THE BLASCHKE CONDITION FOR BOUNDED HOLOMORPHIC FUNCTIONS

BY

PAK SOONG CHEE

1. Introduction. Let $U$ be the unit disc in the complex plane $\mathbb{C}$ and $H^\infty(U)$ the space of all bounded holomorphic functions in $U$. Let $f \in H^\infty(U), f \neq 0$, and let $\alpha_1, \alpha_2, \ldots$ be the zeros of $f$, listed according to multiplicities. For $0 < r < 1$, let $n(r)$ be the number of $\alpha$'s with $|\alpha| \leq r$. Then it is well known that the Blaschke condition

(1) $\int_0^1 n(r) \, dr < \infty$

is satisfied. This is a consequence of Jensen’s formula in the form:

(2) $\int_0^r \frac{n(x)}{x} \, dx = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta - \log |f(0)|$

(see e.g. [11, §3.61]).

The aim of the present paper is to study the generalization of (1) to several variables. We show that the $N$-dimensional volume (given in terms of the Hausdorff measure) of the zero-set $Z(f)$ of a bounded holomorphic function $f$ in the unit polydisc $U^{N+1}$ or the unit ball $B_{N+1}$ in $\mathbb{C}^{N+1}$ satisfies the generalized condition (Theorem 6.3).

In one variable, the condition (1) is also sufficient for the set $\{\alpha_n\}$ to be the zero-set of a bounded holomorphic function in $U$ (see [7, Theorem 15.21]). For more than one variable, this is no longer so, and we give two examples in §7. In this direction, Professor Rudin was the first to obtain a global condition sufficient for a subvariety $E$ in $U^N$ to be the zero-set of a bounded holomorphic function, namely the condition $\text{dist}(E, T^N) > 0$, where $T^N$ is the $N$-dimensional torus (see [6, Theorem 4.8.3]). Recently, Stout [10] has given a different set of sufficient conditions. No sufficient condition seems to be known in $B_{N}$.

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2. Hausdorff measures. A lemma on matrices. For our later applications, we summarize here the definition and some elementary properties of the Hausdorff measure.
Let $A$ be a subset of a metric space $X$. Let $\delta(A)$ denote the diameter of $A$. Write $\delta^p(A) = [\delta(A)]^p$ for $p > 0$; $\delta^0(A) = 1$ if $A \neq \emptyset$, and $\delta^0(\emptyset) = 0$. For $p \geq 0$, $\epsilon > 0$, define

$$H_p(A; \epsilon) = \inf \left\{ \sum_{n=1}^{\infty} \delta^p(A_n) : A \subseteq \bigcup_{n=1}^{\infty} A_n \text{ and } \delta(A_n) < \epsilon \right\},$$

$$H_p(A) = \lim_{\epsilon \to 0^+} C_p H_p(A; \epsilon),$$

where $C_p = \pi^{p/2}/(2^p \Gamma(p/2 + 1))$.

If $N$ is an integer, then $C_N$ is the volume of the ball $\{ x \in \mathbb{R}^N : \sum \lambda_i^2 \leq 1/4 \}$. $H_p(A)$ is called the $p$-dimensional Hausdorff measure or the Hausdorff $p$-measure of $A$. For any $A \subseteq X$, $H_0(A)$ equals the number of points in $A$.

For any $p \geq 0$, $H_p$ is a regular metric outer measure and hence the Borel sets are $H_p$-measurable (see [5, §12]). The Hausdorff measures have the following important elementary properties:

(i) If $H_p(A) < \infty$ and $r > p$, then $H_r(A) = 0$; hence if $A$ is $H_p$-$\sigma$-finite, then $H_r(A) = 0$.

(ii) Let $Y$ be a metric space and $f : X \to Y$ be a Lipschitz map with Lipschitz constant $\lambda$. Then for any $A \subseteq X$ and any $p \geq 0$, $H_p(f(A)) \leq \lambda^p H_p(A)$.

(iii) If $M$ is a $k$-dimensional $C^1$ submanifold of $\mathbb{R}^N$, then volume of $M = H_k(M)$.

The first two properties follow directly from definitions; for (iii), see Stolzenberg [9].

It follows from (i) that the singular locus $S$ of a pure $k$-dimensional analytic subvariety in $\mathbb{C}^N$ has Hausdorff $2k$-measure zero, since $S$ is the countable union of manifolds of real dimension at most $2(k - 1)$. By (iii) we see that the Lebesgue measure on $\mathbb{R}^N$ is equal to the Hausdorff $N$-measure on $\mathbb{R}^N$.

We insert here a lemma on matrices which must be well known. We include a proof for lack of a suitable reference. First, some definitions.

Let $A = (a_{mj})$ be any complex matrix with $n$ rows and $k$ columns (an $n \times k$ matrix). Let $a_{mj} = b_{mj} + ic_{mj}$. Then $\bar{A}$ will denote the real $2n \times 2k$ matrix obtained from $A$ by replacing each $a_{mj}$ by the $2 \times 2$ matrix

$$\begin{pmatrix} b_{mj} & c_{mj} \\ -c_{mj} & b_{mj} \end{pmatrix}.$$

It is easy to check that

(i) $(A + B)^- = \bar{A} + \bar{B}$,

(ii) $(A)^- = \bar{A}$, where $\bar{A} = (\bar{a}_{mj})$,

(iii) $(AB)^- = \bar{A}\bar{B}$, whenever the product $AB$ is defined,

(iv) $U$ is unitary implies $\bar{U}$ is orthogonal.

The first three statements can be verified by writing out both sides. The fourth follows from (ii) and (iii).
Let $k \leq n$, and let $A$ be any complex $n \times k$ matrix. Define

\[(5) \Delta(A) = \sum_x |\det A_x|^2,\]

\[(6) \Delta(\tilde{A}) = \left\{ \sum_y |\det \tilde{A}_y|^2 \right\}^{1/2},\]

where $A_x$ runs over all $k \times k$ submatrices of $A$ and $\tilde{A}_y$ runs over all $2k \times 2k$ submatrices of $\tilde{A}$.

If $A$ and $B$ are two $n \times n$ matrices, then $\det (BA) = \det B \cdot \det A$. This has the following generalization.

**Cauchy-Binet Theorem.** Let $k \leq n$. Let $A$ be an $n \times k$ matrix and $B$ a $k \times n$ matrix. Then $\det (BA)$ is equal to the sum of all the $(\binom{k}{2})$ products which can be made by taking a minor of order $k$ from certain $k$ columns of $B$ and a minor of order $k$ from the corresponding rows of $A$. (See Aitken [1, §36].)

