NONLINEAR EVOLUTION EQUATIONS AND PRODUCT INTEGRATION IN BANACH SPACES

BY

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Abstract. The method of product integration is used to obtain solutions to the nonlinear evolution equation $g' = Ag$ where $A$ is a function from a Banach space $S$ to itself and $g$ is a continuously differentiable function from $[0, \infty)$ to $S$. The conditions required on $A$ are that $A$ is dissipative on $S$, the range of $(e - \varepsilon A) = S$ for all $\varepsilon \geq 0$, and $A$ is continuous on $S$.

1. Introduction. Let $S$ be a Banach space and let $A$ be a mapping from a subset of $S$ to $S$. An evolution equation is a system $g' = A(g), g(0) = p$, where $g$ is a continuous function from $[0, \infty)$ to $S$ and $p$ is a point in $S$. In [3] F. Browder has considered nonlinear evolution equations in which $S$ is the Hilbert space and $A$ is continuous, bounded, and dissipative on $S$. In recent articles Y. Komura [12], T. Kato [10], and M. Crandall and A. Pazy [5] have considered nonlinear evolution equations in which $S$ is the Hilbert space and $A$ is maximal dissipative, not necessarily continuous, and is the infinitesimal generator of a semigroup of nonlinear nonexpansive transformations on $S$.

The object of this paper is to obtain solutions to an evolution system in a general Banach space using the method of product integration. A definition of product integration is given as follows:

Suppose that $p$ is in $S$, $x > 0$, and $z$ is a point in $S$ such that if $c > 0$ there exists a chain $\{s_i\}_{i=0}^n$ from 0 to $x$ such that if $\{n_i\}_{i=0}^n$ is a refinement of $\{s_i\}_{i=0}^n$ then

\[ z - \prod_{i=1}^n (e^{-(t_i-t_{i-1})}A)^{-1}p \leq c. \]

(Note that $e$ denotes the identity map on $S$, $(e-(t_i-t_{i-1})A)^{-1}$ denotes the inverse map of $(e-(t_i-t_{i-1})A)^{-1}$, $\prod_{i=1}^j (e-(t_i-t_{i-1})A)^{-1}p = (e-(t_j-t_0)A)^{-1}p$, and if $j$ is an integer in $[2, n]$)

\[ \prod_{i=1}^j (e-(t_i-t_{i-1})A)^{-1} = (e-(t_j-t_{j-1})A)^{-1} \prod_{i=1}^{j-1} (e-(t_i-t_{i-1})A)^{-1}p, \]

where the product operation is composition of mappings.) Then $z$ is said to be the product integral of $A$ with respect to $p$ from 0 to $x$ and is denoted by $\prod_{0}^x (e-dIA)^{-1}p$.

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In [1] G. Birkhoff and in [16] J. Neuberger have used product integration to solve evolution systems where the mapping $A$ is Lipschitz continuous. In this paper the product integration method will be extended to mappings not necessarily Lipschitz continuous.

2. An existence theorem. Let $A$ be a mapping from a subset of $S$ to $S$ such that the following are true:

(I) $A$ is dissipative on its domain $D_A$, i.e., if $u, v \in D_A$ and $\epsilon \geq 0$ then 
$$
\|(e-\epsilon A)u-(e-\epsilon A)v\| \geq \|u-v\|.
$$

(II) There is an open subset $C$ of $S$ such that $C \subseteq D_A$ and a positive number $\alpha$ such that if $0 \leq \epsilon < \alpha$ then $C \subseteq R_{(e-\epsilon A)}$ (where $R_{(e-\epsilon A)}$ denotes the range of $(e-\epsilon A)$).

(III) $A$ is continuous on $C$.

