphic functions.

**Example 1.** Let $\alpha_n = 1 - 1/n$ and

$$V = \{(\alpha_n, w): |\alpha_n|^2 + |w|^2 < 1, n = 1, 2, 3, \ldots\}.$$

Then (by Cartan's Theorem B) $V$ is the zero-set of a holomorphic function in $B_2$. But $V$ is a $D$-set for bounded holomorphic functions in $B_2$, although it satisfies the Blaschke condition.

An easy calculation shows that $H_2(V_r) = \pi \sum_{|\alpha_n| \leq r} (r^2 - \alpha_n^2)$. Hence

$$\int_0^1 H_2(V_r) \, dr = \pi \int_0^1 \sum_{|\alpha_n| \leq r} (r^2 - \alpha_n^2) \, dr = \pi \sum_{n=1}^\infty \int_0^1 (r^2 - \alpha_n^2) \, dr$$

$$= \frac{\pi}{3} \sum_{n=1}^\infty (1 - \alpha_n^2)(1 + 2\alpha_n) = \frac{\pi}{3} \sum_{n=1}^\infty \frac{1}{n^2} \left(3 - \frac{2}{n}\right) < \infty.$$

Now suppose $f \in H^\alpha(B_2)$ and $f = 0$ on $V$. We shall show that then $f \equiv 0$.

For each $c \in C$, let $D(c)$ be the disc in the $z$-plane passing through the point $z = 1$ and having center at $z = |c|^2/(1 + |c|^2)$. $D(c) \subset U$ for all $c$. Let

$$P(c) = \{(z, c(1-z)): z \in D(c)\}.$$

Then $P(c)$ is a disc imbedded in $B_2$ and its boundary passes through the point $(1, 0)$ for all $c$.

For each $c$, let $f_c(z) = f(z, c(1-z))$, $z \in D(c)$. When $n$ is sufficiently large, $\alpha_n \in D(c)$ and $f_c(\alpha_n) = 0$. Therefore the zero-set of $f_c$ violates the Blaschke condition. Since $f_c$ is bounded, $f_c \equiv 0$, i.e. $f|_{P(c)} \equiv 0$ for all $c$. Since $B_2 = \bigcup_{c \in C} P(c)$, we have $f \equiv 0$.

**Example 2.** Fix $\delta$, $\frac{1}{2} < \delta < 1$. Let $\alpha_n = 1 - 1/n^\delta$ and

$$V = \{(z, 2\alpha_n - z): |z| < 1, |2\alpha_n - z| < 1, n = 1, 2, 3, \ldots\}.$$

Then $V$ is the zero-set of a holomorphic function in $U^2$. We shall show that it satisfies the Blaschke condition and is a $D$-set for bounded holomorphic functions.

For each $n$, the area of the set $\{(z, 2\alpha_n - z): |z| \leq r, |2\alpha_n - z| \leq r\}$ is $\leq 2\pi(r^2 - \alpha_n^2)$. So $H_2(V_r) \leq 2\pi \sum_{|\alpha_n| \leq r} (r^2 - \alpha_n^2)$. The computation in Example 1 shows that

$$\int_0^1 H_2(V_r) \, dr \leq \frac{2\pi}{3} \sum_{n=1}^\infty (1 - \alpha_n^2)(1 + 2\alpha_n)$$

$$= \frac{2\pi}{3} \sum_{n=1}^\infty \frac{1}{n^{2\delta}} (1 + 2\alpha_n)$$

$$< \infty \quad \text{since } 2\delta > 1.$$

Let $f \in H^\alpha(U^2)$ and $f \equiv 0$ on $V$. Let

$$A = \{c \in C: \text{Re } c > 1, |\arg c| < (1 - \delta)\pi/2\}.$$
For $c \in A$, the boundary of the disc $U(c)$ of radius $1/|c|$ and center at $1-1/c$ makes an angle $k\pi/2$ with the real axis, where $\delta<k<1$. The real axis divides $U(c)$ into two regions; let $U_1(c)$ be the smaller one. Let $D(c)$ be the region formed by $U_1(c)$ and its reflection in the real axis. Then $D(c)$ is contained in the unit disc $U$ and is bounded by two circular arcs meeting at an angle $k\pi$ at the point $z=1$ and $z=z_0$, where $z_0=1-2\Re c/|c|^2$. Let $P(c)=\{(z, c(z-1)+1): z \in D(c)\}$. For every $c \in A$, $P(c)$ is a subset of $U^2$ such that the point $(1, 1)$ lies on its boundary.

Fix $c \in A$. Define $f_c(z)=f(z, c(z-1)+1)$, $z \in D(c)$. Let $\tilde{a}_n=1-\frac{2}{1+c}(1/n^4)$. For all sufficiently large $n$, $\tilde{a}_n \in D(c)$ and since $f=0$ on $V$, $f_c(\tilde{a}_n)=0$. Under the mapping $\varphi_c(z)=((1-z)/(z-z_0))^{1/k}$, $D(c)$ is mapped onto the right half-plane. Let $\beta_n=\varphi_c(\tilde{a}_n)$. Then it is easy to check that for fixed $c \in A$, $\Re \beta_n \geq \gamma n^{-\delta/k}$ for sufficiently large $n$, where $\gamma$ is positive and does not depend on $n$. Since $\delta/k<1$, it follows that

$$\sum \Re \beta_n = \infty.$$ 

Thus the function $f_c=f_c \cdot \varphi_c^{-1}$ is a bounded holomorphic function in the right half-plane whose zeros $\beta_n$ satisfy (22). So $f_c \equiv 0$ which implies that $f_c \equiv 0$, i.e. $f|_{P(c)} \equiv 0$ for all $c \in A$. Let $P=\bigcup_{c \in A} P(c)$. Then $P$ contains an open subset of $U^2$ since the open subset $D \times \Delta$ of $C^2$, where $D=D(1+i)$ and $\Delta=A \cap \{c: |c-1|<1\}$, is mapped into $P$ by $\Phi_c(z, c) \to (z, c(z-1)+1)$ which is nonsingular when $z \neq 1$. So $f=0$ on $P$ implies $f \equiv 0$ in $U^2$.

Added in proof. Recently the author has extended Theorem 6.3 to wider classes of functions, namely the Nevanlinna classes on $U^N$ and $B^N$.

References


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