**Lemma 2.1.** Let $A$ be any complex $n \times k$ matrix, $k \leq n$. Then $\Delta(A) = \Delta(\tilde{A})$.

**Proof.** Let $A'$ denote the transpose of $A$. Then by the Cauchy-Binet Theorem,

$$\Delta(A) = \det (\tilde{A}'A), \quad \Delta(\tilde{A}) = \{\det (\tilde{A}'\tilde{A})\}^{1/2}.$$ 

Therefore if $U$ is a unitary $n \times n$ matrix, and $\bar{U}$ the associated orthogonal matrix, then

$$\Delta(A) = \Delta(UA), \quad \Delta(\tilde{A}) = \Delta(\bar{U}\tilde{A}).$$

Given $A$, we choose a unitary $n \times n$ matrix $U$ such that the bottom $(n-k)$ rows of $U$ are orthogonal to each of the $k$ columns of $A$. Then the bottom $(n-k)$ rows of $UA$ are zero. Let $B$ be the $k \times k$ submatrix of $UA$ formed by the first $k$ rows. Then

$$\Delta(A) = \Delta(UA) = |\det B|^2.$$ 

Now $\bar{U}\tilde{A} = (UA)^\sim$ has the bottom $2(n-k)$ rows equal to zero and the $2k \times 2k$ submatrix formed by the first $2k$ rows of $\bar{U}\tilde{A}$ is $\bar{B}$. Therefore

$$\Delta(\tilde{A}) = \Delta(\bar{U}\tilde{A}) = |\det \bar{B}|.$$ 

It remains to show that $|\det \bar{B}| = |\det B|^2$.

Assume first that $B$ has $k$ distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ and that $\lambda_i \neq \lambda_j$ for all $i, j$. To each $\lambda_i$ corresponds an eigenvector $(z_1, \ldots, z_k)$ of $B$. It is easy to verify that then $(z_1, iz_1, \ldots, z_k, iz_k)$ is an eigenvector of $\bar{B}$ with the same eigenvalue $\lambda_i$. Since $\bar{B}$ is real $\lambda_i$ is also an eigenvalue of $\bar{B}$. Thus $\lambda_1, \lambda_2, \ldots, \lambda_k, \lambda_k, \lambda_k$ are $2k$ distinct eigenvalues of $\bar{B}$, so there are no others. Hence

$$\det \bar{B} = \lambda_1 \lambda_2 \cdots \lambda_k \lambda_k = |\lambda_1 \cdots \lambda_k|^2 = |\det B|^2.$$ 

The general case follows by continuity.
3. **The volume of an analytic variety.** A pure \( k \)-dimensional analytic subvariety in \( \mathbb{C}^N \) is the closure of a complex \( k \)-dimensional manifold, viz., the set of its regular points. A complex analytic manifold is also a real analytic manifold. To get a formula for the volume of an analytic subvariety, we shall use the following facts concerning the volume of manifolds in \( \mathbb{R}^N \). For an account of these, see Schwartz [8, Chapter IV, §10].

Let \( M \) be an open subset of a \( C^1 \) submanifold of dimension \( k \) in \( \mathbb{R}^n \). Suppose \( M \) is homeomorphic to an open subset \( \Omega \) in \( \mathbb{R}^k \) under the map \( \Phi: \Omega \rightarrow M \), where \( \Phi \) and its inverse \( \Phi^{-1} \) are both of class \( C^1 \). Let \( L \) be the Jacobian matrix of \( \Phi \). Define \( J\Phi = (\Delta(L))^{1/2} \) where \( \Delta(L) \) is given by formula (5). Then by Theorem 107 of [8, p. 688], the \( k \)-dimensional volume of \( M \) is given by

\[
H_k(M) = \int_{\Omega} J\Phi(x) \, dH_k(x).
\]

**Theorem 3.1.** Let \( \Omega \) be an open subset of \( \mathbb{R}^k \) and let \( \Phi: \Omega \rightarrow \mathbb{R}^n \) be a \( C^1 \) map from \( \Omega \) into \( \mathbb{R}^n \). Then for any Lebesgue measurable subset \( A \) of \( \Omega \),

\[
\int_{\Phi(A)} H_0(A \cap \Phi^{-1}(y)) \, dH_k(y) = \int_A J\Phi(x) \, dH_k(x).
\]

This is proved in Federer [3, Theorem 4.5], for Lipschitz maps and the measure \( \mathcal{L}^k_\beta \). The same proof works for \( C^1 \) maps and Hausdorff measures.

Now let \( V \) be a pure \( k \)-dimensional analytic subvariety in a domain \( \Omega \) in \( \mathbb{C}^n \). Let \( S \) be the singular locus of \( V \). Then as we have seen \( H_{2k}(S) = 0 \) and so \( H_{2k}(V) = H_{2k}(V - S) \). We shall establish the following integral geometric formula for the volume of an analytic variety:

**Theorem 3.2.** Let \( V \) be a pure (complex) \( k \)-dimensional analytic subvariety in a domain \( \Omega \) in \( \mathbb{C}^n \). Let the \( k \)-dimensional coordinate subspaces of \( \mathbb{C}^n \) be enumerated in some order. Let \( \pi_j \) be the projection from \( \mathbb{C}^n \) onto the \( j \)th subspace and write \( \tilde{z}_j = \pi_j(z) \). Then

\[
H_{2k}(V) = \sum_{j=1}^{m} \int_{\pi_j} H_0(V \cap \pi_j^{-1}(\tilde{z}_j)) \, dH_{2k}(\tilde{z}_j),
\]

where \( m = \binom{n}{k} \).

**Proof.** Let \( R = V - S \) be the set of regular points of \( V \). We shall show that (9) holds for \( R \). Since both sides of (9) are regular Borel measures, it is sufficient to show that it holds for \( R \cap K \), \( K \) any compact subset of \( \Omega \).

For each \( z \in R \), we can find a neighborhood \( B \) of arbitrarily small diameter which is holomorphically homeomorphic to a closed polydisc \( A \) in \( \mathbb{C}^k \). The sets \( B \cap K \) form a covering of \( R \cap K \) in the sense of Vitali. By the classical covering theorem of Vitali, there exists a countable disjoint family \( \{B_j \cap K\} \) such that

\[
H_{2k}\left( R \cap K - \bigcup_{j=1}^{\infty} B_j \cap K \right) = 0.
\]