Note that by (I) if $\epsilon > 0$ then $(e-\epsilon A)$ is 1-1 on $D_A$ and by (II) if $0 \leq \epsilon < \alpha$ and $q \in C$ then $q \in D_{(e-\epsilon A)^{-1}} = R_{(e-\epsilon A)}$. If $0 \leq \epsilon < \alpha$ and $q \in R_{(e-\epsilon A)}$ let $L(\epsilon)q = (e-\epsilon A)^{-1}q$. By (I) $L(\epsilon)$ is nonexpansive on $R_{(e-\epsilon A)}$, i.e., if $u, v \in R_{(e-\epsilon A)}$ then

$$
\|L(\epsilon)u-L(\epsilon)v\| \leq \|u-v\|.
$$

Theorem. Let $A$ satisfy conditions (I), (II), and (III). If $p \in C$ and

$$
\gamma_p = \min \{\text{dist}(p, \partial C)/\|Ap\|, \alpha\},
$$

then there is a continuously differentiable function $g_p$ from $[0, \gamma_p)$ to $S$ such that $g_p(0)=p$ and if $0 \leq x < \gamma_p$, $g'_p(x) = Ag_p(x)$ and $g_p(x) = \prod_{\epsilon=\gamma_p}^{\gamma_p} (e-dIA)^{-1}p$.

The theorem will be proved by means of a sequence of lemmas each of which is under the hypothesis of the theorem.

Lemma 1.1. If $q \in C$ and $0 \leq x, y < \alpha$, then 

$$
\|L(x)q-L(y)q\| \leq |x-y| \cdot \|Aq\|.
$$

Proof. Using (2) we have that

$$
\|L(x)q-L(y)q\| = \|L(x)q-L(x)(e-xA)L(y)q\|
\leq \|q-(e-xA)L(y)q\|
= \|q-[(x/y)(e-yA)L(y)q+(1-x/y)L(y)q]\|
= |1-x/y| \|q-L(y)q\|
\leq |1-x/y| \|(e-yA)q-q\|
= |x-y| \|Aq\|.
$$

Lemma 1.2. Let $q \in C$, let $0 < x < \gamma_q$, and let $\{s_i\}_{i=0}^{m}$ be a chain from $0$ to $x$. If $j$ is an integer in $[1, m]$ then

$$
\prod_{i=1}^{j-1} L(s_i-s_{i-1})q \in C,
$$

(3)

$$
\prod_{i=1}^{j} L(s_i-s_{i-1})q \in C.
$$

(4)
\[ A \prod_{i=1}^{j} L(s_i - s_{i-1})q \leq Aq. \]

(Note that \( \prod_{i=1}^{j} L(s_i - s_{i-1}) \) denotes the identity map, i.e., \( \prod_{i=1}^{j} L(s_i - s_{i-1})q = q \).

**Proof.** The proof is by induction. For \( j = 1 \), \( \prod_{i=1}^{1} L(s_i - s_{i-1})q = q \in C \),

\[ \left\| A \prod_{i=1}^{1} L(s_i - s_{i-1})q - q \right\| \leq s_1 \cdot \| Aq \| \]

(by Lemma 1.1), and

\[ A \prod_{i=1}^{1} L(s_i - s_{i-1})q = \| 1/s_1 [L(s_1 - s_0)q - q] \| \leq Aq. \]

Suppose that \( j \) is an integer in \( [1, m - 1] \), \( \prod_{i=1}^{j} L(s_i - s_{i-1})q \in C \),

\[ \left\| A \prod_{i=1}^{j} L(s_i - s_{i-1})q - q \right\| \leq s_{j+1} \cdot \| Aq \| , \]

and \( \| A \prod_{i=1}^{j} L(s_i - s_{i-1})q \| \leq \| Aq \| . \) Then,

\[ A \prod_{i=1}^{j} L(s_i - s_{i-1})q \in C \subseteq D_{t_{j+1} - t_j}. \]

Further,

\[ \left\| A \prod_{i=1}^{j+1} L(s_i - s_{i-1})q - q \right\| = \left\| \sum_{i=1}^{j+1} \left[ \sum_{k=i}^{j+1} L(s_k - s_{k-1})q - \sum_{k=i}^{j+1} L(s_{k} - s_{k-1})q \right] \right\| \]

(note that \( \sum_{k=j+1}^{j+2} L(s_k - s_{k-1}) \) is the identity map)

\[ \leq \sum_{i=1}^{j+1} \| L(s_i - s_{i-1})q - q \| \]

\[ \leq s_{j+1} \cdot \| Aq \| .\]

Moreover,

\[ \left\| A \prod_{i=1}^{j+1} L(s_i - s_{i-1})q \right\| = \left\| \left( \frac{1}{s_{j+1} - s_j} \right) \left[ \prod_{i=1}^{j+1} L(s_i - s_{i-1})q - \prod_{i=1}^{j} L(s_i - s_{i-1})q \right] \right\| \]

\[ \leq \left\| A \prod_{i=1}^{j} L(s_i - s_{i-1})q \right\| \]

\[ \leq \| Aq \|. \]

**Lemma 1.3.** Let \( q \in C \), let \( 0 < x < \gamma_q \), and let \( \{t_i\}_{i=0}^{n} \) be a chain from \( 0 \) to \( x \). If \( j \) is an integer in \( [1, n] \) then

\[ \prod_{i=j}^{n} L(t_i - t_{i-1})q - q = \sum_{i=j}^{n} (t_i - t_{i-1}) A \prod_{k=i}^{j} L(t_k - t_{k-1})q. \]
Proof.

\[
\prod_{i=1}^{n} L(t_i - t_{i-1}) q - q = \sum_{i=1}^{n} \left[ \prod_{k=j}^{i-1} L(t_k - t_{k-1}) q - \prod_{k=j}^{i-1} L(t_k - t_{k-1}) q \right]
\]

\[
= \sum_{i=1}^{n} (t_i - t_{i-1}) AL(t_i - t_{i-1}) \prod_{k=j}^{i-1} L(t_k - t_{k-1}) q
\]

\[
= \sum_{i=1}^{n} (t_i - t_{i-1}) A \prod_{k=j}^{i-1} L(t_k - t_{k-1}) q.
\]

Let \( p \in C \), let \( c > 0 \), and let \( m \) be a nonnegative integer. The number-sequence \( \{s_i\}_{i=0}^{m} \) is said to have property \( P_c \) provided that the following are true: (i) \( s_0 = 0 \), \( s_m < \gamma_p \) (ii) \( \{s_i\}_{i=0}^{m} \) is increasing, and (iii) if \( h \) is an integer in \([0, m-1]\), \( s_h \leq x \leq s_{h+1} \), \( \{t_i\}_{i=0}^{m} \) is a chain from \( s_h \) to \( x \), and \( j \) is an integer in \([0, n]\), then

\[
\left| \prod_{k=1}^{n} L(t_k - t_{k-1}) \prod_{i=1}^{k} L(s_i - s_{i-1}) p \right| \leq c.
\]

**Lemma 1.4.** Let \( p \in C \), let \( c > 0 \), and let \( \{s_i\}_{i=0}^{m} \) have property \( P_c \). There is a number \( s_{m+1} \) such that \( s_m < s_{m+1} < \gamma_p \) and \( \{s_i\}_{i=0}^{m+1} \) has property \( P_c \).

**Proof.** Lemma 1.4 follows from Lemma 1.2 and the continuity of \( A \) at \( \prod_{i=1}^{n} L(s_i - s_{i-1}) p \).

**Lemma 1.5.** Let \( p \in C \), let \( c > 0 \), and let \( \{s_i\}_{i=0}^{m} \) have property \( P_c \). Suppose that \( y \) is a number such that \( s_m < y < \gamma_p \) and if \( s_{m+1} \) is a number such that \( s_m < s_{m+1} < y \) then \( \{s_i\}_{i=0}^{m+1} \) has property \( P_c \). Then, if \( s_{m+1} = y \), \( \{s_i\}_{i=0}^{m+1} \) has property \( P_c \).