Hence it suffices to prove the formula (9) for each set \( B_j \cap K \).
Thus let $B$ be a subset of $R$ which is holomorphically homeomorphic to a closed polydisc $A$ in $C^{k}$, under the map

$$F: A \rightarrow B, \quad F(x) = (f_{1}(x), \ldots, f_{k}(x)).$$

Let $L$ be the complex Jacobian matrix of $F$ and $\tilde{L}$ the associated real Jacobian matrix. Let $JF = \Delta(L)$ and $J\tilde{F} = \Delta(\tilde{L})$ as given by (5) and (6). Then by (7),

$$H_{2k}(B) = \int_{A} JF(x) \, dH_{2k}(x).$$

This holds since the boundary of $A$ has measure zero and so does its image, the boundary of $B$, on account of property (ii) of the Hausdorff measures. By Lemma 2.1, $JF = J\tilde{F}$ and so

$$H_{2k}(B) = \int_{A} JF(x) \, dH_{2k}(x) = \sum_{j=1}^{m} \int_{\Lambda_{j}} |\det L_{j}(x)|^{2} \, dH_{2k}(x),$$

where the $L_{j}$'s are the $k \times k$ submatrices of $L$. With $\pi_{j}$ as defined in the statement of the theorem, each $L_{j}$ can be regarded as the Jacobian matrix of the map $F_{j} = \pi_{j} \circ F: A \rightarrow C^{N}$ mapping $A$ into the $j$th $k$-dimensional coordinate subspace of $C^{N}$. $F$ being a homeomorphism implies

$$H_{0}(A \cap F_{j}^{-1}(\xi_{j})) = H_{0}(B \cap \pi_{j}^{-1}(\xi_{j})).$$

So since $JF_{j} = |\det L_{j}|^{2}$, Theorem 3.1, formula (8) gives

$$\int_{A} |\det L_{j}(x)|^{2} \, dH_{2k}(x) = \int_{\pi_{j}B} H_{0}(B \cap \pi_{j}^{-1}(\xi_{j})) \, dH_{2k}(\xi_{j}).$$

Substituting in (10) we get

$$H_{2k}(B) = \sum_{j=1}^{m} \int_{\pi_{j}\Omega} H_{0}(B \cap \pi_{j}^{-1}(\xi_{j})) \, dH_{2k}(\xi_{j}).$$

Noting that $H_{0}(B \cap \pi_{j}^{-1}(\xi_{j})) = 0$ if $\xi_{j} \notin \pi_{j}\Omega$, we may replace the domain of integration $\pi_{j}B$ by $\pi_{j}\Omega$ and so complete the proof of (9) for the set $B$.

For later application, we give here a generalization of (9):

**Theorem 3.3.** With the notation as in Theorem 3.2, let $f$ be a nonnegative Borel function which vanishes outside $V$. Then

$$\int_{\Omega} f(z) \, dH_{2k}(z) = \sum_{j=1}^{m} \int_{\pi_{j}\Omega} dH_{2k}(\xi_{j}) \int_{\pi_{j}^{-1}(\xi_{j})} f(z) \, dH_{0}(z).$$

**Proof.** The proof of Theorem 3.2 shows that (9) holds for any open subset of $V$. Hence since both sides of (9) are regular Borel measures, it holds for any Borel subset $A$ of $V$. Thus (11) holds if $f = \chi_{A}$, the characteristic function of $A$; hence it holds if $f$ is any nonnegative simple Borel function vanishing outside $V$. If $f$ is any nonnegative Borel function, then there is an increasing sequence of nonnegative
simple Borel functions $s_n$ such that $\lim s_n = f$. Since (11) holds for each $s_n$, the monotone convergence theorem shows that it holds for $f$.

4. The multiplicity function. Let $f$ be a holomorphic function in a domain $\Omega$ in $\mathbb{C}^N$. For each $a \in \Omega$, we define the zero-multiplicity $\mu(a) = \mu_f(a)$ of $f$ at $a$ as follows: If $f \equiv 0$, then $\mu(a) = \infty$. If $f \not\equiv 0$, then $f$ has an expansion of the form

$$f(z) = f_m(z-a) + f_{m+1}(z-a) + \cdots$$

in a neighborhood of $a$, where $f_j$ is a homogeneous polynomial of degree $j$ and $f_n \not\equiv 0$. Define $\mu(a) = m$.

The following observations can be made:

(i) If $f = gh$, then $\mu_f = \mu_g + \mu_h$.

(ii) $\mu(a)$ does not depend on the choice of coordinates at $a$.

(iii) If $0 \in \Omega$ and $\mu(0) = m > 0$, then by the Weierstrass preparation theorem, there is a coordinate system $z_1, \ldots, z_N$ such that

$$f(z) = u(z)W$$

in a neighborhood of 0, where $u$ has no zeros in that neighborhood and $W$ is a Weierstrass polynomial of degree $m$ in $z_N$.

(iv) The converse of (iii) is also true, viz., if (12) holds, then $\mu(0) = m$, the degree of $W$ in $z_N$.

The first two observations follow easily from definition (see [6, 1.1.6]); (iv) is a consequence of the uniqueness of the Weierstrass polynomial for $f$.

**Proposition 4.1.** Let $f$ be a holomorphic function in a domain $\Omega$ in $\mathbb{C}^N$, $N > 1$. Then $\mu$ is constant on each (connectivity) component of the set of regular points of $V = Z(f)$.

**Proof.** Without loss of generality, let 0 be a regular point of $V$. Let $\mu(0) = m$. If $m = \infty$, then $f \equiv 0$ and the proposition is trivial. Suppose $m < \infty$. At a regular point, $V$ is an $(N-1)$-dimensional manifold. By the definition of a manifold (see [4, I.B. 8]), there exist a neighborhood $U_0$ of 0 and a mapping $F: U_0 \to \mathbb{C}$ which is nonsingular at 0 such that $V \cap U_0 = Z(F)$. $F$ being nonsingular at 0 implies that $\partial F(0)/\partial z_j \neq 0$ for some $j$, $1 \leq j \leq N$. By renaming the coordinates, we may assume that $j = N$ and $\partial F(0)/\partial z_N \neq 0$. By the implicit function theorem, there is a disc $D$ containing $0 \in \mathbb{C}$ and there is a function $\varphi(z')$ holomorphic in $D^{N-1}$ such that $(z', z_N) \in D^{N-1} \times D$ and $f(z', z_N) = 0$ if and only if $z_N = \varphi(z')$. Define $p(z) = z_N - \varphi(z')$, $z \in D_N$. Then in a possibly smaller neighborhood of $0 \in \mathbb{C}^N$ (again denoted by $U_0$) we have

$$Z(f) \cap U_0 = Z(F) \cap U_0 = Z(p) \cap U_0.$$ 

Thus in $U_0$, $f(z) = 0$ if and only if $p(z) = 0$. Since $p(0', z_N) \neq 0$, $f(0', z_N) \neq 0$. Therefore by the Weierstrass preparation theorem, $f = uW$ in a neighborhood $U_1$ of 0, where $u(z) \neq 0$ for all $z \in U_1$ and $W$ is a Weierstrass polynomial of degree $m$ in $z_N$. Let
Let \( W = \prod_{i=1}^{s} p_i^{n_i} \) be the factorization of \( W \) into irreducible factors. By taking \( U_1 \) small enough, we may assume that each \( p_i \) is holomorphic in \( U_1 \) and \( U_1 \subseteq U_0 \). Then

\[
Z(p) \cap U_1 = Z(f) \cap U_1 = Z(W) \cap U_1 = \bigcup_{i=1}^{s} Z(p_i).
\]

Since \( Z(p) \) is an irreducible variety at 0, the uniqueness of the irreducible decomposition of analytic varieties shows that \( s = 1 \) and \( p_1 = p \). Thus \( W = p^{n_1} \). Since \( W \) is of degree \( m \) and \( p \) is of degree 1 in \( z_N \), we must have \( n_1 = m \). So \( f = w p^m \) in \( U_1 \).