**Proof.** Let \( q = \prod_{i=1}^{n} L(s_i - s_{i-1}) p \), let \( \{t_i\}_{i=0}^{n} \) be a chain from \( s_m \) to \( y \), and let \( d > 0 \). There is a positive number \( b \) such that if \( u \in C \) and \( \|u - \prod_{i=1}^{n} L(t_i - t_{i-1}) q\| < b \) then

\[
Au - A \prod_{i=1}^{n} L(t_i - t_{i-1}) q < d.
\]

There is a positive number \( r \) such that \( t_{n-1} < r < t_n = y \) and \( t_n - r < b / \|A p\| \). By Lemmas 1.1 and 1.2

\[
\|L(r - t_{n-1}) \prod_{i=1}^{n-1} L(t_i - t_{i-1}) q - \prod_{i=1}^{n} L(t_i - t_{i-1}) q\| \leq (t_n - r) \cdot \|A p\| < b.
\]

Then, if \( j \) is an integer in \([0, n-1]\)

\[
A \prod_{i=1}^{j} L(t_i - t_{i-1}) q - A \prod_{i=1}^{n} L(t_i - t_{i-1}) q \leq A \prod_{i=1}^{j} L(t_i - t_{i-1}) q - AL(r - t_{n-1}) \prod_{i=1}^{n-1} L(t_i - t_{i-1}) q
\]

\[
+ AL(r - t_{n-1}) \prod_{i=1}^{n-1} L(t_i - t_{i-1}) q - A \prod_{i=1}^{n} L(t_i - t_{i-1}) q
\]

\[
< c + d.
\]

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Then, if \( j \) is an integer in \([0, n]\)

\[
\left\| A \sum_{i=1}^{j} L(t_i-t_{i-1})q - A \sum_{i=1}^{n} L(t_i-t_{i-1})q \right\| \leq c
\]

and so the lemma is established.

**Lemma 1.6.** Let \( p \in C \), let \( c > 0 \), and suppose that \( \{s_i\}_{i=0}^{n} \) is an infinite increasing number-sequence such that \( \lim \{s_i\}_{i=0}^{n} < \gamma_p \) and if \( n \) is a nonnegative integer \( \{s_i\}_{i=0}^{n} \) has property \( P_c \). Then there is a positive integer \( m \) and a sequence \( \{r_i\}_{i=0}^{m+1} \) such that if \( i \) is an integer in \([0, m]\) \( s_i = r_i \), \( r_{m+1} = \lim \{s_i\}_{i=0}^{n} \), and \( \{r_i\}_{i=0}^{m+1} \) has property \( P_c \).

**Proof.** Let \( q_0 = p \) and if \( n \) is a positive integer let \( q_n = L(s_n-s_{n-1})q_{n-1} \). If \( n \) is a positive integer then \( \|q_n - q_{n-1}\| = \|L(s_n-s_{n-1})q_{n-1} - q_{n-1}\| \leq (s_n-s_{n-1}) \|Ap\| \). Let \( s = \lim \{s_i\}_{i=0}^{n} \), let \( q = \lim \{q_i\}_{i=0}^{n} \), and note that \( q \in C \) since \( \|q - p\| < c \cdot \|Ap\| \) and so \( \|q - p\| < \text{dist}(p, \partial C) \). There is a positive number \( b \) such that if \( u \in C \) and \( \|u - q\| < b \) then \( \|Au - Aq\| < c/2 \). Let \( m \) be a positive integer such that \( \|q - q_m\| < b/2 \) and \( s - s_m < b/2 \|Ap\| \). Let \( 0 < s - s_m \), let \( \{t_i\}_{i=0}^{n} \) be a chain from 0 to \( x \), and let \( j \) be an integer in \([0, n]\). By Lemma 1.2

\[
\left\| A \sum_{i=1}^{j} L(t_i-t_{i-1})q_m - q_m \right\| \leq t_j \|Ap\| < b/2
\]

and so

\[
\left\| A \sum_{i=1}^{j} L(t_i-t_{i-1})q_m - Aq \right\| < c/2.
\]

Then, if \( j \) is an integer in \([0, n]\)

\[
\left\| A \sum_{i=1}^{j} L(t_i-t_{i-1})q_m - A \sum_{i=1}^{n} L(t_i-t_{i-1})q_m \right\|
\]

\[
\leq \left\| A \sum_{i=1}^{j} L(t_i-t_{i-1})q_m - Aq \right\| + \|Aq - A \sum_{i=1}^{n} L(t_i-t_{i-1})q_m \|
\]

\[
\leq c
\]

and so the lemma is established.

**Lemma 1.7.** Let \( p \in C \), let \( c > 0 \), and let \( 0 < x < \gamma_p \). There is a chain \( \{s_i\}_{i=0}^{n} \) from 0 to \( x \) such that \( \{s_i\}_{i=0}^{n} \) has property \( P_c \).

**Proof.** By Lemma 1.4 there is an infinite increasing number-sequence \( \{s_i\}_{i=0}^{n} \) such that \( \lim \{s_i\}_{i=0}^{n} < \gamma_p \) and if \( n \) is a nonnegative integer \( \{s_i\}_{i=0}^{n} \) has property \( P_c \). Let \( M \) denote the set of all such sequences. If \( s = \{s_i\}_{i=0}^{n} \) is in \( M \) let \( z(s) \) denote the limit of \( s \). If each of \( s \) and \( t \) belongs to \( M \) define \( s \leq t \) only in case \( s \) is \( t \) or if \( n \) is the greatest nonnegative integer such that if \( i \) is an integer in \([0, n]\) \( s_i = t_i \), then \( z(s) \leq t_{n+1} \). Then, \( \leq \) is a partial ordering of \( M \).

Assume that there exists no member \( s \) of \( M \) such that \( z(s) > x \). Let \( L \) be a linearly ordered subset of \( M \) and let \( \gamma \) be the smallest positive number such that if \( s \) is in
Let \( \{s_i(0)\}_{i=0}^{\infty}, \{s_i(1)\}_{i=0}^{\infty}, \ldots \) be an increasing sequence of points in \( L \) such that \( z(s(0)), z(s(1)), \ldots \) converges to \( y \). For each nonnegative integer \( i \) define \( y_i = \sup_k s_i(k) \). Then, \( y_i \leq y_{i+1} \) and \( \lim_{i \to \infty} y_i = y \).

Suppose there is a positive integer \( n \) such that \( y_n = y \). Then there is a least positive integer \( n \) such that \( y_n = y \) and there must exist an integer \( k \) such that \( s_i(k) = s_j(j) \) for each integer \( i \) in \([0, n-1]\) and \( j \geq k \). In this case \( s_n(k), s_n(k+1), \ldots \) converges to \( y \) and so by Lemma 1.5 \( \{s_i\}_{i=0}^{\infty} \), \( s_k = s_i(k) \) for \( i \) in \([0, n-1]\) and \( s_n = y \), has property \( P_c \). Further, since \( y < \gamma_p \), we have by Lemma 1.4 that \( \{s_k\}_{k=0}^{\infty} \) may be extended to a member \( \{s_i\}_{i=0}^{\infty} \) of \( M \) and so \( \{s_i\}_{i=0}^{\infty} \) is an upper bound for \( L \). If there is no positive integer \( n \) such that \( y_n = y \) then \( y_n < y \) for every \( n \), \( \{y_n\}_{n=0}^{\infty} \) is in \( M \), \( \{y_n\}_{n=0}^{\infty} \) is an upper bound for \( L \).

Thus, if \( L \) is a linearly ordered subset of \( M \), then \( L \) is bounded by a member of \( M \). By Zorn’s lemma there exists \( u \in M \) such that \( u \) is maximal. But then we have a contradiction since \( z(u) \leq x < \gamma_p \) and by Lemma 1.6 there exists \( t \in M \) such that \( u < t \). Hence, there exists \( s \in M \) such that \( z(s) > x \) and the lemma is proved.