Now let \( b \in U_1 \cap V \). We claim that \( \mu(b) = m \). Let \( w = z - b, z \in U_1 \). Then \( f(z) = u(z)[p(z)]^m = \tilde{u}(w)[\tilde{p}(w)]^m \), where

\[
\tilde{p}(w) = p(w + b) = w_N + b_N - \varphi(w' + b') = w_N + \varphi(w').
\]

Since \( b \in V, \varphi(0') = b_N - \varphi(b') = 0 \). Therefore \( \tilde{p} \) is a Weierstrass polynomial at \( b \), hence so is \( \tilde{p}^m \). Since \( f = \tilde{a} \tilde{p}^m \), the observation (iv) above shows that \( \mu(b) = m \).

It follows then that \( \mu \) is constant on each connected component of the set of regular points of \( V \). Q.E.D.

This shows that the restriction of \( \mu \) to the regular points of \( V \) is a continuous function. In general we have

**Proposition 4.2.** Let \( f \) be a holomorphic function in a domain \( \Omega \) in \( \mathbb{C}^N \). Then its multiplicity function \( \mu \) is upper semicontinuous in \( \Omega \).

**Proof.** In the definition of \( \mu \), the homogeneous polynomials \( f_i \) are obtained by rearrangement of the Taylor series of \( f \) at \( a \). Thus if \( f \) has a partial derivative of total order \( n \) which is different from zero at \( a \), then \( \mu(a) = n \). Now let \( a \in \Omega \) and \( \mu(a) = m \). Then \( f \) has a partial derivative of total order \( m \) different from zero at \( a \).

By the continuity of the partial derivative, it is different from zero in a neighborhood \( U \) of \( a \). So for any \( b \in U, \mu(b) \leq m \).

Combining this with Theorem 3.3, noting that the zero-set of a nontrivial holomorphic function is a pure dimensional subvariety of codimension one, we get

**Corollary 4.3.** Let \( f \) be a holomorphic function in a domain \( \Omega \) in \( \mathbb{C}^{N+1}, f \neq 0 \). Let \( \mu \) be its multiplicity function. Then

\[
\int_{\Omega} \mu(z) \, dH_{2N}(z) = \sum_{j=1}^{N+1} \int_{\pi_j \Omega} dH_{2N}(\tilde{z}_j) \int_{\pi_j^{-1}(a)} \mu(z) \, dH_0(z_j)
\]

where \( \pi_j: \mathbb{C}^{N+1} \rightarrow \mathbb{C}^{N+1}, \pi_j(z) = (z_1, \ldots, z_{j-1}, 0, z_{j+1}, \ldots, z_{N+1}) = \tilde{z}_j \).

5. **The mean value of a plurisubharmonic function.** We recall the definition of a plurisubharmonic function: Let \( \Omega \) be a domain in \( \mathbb{C}^N \). A function \( u: \Omega \rightarrow [-\infty, \infty) \) is called plurisubharmonic if

(i) \( u \) is upper semicontinuous,

(ii) for any \( z \) and \( w \in \mathbb{C}^N \), the function \( \lambda \rightarrow u(z + \lambda w) \) is subharmonic where it is defined.
For subharmonic functions, the following is well known.

**Theorem 5.1.** Let \( u \) be a subharmonic function in the unit disc \( U \), \( u \neq -\infty \). Let \( m_1 \) be the Lebesgue measure on \( T \) normalized so that \( m_1(T) = 1 \). Let

\[
M_1(r) = \int_T u(r\lambda) \, dm_1(\lambda), \quad 0 \leq r < 1.
\]

Then (i) \( M_1(r) > -\infty \) if \( r > 0 \),

(ii) \( M_1(r) \leq M_1(s) \) if \( r \leq s \),

(iii) \( M_1(r) \) is a convex function of \( \log r \) in the interval \((0, 1)\), i.e. \( M_1(r^a r_2^{1-a}) \leq a M_1(r_1) + (1-a) M_1(r_2) \), whenever \( 0 \leq a \leq 1 \), \( 0 < r_1 \leq r_2 < 1 \).

See [7, Chapter 17] and [12, Chapter 2]. This has been generalized to plurisubharmonic functions (see [12]). We give here a further generalization.

Let \( \Omega \) be a domain in \( \mathbb{C}^n \). Then \( \Omega \) is called a complete circular domain if \( z = (z_1, \ldots, z_n) \in \Omega \) and \( |\lambda| \leq 1 \) imply \( Az = (\lambda z_1, \ldots, \lambda z_n) \in \Omega \). Thus \( U^N \) and \( B_N \) are complete circular domains. A measure \( m \) on \( \Omega \) is said to be circularly invariant if \( m(E) = m(AE) \) for any measurable subset \( E \) of \( \Omega \) and any \( \lambda \in \mathcal{T} \), where \( E_\lambda = \{Az : z \in E\} \). The Lebesgue measure is circularly invariant. Such measures were first considered by Bochner [2]. His method can be applied to prove the following.

**Theorem 5.2.** Let \( \Omega \) be a bounded complete circular domain in \( \mathbb{C}^n \). Let \( m \) be a positive circularly invariant measure on \( \Omega \) normalized so that \( m(\Omega) = 1 \). Let \( u \) be a plurisubharmonic function in \( \Omega \), \( u \neq -\infty \), and let

\[
M(r) = \int_\Omega u(rz) \, dm(z), \quad 0 \leq r \leq 1.
\]

Then (i) \( M(r) > -\infty \) if \( r > 0 \) and \( m \) is the Lebesgue measure,

(ii) \( M(r) \leq M(s) \) if \( r \leq s \),

(iii) \( M(r) \) is a convex function of \( \log r \) in the interval \((0, 1)\).