**Lemma 1.8.** Let \( p \in C \), let \( c > 0 \), and let \( 0 < x < \gamma_p \). There is a chain \( \{s_i\}_{i=0}^{\infty} \) from 0 to \( x \) such that if \( \{t_i\}_{i=0}^{n} \) is a refinement of \( \{s_i\}_{i=0}^{\infty} \) then

\[
\left( \prod_{i=1}^{n} L(t_i - t_{i-1})p - \prod_{i=1}^{m} L(s_i - s_{i-1})p \right) < c.
\]

**Proof.** Let \( \{s_i\}_{i=0}^{\infty} \) be a chain from 0 to \( x \) such that \( \{s_i\}_{i=0}^{\infty} \) has property \( P_c \). Let \( \{t_i\}_{i=0}^{n} \) be a refinement of \( \{s_i\}_{i=0}^{\infty} \), i.e., there is an increasing sequence \( u \) such that \( u_0 = 0 \), \( u_m = n \), and if \( i \) is an integer in \([0, m]\) \( s_i = t_u \). If \( i \) is an integer in \([1, m]\) let \( K_i = \prod_{j=u_i+1}^{u_i+1} L(t_j - t_{j-1}) \), let \( J_i = \prod_{j=1}^{u_i} L(s_j - s_{j-1}) \), let \( K_{m+1} = e \), and let \( J_0 = e \). Then,

\[
\left( \prod_{i=1}^{n} L(t_i - t_{i-1})p - \prod_{i=1}^{m} L(s_i - s_{i-1})p \right) = \prod_{i=1}^{m} K_i p - J_m p
\]

\[
= \sum_{i=1}^{m} \left[ \prod_{j=1}^{m} K_j p - \prod_{j=i+1}^{m} K_j p \right]
\]

\[
\leq \sum_{i=1}^{m} \|K_j p - J_{i+1} p\|
\]

\[
= \sum_{i=1}^{m} \|K_j p - L(s_i - s_{i-1}) J_{i+1} p\|
\]

\[
\leq \sum_{i=1}^{m} \| (e - (s_i - s_{i-1}) A) K_i p - J_{i+1} p \|
\]

\[
= \sum_{i=1}^{m} \left[ \prod_{j=u_i+1}^{u_i+1} L(t_j - t_{j-1}) J_{i+1} p - J_{i+1} p \right] - (s_i - s_{i-1}) A K_i p - J_{i+1} p
\]
\[
\begin{align*}
\sum_{i=1}^{m} \left| \sum_{j=u_{i-1}+1}^{u_{i}} (t_{j}-t_{j-1}) \left[ A \prod_{k=u_{i-1}+1}^{j} L(t_{k}-t_{k-1})J_{r-1}p - AK_{r-1}p \right] \right| & \quad \text{(by (6))} \\
\leq \sum_{i=1}^{m} \sum_{j=u_{i-1}+1}^{u_{i}} (t_{j}-t_{j-1}) \left| A \prod_{k=u_{i-1}+1}^{j} L(t_{k}-t_{k-1})J_{r-1}p \right| \\
& \quad - A \prod_{j=u_{i-1}+1}^{u_{i}} L(t_{j}-t_{j-1})J_{r-1}p \\
\leq c \cdot \sum_{i=1}^{m} \sum_{j=u_{i-1}+1}^{u_{i}} (t_{j}-t_{j-1}) \\
= c \cdot x.
\end{align*}
\]

**Proof of the theorem.** Let \( p \in C \). If \( x = 0 \), then \( \prod_{i=1}^{n} (e - dIA)^{-1}p = p \). If \( 0 < x < \gamma_{p} \), then \( \prod_{i=1}^{n} (e - dIA)^{-1}p \) exists by virtue of Lemma 1.8. If \( 0 \leq x < \gamma_{p} \), define \( g_{p}(x) = \prod_{i=1}^{n} (e - dIA)^{-1}p \). By Lemma 1.2 we see that \( g_{p} \) is Lipschitz continuous on \([0, \gamma_{p})\) with Lipschitz constant \( \leq \| Ap \| \), \( g_{p}(x) \in C \) for \( x \in [0, \gamma_{p}) \), and \( \| Ag_{p}(x) \| \leq \| Ap \| \) for \( x \in [0, \gamma_{p}) \). For \( 0 \leq x < \gamma_{p} \), we have that \( \text{dist} (p, \partial C) \leq \text{dist} (g_{p}(x), \partial C) + \| p - g_{p}(x) \| \leq \text{dist} (g_{p}(x), \partial C) + x \| Ap \| \). Hence,