**Proof.** To prove (ii) and (iii), we define for each \( z \in \Omega \), \( u_\lambda(z) = u(\lambda z) \). Then \( u_\lambda \) is a subharmonic function in a neighborhood of \( \mathring{U} \). So by Theorem 5.1, we have for all \( z \in \Omega \) and \( 0 \leq r \leq s \leq 1 \),

\[
\int_T u(r\lambda z) \, dm_1(\lambda) \leq \int_T u(s\lambda z) \, dm_1(\lambda).
\]

Since \( m \) is circularly invariant,

\[
\int_\Omega u(rz) \, dm(z) = \int_\Omega u(r\lambda z) \, dm(\lambda) \quad \text{for all } \lambda \in \mathcal{T}.
\]

Hence integrating over \( T \), we get

\[
\int_\Omega u(rz) \, dm(z) = \int_T dm_1(\lambda) \int_\Omega u(r\lambda z) \, dm(z) = \int_\Omega dm(z) \int_T u(r\lambda z) \, dm_1(\lambda)
\]

by Fubini's theorem.
From (14), we get by integrating over $\Omega$,

$$
\int_{\Omega} dm(z) \int_{\tau} u(\lambda z) \, dm_1(\lambda) \leq \int_{\Omega} dm(z) \int_{\tau} u(x\lambda z) \, dm_1(\lambda).
$$

Hence substitution of (15) gives

$$
\int_{\Omega} u(rz) \, dm(z) \leq \int_{\Omega} u(sz) \, dm(z), \quad 0 \leq r \leq s \leq 1,
$$

which is (ii).

Let $M_z(r) = \int_{\tau} u(r\lambda z) \, dm_1(\lambda)$, $z \in \Omega$. Then

$$
M_z(r^a r_2^{1-a}) \leq a M_z(r_1) + (1-a) M_z(r_2)
$$

whenever $0 \leq a \leq 1$, $0 < r_1 \leq r_2 < 1$. Integrating over $\Omega$ and noting that by (15), $M(r) = \int_{\Omega} M_z(r) \, dm(z)$, we get

$$
M(r^a r_2^{1-a}) \leq a M(r_1) + (1-a) M(r_2).
$$

This proves (iii).

To prove (i), let $m$ be the Lebesgue measure on $\Omega$. Then since the Jacobian of the transformation $z \rightarrow rz$ is $r^{2N}$,

$$
M(r) = \frac{1}{r^{2N}} \int_{r\Omega} u(z) \, dm(z).
$$

We note further that the proof of (ii) gives the following: If the ball $B = B(a, r)$ of center $a$ and radius $r$ is contained in $\Omega$, then

$$
u(a) \leq \frac{1}{m(B)} \int_B u(z) \, dm(z).
$$

Now suppose $M(r_0) = -\infty$ for some $r_0 > 0$. Then there exist an $a \in r_0 \Omega$ and a number $r_1 > 0$ such that $B(a, 3r_1) \subseteq \Omega$ and

$$
\int_{B(a, r_1)} u(z) \, dm(z) = -\infty.
$$

By (16), $u(a) = -\infty$. Let $z' \in B(a, r_1)$. Then

$$
B(a, r_1) \subseteq B(z', 2r_1) \subseteq B(a, 3r_1).
$$

Therefore

$$
\int_{B(z', 2r_1)} u(z) \, dm(z) = -\infty,
$$

which implies as before $u(z') = -\infty$. So $\Omega_0$, the interior of the set $\{z \in \Omega : u(z) = -\infty\}$, is a nonempty open set. If $z$ is a limit point of $\Omega_0$ in $\Omega$, then by the above argument, we also have $u(z) = -\infty$ and $z \in \Omega_0$; hence $\Omega_0$ is also closed in $\Omega$. Since $\Omega$ is connected, $\Omega_0 = \Omega$ and $u = -\infty$, a contradiction. Q.E.D.

If $f$ is a holomorphic function in $\Omega$, then $\log |f|$ is a plurisubharmonic function in $\Omega$. Thus we have the following corollary which will be used in the proof of the Blaschke condition.
Corollary 5.3. Let $\Omega = U^N$ or $B_N$ and let $m$ be the Lebesgue measure on $\Omega$ normalized so that $m(\Omega) = 1$. Let $f \in H(\Omega)$, $f \neq 0$. Then
\[
\int_{\Omega} \log |f(rz)| \, dm(z) > -\infty \quad \text{if } 0 < r \leq 1.
\]
If $f(0) \neq 0$, then
\[
\int_{\Omega} \log |f(z)| \, dm(z) \geq \log |f(0)|.
\]
If $f \in H^\infty(\Omega)$, then $\log |f| \in L^1(m)$.

6. The Blaschke condition. We now show that the generalized Blaschke condition holds for bounded holomorphic functions in several complex variables. We begin with a (well-known) lemma.

Lemma 6.1. Let $X$ be a Lebesgue measurable subset of $\mathbb{R}^N$, $I$ an interval in $\mathbb{R}$. For each positive integer $k$, let $m_k$ be the Lebesgue measure on $\mathbb{R}^k$. Let $f: I \times X \to [-\infty, \infty]$ be a function satisfying the conditions
(i) for each $t \in I$, $x \mapsto f(t, x)$ is Lebesgue measurable,
(ii) for each $x \in X$, $t \mapsto f(t, x)$ is increasing.

Then $f$ is a Lebesgue measurable function in $I \times X$.

Proof. In what follows, measurable will mean Lebesgue measurable. It is sufficient to prove that
\[
A = \{(t, x) \in I \times X : f(t, x) > \alpha\}
\]
is measurable for every real number $\alpha$. Since $I$ is $\sigma$-compact and $X$ is the union of an $F_\sigma$ and a set of measure zero, we may assume that they are compact.

Let $\varepsilon > 0$ be given. Choose points $t_0, t_1, \ldots, t_n \in I$ such that $t_0 < t_1 < \cdots < t_n$, $[t_0, t_n] = I$ and $m_1(I_i) \leq \varepsilon$ where $I_i = [t_i, t_{i+1}]$, $0 \leq i \leq n-1$. Let $A_i = \{x : f(t_i, x) > \alpha\}$. By condition (i), each $A_i$ is measurable. By (ii), $A_i \subseteq A_j$ if $i \leq j$. Let $B = \bigcup_{i=0}^{n-1} (I_i \times A_i)$, $C = \bigcup_{i=0}^{n-1} (I_i \times A_{i+1})$. Then $B$ and $C$ are measurable subsets of $I \times X$ and by the condition (ii), it is easy to check that $B \subseteq A \subseteq C$. Now
\[
C - B = \bigcup_{i=0}^{n-1} (I_i \times (A_{i+1} - A_i)).
\]
Since $(A_{i+1} - A_i) \cap (A_i - A_{i-1}) = \emptyset$ for all $i$, we have
\[
m_{N+1}(C - B) = \sum_{i=0}^{n-1} m_1(I_i) m_N(A_{i+1} - A_i)
\]
\[
\leq \varepsilon \sum_{i=0}^{n-1} m_N(A_{i+1} - A_i)
\]
\[
= \varepsilon m_N \left( \bigcup_{i=0}^{n-1} (A_{i+1} - A_i) \right)
\]
\[
\leq \varepsilon m_N(X).
\]
Let \( \varepsilon = 1, \frac{1}{2}, \frac{1}{3}, \ldots \). Then we see that there is an increasing sequence \( B_k \) and a decreasing sequence \( C_k \) of measurable sets such that with \( E = \bigcup_{k=1}^{\infty} B_k \), \( F = \bigcup_{k=1}^{\infty} C_k \), we have \( E \subseteq A \subseteq F \) and \( m_{N+1}(F - E) = 0 \). Hence \( A \) is Lebesgue measurable.

For our application, we note that the Lebesgue measure in \( \mathbb{R}^N \) coincides with the Hausdorff measure \( H_N \) in \( \mathbb{R}^N \). For what follows, we shall use the following notation: \( N \) will denote a positive integer. For \( j = 1, 2, \ldots, N+1 \), \( \pi_j \) will denote the projection on \( \mathbb{C}^{N+1} \) defined by

\[
\pi_j(z) = (z_1, \ldots, z_{j-1}, 0, z_{j+1}, \ldots, z_{N+1})
\]

and we write \( \hat{z}_j = \pi_j(z) \). \( \Omega \) will denote \( U^{N+1} \) or \( B_{N+1} \) and for \( 0 < r < 1 \), \( \Omega(r) \) is the corresponding domain of radius \( r \).

**Lemma 6.2.** Let \( f \in H(\Omega) \), \( f \neq 0 \) and \( \mu \) its multiplicity function. Let \( V_r = Z(f) \cap \Omega(r) \). Then for each \( j, 1 \leq j \leq N+1 \), the function

\[
F(r, \hat{z}_j) = \int_{V_r \cap \pi_j^{-1}(\hat{z}_j)} \mu(z) \, dH_0(z)
\]

is Lebesgue measurable in \((0, 1) \times \pi_j \Omega \).

**Proof.** Fix \( j, 1 \leq j \leq N+1 \). Clearly \( F(r, \hat{z}_j) \) is an increasing function of \( r \) for each \( \hat{z}_j \). Thus in view of Lemma 6.1, we need only show that \( F_r : \hat{z}_j \to F(r, \hat{z}_j) \) is Lebesgue measurable for each \( r \).

Fix \( r \in (0, 1) \). Let \( S \) be the singular locus of \( Z(f) \). Then \( H_{2N}(S) = 0 \); hence by property (ii) of the Hausdorff measures, \( H_{2N}(\pi_j S) = 0 \). We shall show that \( F_r \) is a Borel function on \( R = \pi_j \Omega - \pi_j S \). This will imply that \( F_r \) is Lebesgue measurable on \( \pi_j \Omega \).

The value \( n = F(r, \hat{z}_j) \) is a nonnegative integer or \( \infty \). Suppose \( n \neq 0 \) or \( \infty \). Then \( V_r \cap \pi_j^{-1}(\hat{z}_j) \) consists of only a finite number of points. If \( \hat{z}_j \in R \), then each point of \( V_r \cap \pi_j^{-1}(\hat{z}_j) \) is a regular point of \( Z(f) \). By Proposition 4.1, \( \mu(z) \) is constant in a neighborhood of each such point. So \( F_r \) is constant in a neighborhood of \( \hat{z}_j \). If \( n = 0 \), then \( f \) has no zeros on \( V_r \cap \pi_j^{-1}(\hat{z}_j) \). By the continuity of \( f \), it has no zeros in a neighborhood of \( V_r \cap \pi_j^{-1}(\hat{z}_j) \). Thus for each integer \( n, 0 \leq n < \infty \), the set \( A_n = \{ \hat{z}_j \in R : F(r, \hat{z}_j) = n \} \) is an open set in \( R \). Since \( A_\infty = \{ \hat{z}_j \in R : F(r, \hat{z}_j) = \infty \} = R - \bigcup_{n=1}^{\infty} A_n \), we see that \( A_\infty \) is a closed set of \( R \). This shows that \( F_r \) is a Borel function on \( R \) and the proof is complete.

**Theorem 6.3.** Let \( f \in H^\infty(\Omega) \), \( f \neq 0 \) and \( |f| \leq 1 \). Let \( \mu \) be its multiplicity function. Then

\[
\int_0^1 dr \int_{\Omega(r)} \mu(z) \, dH_{2N}(z) < \infty.
\]

If \( f(0) \neq 0 \), then

\[
\int_0^1 dr \int_{\Omega(r)} \mu(z) \, dH_{2N}(z) \leq c(\Omega) \log \frac{1}{|f(0)|}
\]

where \( c(U^{N+1}) = (N+1)\pi^N \), \( c(B_{N+1}) = (N+1)\pi^N/N! \).
Proof. Assume first that \( f(\hat{z}_j) \neq 0 \) for all \( j \), \( 1 \leq j \leq N + 1 \). Let \( f_j(z_j) = f(z) \) and let \( \mu_j(z_j) \) be the zero-multiplicity of \( f_j \) at \( z_j \). It is easily seen that \( \mu_j(z_j) \geq \mu(z) \). Let \( n_j(r) \) be the number of zeros of \( f_j \) in \( \pi^{-1}(\hat{z}_j) \cap \overline{D}(r) \), counting multiplicities. Let \( V_r = Z(f) \cap \overline{D}(r) \). By Corollary 4.3,

\[
\int_{\overline{D}(r)} \mu(z) \, dH_{2N}(z) = \sum_{j=1}^{N+1} \int_{\overline{D}(r) \cap \pi^{-1}(\hat{z}_j)} \mu(z) \, dH_0(z_j).
\]

Since \( \hat{z}_j \notin \pi_j \Omega \) implies \( V_r \cap \pi_j^{-1}(\hat{z}_j) = \emptyset \), this can be written

\[
\int_{\overline{D}(r)} \mu(z) \, dH_{2N}(z) = \sum_{j=1}^{N+1} \int_{\pi_j \Omega \setminus \hat{z}_j} \mu(z) \, dH_0(z_j)
\]

(19)

where

\[
F(r, \hat{z}_j) = \int_{\pi_j \Omega \setminus \hat{z}_j} \mu(z) \, dH_0(z_j).
\]

By Lemma 6.2, \( F \) is Lebesgue measurable in \((0, 1) \times \pi_j \Omega\) and Fubini's theorem applies to give

\[
\int_0^1 dr \int_{\pi_j \Omega} F(r, \hat{z}_j) \, dH_{2N}(\hat{z}_j) = \int_{\pi_j \Omega} dH_{2N}(\hat{z}_j) \int_0^1 F(r, \hat{z}_j) \, dr.
\]