\[
\text{dist} (p, \partial C)/\| Ap \| \leq \text{dist} (g_{p}(x), \partial C)/\| Ap \| + x
\]
and so \( \gamma_{p} - x \leq \gamma_{p}(x) \). Thus, if \( 0 \leq x < \gamma_{p} \) and \( 0 \leq y < \gamma_{p} - x \), one sees that \( g_{p}(x)(y) = g_{p}(x + y) \). To show that \( g' = Ag_{p} \), let \( 0 \leq x < \gamma_{p} \) and let \( c > 0 \). By Lemma 1.2 there is a positive number \( z < \gamma_{p} - x \) such that if \( 0 < y < z \) and \( \{s_{i}\}_{i=0}^{n} \) is a chain from 0 to \( y \), then

\[
\left| A \prod_{i=1}^{m} L(s_{i}-s_{i-1})g_{p}(x) - Ag_{p}(x) \right| < c/2.
\]

Let \( 0 < y < z \). There is a chain \( \{t_{i}\}_{i=0}^{n} \) from 0 to \( y \) such that

\[
\left| \prod_{i=1}^{m} L(t_{i}-t_{i-1})g_{p}(x) - g_{p}(x)(y) \right| < c \cdot y/2.
\]

Then,

\[
\left| \frac{1}{y} \left[ g_{p}(x+y) - g_{p}(x) \right] - Ag_{p}(x) \right|
\]

\[
< \frac{c}{2} + \frac{1}{y} \left| \left( \prod_{i=1}^{m} L(t_{i}-t_{i-1})g_{p}(x) - g_{p}(x) \right) - yAg_{p}(x) \right|
\]

\[
= \frac{c}{2} + \frac{1}{y} \left| \sum_{i=1}^{n} (t_{i}-t_{i-1})A \prod_{j=1}^{i} L(t_{j}-t_{j-1})g_{p}(x) - yAg_{p}(x) \right|
\]

\[
\leq \frac{c}{2} + \frac{1}{y} \left| \sum_{i=1}^{n} (t_{i}-t_{i-1}) \right| A \prod_{j=1}^{i} L(t_{j}-t_{j-1})g_{p}(x) - Ag_{p}(x) \right|
\]

\[
< c
\]
and so $g_p^+(x) = Ag_p(x)$. Thus, $g_p^+ = Ag_p$ on $[0, \gamma_p)$ and so $g_p$ has a continuous right derivative on $[0, \gamma_p)$. Then $g_p$ has a continuous derivative on $[0, \gamma_p)$ and so the theorem is proved.

**Corollary.** Let $A$ be a mapping from the Banach space $S$ to $S$ such that the following are true:

(I') $A$ is dissipative on $S$, i.e., if $u, v \in D_A$ and $\epsilon \geq 0$ then $\|\epsilon - \epsilon A\|u - (\epsilon - \epsilon A)v\| \geq \|u - v\|$.

(II') $R(\epsilon - \epsilon A) = S$ for each $\epsilon \geq 0$.

(III') $A$ is continuous on $S$. If $p \in S$ then there is a continuously differentiable function $g_p$ from $[0, \infty)$ to $S$ such that $g_p(0) = p$ and if $x \geq 0$, $g_p(x) = T^{-1}e^{\epsilon - \epsilon A}p$.

**Proof.** The proof follows immediately from the theorem if one observes that $a = +\infty$ and $\text{dist}(p, \partial S) = +\infty$.

It may be noted that a result of J. Dorroh [8] can be used to show that the solutions of $g_p^+ = Ag_p$, $g_p(0) = p$ in the corollary are unique. In [15] G. Minty has shown that if $S$ is the Hilbert space then (I') and (III') imply (II'). More generally, it has been shown recently by T. Kato in [11] that (I') and (III') imply (II') in the case that $S^*$ is uniformly convex. If $S$ is a general Banach space F. Browder has shown in [4] that (I') and (III') imply (II') in the case that $A$ is locally uniformly continuous.