Since \( \mu(z) \leq \mu_j(z_j) \), we have

\[
F(r, \hat{z}_j) \leq \int_{\pi_j \Omega \setminus \hat{z}_j} \mu_j(z_j) \, dH_0(z_j) = n_j(\rho)
\]

where \( \rho = r \) if \( \Omega = U^{N+1} \) and \( \rho = (r^2 - \| \hat{z}_j \|^2)^{1/2} \) if \( \Omega = B_{N+1} \) (\( \| \cdot \| \) is Euclidean norm of \( \hat{z}_j \)). Noting that \( dr/\rho = \rho/\sqrt{r} \leq 1 \), we get by Jensen's formula (2),

\[
\int_0^1 n_j(\rho) \, dr = \int_0^1 n_j(r) \, dr \leq \log \frac{1}{|f(\hat{z}_j)|} \quad \text{if } \Omega = U^{N+1};
\]

and

\[
\int_0^1 n_j(\rho) \, dr = \int_0^a n_j(\rho) \frac{\rho}{r} \, d\rho \leq \int_0^a n_j(\rho) \, d\rho, \quad (a = (1 - \| \hat{z}_j \|^2)^{1/2})
\]

\[
\leq \log \frac{1}{|f(\hat{z}_j)|} \quad \text{if } \Omega = B_{N+1}.
\]

Thus

\[
\int_0^1 F(r, \hat{z}_j) \, dr \leq \log \frac{1}{|f(\hat{z}_j)|}.
\]

Integrating (19) with respect to \( r \) and substituting (20) and (21), we get

\[
\int_0^1 dr \int_{\overline{D}(r)} \mu(z) \, dH_{2N}(z) \leq \sum_{j=1}^{N+1} \int_{\pi_j \Omega} \log \frac{1}{|f(\hat{z}_j)|} \, dH_{2N}(\hat{z}_j).
\]

Since \( f(\hat{z}_j) \neq 0 \) for all \( j \), Corollary 5.3 shows that each integral on the right is finite and (17) is proved.
If $f(0) \neq 0$, Corollary 5.3 gives
\[ \int_{\tau_{2N}} \log \frac{1}{|f(\xi)|} \, dH_{2N}(\xi) \leq H_{2N}(\pi, \Omega) \log \frac{1}{|f(0)|}. \]

Hence,
\[ \int_{0}^{1} \int_{\tilde{\Omega}(r)} \mu(z) \, dH_{2N}(z) \leq \left( \sum_{j=1}^{N+1} H_{2N}(\pi, \Omega) \right) \log \frac{1}{|f(0)|}. \]

Putting $c(\Omega) = \sum_{j=1}^{N+1} H_{2N}(\pi, \Omega)$, we get (18).

The case when $f(\xi_j) = 0$ for some $j$ can be reduced to the first case as follows. We do this separately for $U_{N+1}$ and $B_{N+1}$.

For $U_{N+1}$. If $f(\xi_j) = 0$ for some $j$, then there is a positive integer $\alpha_j$ such that $g_j(z) = f(z)/z_{\xi_j}^\alpha$ is holomorphic in $U_{N+1}$ and $g_j(\xi_j) \neq 0$. Doing this for all $j$, we get nonnegative integers $\alpha_j$ such that

\[ f(z) = z_{\xi_1}^{\alpha_1} \cdots z_{\xi_{N+1}}^{\alpha_{N+1}} g(z) \]

where $g$ is holomorphic in $U_{N+1}$ and $g(\xi_j) \neq 0$ for all $j$. Since $|f(z)| \leq 1$ as $z$ tends to $T_{N+1}$, the same is true for $g$, so that $|g| \leq 1$ in $U_{N+1}$ by the maximum modulus theorem. Thus the first part of the proof applies to $g$. An easy computation shows that each factor $z_{\xi_i}^{\alpha_i}$ contributes $\alpha_i \pi^N/(2N+1)$ to the integral in (17). Thus with $\mu_g = \text{multiplicity function of } g$,

\[ \int_{0}^{1} \int_{\tilde{\Omega}(r)} \mu(z) \, dH_{2N}(z) = \int_{0}^{1} \int_{\tilde{\Omega}(r)} \mu_g(z) \, dH_{2N}(z) + \frac{\pi^N}{2N+1} \left( \sum_{j=1}^{N+1} \alpha_j \right) < \infty. \]

For $B_{N+1}$, we have the following lemma.

\textbf{Lemma 6.4.} Let $f \in H(B_N), f \neq 0$. Then there exists a coordinate system $z_1, \ldots, z_N$ such that $f(\xi_j) \neq 0$ for all $j$.

\textbf{Proof.} Without loss of generality, we may assume that $f \in H(\tilde{B}_N)$. Let $e_1, \ldots, e_N$ be $N$ points on $S^{2N-1}$ which form an orthogonal basis for $C^N$. We shall show that there exists a unitary transformation $A$ such that $f(Ae_1)f(Ae_2) \cdots f(Ae_N) \neq 0$. Then $Ae_1, \ldots, Ae_N$ will give the required coordinate system.

If $f(e_1) \neq 0$, we take $A_1 = I$ the identity transformation. If $f(e_1) = 0$, then since $f$ is not identically zero on $S^{2N-1}$, there is an $\bar{e}_1 \in S^{2N-1}$ such that $f(\bar{e}_1) \neq 0$. Since the unitary transformations are transitive on $S^{2N-1}$, we can find a unitary transformation $A_1$ such that $A_1e_1 = \bar{e}_1$. By the continuity of $f$, there is a neighborhood $W_1$ of $\bar{e}_1$ such that $f(\xi) \neq 0$ for all $\xi \in W_1$.

If $f(A_1e_2) \neq 0$, we take $A_2 = I$. If $f(A_1e_2) = 0$, then since $Z(f)$ is nowhere dense on $S^{2N-1}$, there is an $\bar{e}_2$ arbitrarily close to $A_1e_2$ such that $f(\bar{e}_2) \neq 0$. Then $\bar{e}_2 = A_2A_1e_2$ for some unitary transformation $A_2$. We can choose $\bar{e}_2$ so close to $A_1e_2$ that $A_2\bar{e}_1 \in W_1$. Fix such an $A_2$. Then there exists a neighborhood $W_2$ of $\bar{e}_2$ such that $f(\xi) \neq 0$ for all $\xi \in W_2$.

Continuing the process $N$ times, we get unitary transformations $A_1, \ldots, A_N$ such that if $A = A_NA_{N-1}, \ldots, A_1$, then $f(Ae_j) \neq 0$ for all $j$. This completes the proof of the lemma and that of the theorem.
7. **Examples.** In contrast to the theorem in one variable, the Blaschke condition is not sufficient for an analytic subvariety to be the zero-set of a bounded holomorphic function in $U^2$ or $B_2$. In fact there are analytic subvarieties which satisfy the Blaschke condition and which are determining sets for bounded holomorphic functions.

**Example 1.** Let $\alpha_n = 1 - 1/n$ and

$$V = \{(\alpha_n, w): |\alpha_n|^2 + |w|^2 < 1, n = 1, 2, 3, \ldots\}.$$ 

Then (by Cartan’s Theorem B) $V$ is the zero-set of a holomorphic function in $B_2$. But $V$ is a $D$-set for bounded holomorphic functions in $B_2$, although it satisfies the Blaschke condition.

An easy calculation shows that $H_2(V_r) = \pi \sum_{|\alpha_n| \leq r} (r^2 - \alpha_n^2)$. Hence

$$\int_0^1 H_2(V_r) \, dr = \pi \int_0^1 \sum_{|\alpha_n| \leq r} (r^2 - \alpha_n^2) \, dr = \pi \sum_{n=1}^\infty \int_0^1 (r^2 - \alpha_n^2) \, dr$$

$$= \frac{\pi}{3} \sum_{n=1}^\infty (1 - \alpha_n)^2 (1 + 2\alpha_n) = \frac{\pi}{3} \sum_{n=1}^\infty \frac{1}{n^2} \left(3 - \frac{2}{n}\right) < \infty.$$

Now suppose $f \in H^\infty(B_2)$ and $f = 0$ on $V$. We shall show that then $f \equiv 0$.

For each $c \in C$, let $D(c)$ be the disc in the $z$-plane passing through the point $z = 1$ and having center at $z = |c|^2/(1 + |c|^2)$. $D(c) \subset U$ for all $c$. Let

$$P(c) = \{(z, c(1-z)): z \in D(c)\}.$$ 

Then $P(c)$ is a disc imbedded in $B_2$ and its boundary passes through the point $(1, 0)$ for all $c$.

For each $c$, let $f_c(z) = f(z, c(1-z)), z \in D(c)$. When $n$ is sufficiently large, $\alpha_n \in D(c)$ and $f_c(\alpha_n) = 0$. Therefore the zero-set of $f_c$ violates the Blaschke condition. Since $f_c$ is bounded, $f_c \equiv 0$, i.e. $f|_{P(c)} \equiv 0$ for all $c$. Since $B_2 = \bigcup_{c \in C} P(c)$, we have $f \equiv 0$.

**Example 2.** Fix $\delta, 0 < \delta < 1$. Let $\alpha_n = 1 - 1/n^8$ and

$$V = \{(z, 2\alpha_n - z): |z| < 1, |2\alpha_n - z| < 1, n = 1, 2, 3, \ldots\}.$$ 

Then $V$ is the zero-set of a holomorphic function in $U^2$. We shall show that it satisfies the Blaschke condition and is a $D$-set for bounded holomorphic functions.

For each $n$, the area of the set $\{(z, 2\alpha_n - z): |z| \leq r, |2\alpha_n - z| \leq r\}$ is $\leq 2\pi(r^2 - \alpha_n^2)$. So $H_2(V_r) \leq 2\pi \sum_{|\alpha_n| \leq r} (r^2 - \alpha_n^2)$. The computation in Example 1 shows that

$$\int_0^1 H_2(V_r) \, dr \leq \frac{2\pi}{3} \sum_{n=1}^\infty (1 - \alpha_n)^2 (1 + 2\alpha_n)$$

$$= \frac{2\pi}{3} \sum_{n=1}^\infty \frac{1}{n^{28}} (1 + 2\alpha_n)$$

$$< \infty \quad \text{since} \quad 2\delta > 1.$$

Let $f \in H^\infty(U^2)$ and $f = 0$ on $V$. Let

$$A = \{c \in C: \Re c > 1, |\arg c| < (1 - \delta)\pi/2\}.$$
For \( c \in A \), the boundary of the disc \( U(c) \) of radius \( 1/|c| \) and center at \( 1 - 1/c \) makes an angle \( k\pi/2 \) with the real axis, where \( \delta < k < 1 \). The real axis divides \( U(c) \) into two regions; let \( U_1(c) \) be the smaller one. Let \( D(c) \) be the region formed by \( U_1(c) \) and its reflection in the real axis. Then \( D(c) \) is contained in the unit disc \( U \) and is bounded by two circular arcs meeting at an angle \( k\pi \) at the point \( z = 1 \) and \( z = z_0 \), where \( z_0 = 1 - 2 \text{Re} \ c/|c|^2 \). Let \( P(c) = \{(z, \ c(z-1)+1) : z \in D(c)\} \). For every \( c \in A \), \( P(c) \) is a subset of \( U^2 \) such that the point \((1, 1)\) lies on its boundary.

Fix \( c \in A \). Define \( f_c(z) = f(z, c(z-1)+1), \ z \in D(c) \). Let \( \tilde{a}_n = 1 - (2/(1+c))(1/n^k) \). For all sufficiently large \( n \), \( \tilde{a}_n \in D(c) \) and since \( f = 0 \) on \( V \), \( f_c(\tilde{a}_n) = 0 \). Under the mapping \( \varphi_c(z) = ((1-z)/(z-z_0))^{1/k}, \ D(c) \) is mapped onto the right half-plane. Let \( \beta_n = \varphi_c(\tilde{a}_n) \). Then it is easy to check that for fixed \( c \in A \), \( \text{Re} \ \beta_n \geq \gamma n^{-\delta/k} \) for sufficiently large \( n \), where \( \gamma \) is positive and does not depend on \( n \). Since \( \delta/k < 1 \), it follows that

\[
\sum \text{Re} \ \beta_n = \infty.
\]

Thus the function \( f' = f_c \cdot \varphi_c^{-1} \) is a bounded holomorphic function in the right half-plane whose zeros \( \beta_n \) satisfy (22). So \( f' \equiv 0 \) which implies that \( f_c \equiv 0 \), i.e. \( f|_{P(c)} \equiv 0 \) for all \( c \in A \). Let \( P = \bigcup_{c \in A} P(c) \). Then \( P \) contains an open subset of \( U^2 \) since the open subset \( D \times \Delta \) of \( C^2 \), where \( D = D(1+i) \) and \( \Delta = A \cap \{c : |c-1| < 1\} \), is mapped into \( P \) by \( \Phi_c(z, c) \rightarrow (z, c(z-1)+1) \) which is nonsingular when \( z \neq 1 \). So \( f = 0 \) on \( P \) implies \( f \equiv 0 \) in \( U^2 \).

**Added in proof.** Recently the author has extended Theorem 6.3 to wider classes of functions, namely the Nevanlinna classes on \( U^N \) and \( B^N \).

**References**