By virtue of the corollary one may define for each $x \geq 0$ the transformation $T(x)$ from $S$ to $S$ as follows: $T(x)p = g_p(x)$ for each $p \in S$. Then $T$ is a strongly continuous semigroup of nonlinear nonexpansive transformations on $S$, i.e.,

(i) $T(x + y) = T(x)T(y)$ for $x, y \geq 0$,

(ii) $T(0) = e$,

(iii) $\|T(x)p - T(x)q\| \leq \|p - q\|$ for $x \geq 0$ and $p, q \in S$ and

(iv) $g_p(x) = T(x)p$ is continuous for $p$ fixed and $x \geq 0$.

Further, $A$ is the infinitesimal generator of $T$, i.e., $Ap = g_p^+(0)$ for each $p \in S$. In [2], [14], [17], [18], and [19] representations are given for nonlinear nonexpansive semigroups of transformations in terms of their infinitesimal generators using product integrals.

3. **Examples.** In conclusion we give some examples. In [6] a well-known example is given by J. Dieudonné of a continuous mapping $A$ from a Banach space $S$ to $S$ for which there is no solution to the equation $g' = Ag$ and $g(0) = 0$. This example is given in a Banach space which is not reflexive. Recently, J. Yorke [20] has given an example of a continuous mapping $A$ from a Hilbert space to itself for which no solution exists to $g' = Ag$, $g(0) = 0$.

In the examples below the mapping $A$ satisfies conditions (I'), (II'), and (III') of the corollary.

**Example 1.** Let $S = E_1$ and let $A$ be a continuous nonincreasing function from $E_1$ to $E_1$. 

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Example 2. Let \( S = C_{[0,1]} \), i.e., \( S \) is the Banach space of continuous real-valued functions on \([0, 1]\) with supremum norm. Let \( F \) be a continuous increasing function from \( E_1 \) onto \( E_1 \) such that \( F' \) is continuous and nonincreasing on \( E_1 \). Define the mapping \( A \) on \( C_{[0,1]} \) as follows:

\[
Af = F'(F^{-1}[f]) \quad \text{for each } f \in C_{[0,1]}.
\]

The solutions \( g_f \) of the corollary are then given by \( g_f(x) = F[x + F^{-1}[f]] \) for \( x \geq 0 \).

In both Examples 1 and 2 \( A \) may be neither linear nor Lipschitz continuous. In both, however, \( A \) is locally uniformly continuous. In Example 3 the mapping \( A \) is not locally uniformly continuous.

Example 3. Let \( S = (c_0) \), i.e., \( S \) is the Banach space of real-number sequences \( x = (x_n) \) converging to 0 with \( \|x\| = \sup_n |x_n| \). If each of \((a, b)\) and \((c, d)\) is a point in the plane define the function \( F_{[(a, b), (c, d)]} \) from \([a, c]\) to \([b, d]\) by

\[
F_{[(a, b), (c, d)]}(x) = b + \left( \frac{d - b}{c - a} \right)(x - a) \quad \text{for } x \in [a, c].
\]

For each positive integer \( n \) define the function \( A_n \) from \( E_1 \) to \( E_1 \) as follows:

\[
A_n(x) = \begin{cases} 
1 & \text{if } x < -1 \\
0 & \text{if } x \geq 0 \\
F_{[(1/k, 1/k), (-1/k + 1/n, 1/k - 1/(k + 1), 1/(k + 1))]}(x) & \text{if } x \in \left[ -\frac{1}{k} + \frac{1}{n}, \frac{1}{k} - \frac{1}{k + 1} \right] 
\end{cases}
\]

Define the mapping \( A \) from \((c_0)\) to \((c_0)\) by \( Ax = (A_n(x_n)) \) for each \( x = (x_n) \in (c_0) \).

One sees that \( A \) satisfies conditions (I'), (II'), and (III'), since for each positive integer \( n \) \( A_n \) is nonincreasing and continuous. Moreover, there is no neighborhood about \( 0 \) on which \( A \) is uniformly continuous.

References


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