NONLINEAR EVOLUTION EQUATIONS AND PRODUCT INTEGRATION IN BANACH SPACES

BY
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Abstract. The method of product integration is used to obtain solutions to the nonlinear evolution equation \( g' = Ag \) where \( A \) is a function from a Banach space \( S \) to itself and \( g \) is a continuously differentiable function from \([0, \infty)\) to \( S \). The conditions required on \( A \) are that \( A \) is dissipative on \( S \), the range of \((e - \varepsilon A) = S\) for all \( \varepsilon \geq 0 \), and \( A \) is continuous on \( S \).

1. Introduction. Let \( S \) be a Banach space and let \( A \) be a mapping from a subset of \( S \) to \( S \). An evolution equation is a system \( g' = A(g) \), \( g(0) = p \), where \( g \) is a continuous function from \([0, \infty)\) to \( S \) and \( p \) is a point in \( S \). In [3] F. Browder has considered nonlinear evolution equations in which \( S \) is the Hilbert space and \( A \) is continuous, bounded, and dissipative on \( S \). In recent articles Y. K"omura [12], T. Kato [10], and M. Crandall and A. Pazy [5] have considered nonlinear evolution equations in which \( S \) is the Hilbert space and \( A \) is maximal dissipative, not necessarily continuous, and is the infinitesimal generator of a semigroup of nonlinear nonexpansive transformations on \( S \).

The object of this paper is to obtain solutions to an evolution system in a general Banach space using the method of product integration. A definition of product integration is given as follows:

Suppose that \( p \) is in \( S \), \( x > 0 \), and \( z \) is a point in \( S \) such that if \( c > 0 \) there exists a chain \( \{s_i\}_{i=0}^{n} \) from 0 to \( x \) such that if \( \{s_i\}_{i=0}^{n} \) is a refinement of \( \{s_i\}_{i=0}^{n} \) then

\[
\left| z - \prod_{i=1}^{n} \left( e - (t_i - t_{i-1}) A \right)^{-1} p \right| < c.
\]

(Note that \( e \) denotes the identity map on \( S \), \( (e - (t_i - t_{i-1}) A)^{-1} \) denotes the inverse map of \( e - (t_i - t_{i-1}) A \), \( \prod_{i=1}^{n} (e - (t_i - t_{i-1}) A)^{-1} p = (e - (t_1 - t_0) A)^{-1} p \), and if \( j \) is an integer in \([2, n]\)

\[
\prod_{i=1}^{j} (e - (t_i - t_{i-1}) A)^{-1} p = (e - (t_j - t_{j-1}) A)^{-1} \prod_{i=1}^{j} (e - (t_i - t_{i-1}) A)^{-1} p,
\]

where the product operation is composition of mappings.) Then \( z \) is said to be the product integral of \( A \) with respect to \( p \) from 0 to \( x \) and is denoted by \( \prod_{i=0}^{n} (e - dIA)^{-1} p \).

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In [1] G. Birkhoff and in [16] J. Neuberger have used product integration to solve evolution systems where the mapping $A$ is Lipschitz continuous. In this paper the product integration method will be extended to mappings not necessarily Lipschitz continuous.

2. An existence theorem. Let $A$ be a mapping from a subset of $S$ to $S$ such that the following are true:

(I) $A$ is dissipative on its domain $D_A$, i.e., if $u, v \in D_A$ and $\epsilon \geq 0$ then $\| (e - \epsilon A) u - (e - \epsilon A) v \| \geq \| u - v \|$

(II) There is an open subset $C$ of $S$ such that $C \subseteq D_A$ and a positive number $\alpha$ such that if $0 \leq \epsilon < \alpha$ then $C \subseteq R(e - \epsilon A)$ (where $R(e - \epsilon A)$ denotes the range of $(e - \epsilon A)$).

(III) $A$ is continuous on $C$.

Note that by (I) if $\epsilon > 0$ then $(e - \epsilon A)$ is 1-1 on $D_A$ and by (II) if $0 \leq \epsilon < \alpha$ and $q \in C$ then $q \in D_{e - \epsilon A}$. If $0 \leq \epsilon < \alpha$ and $q \in R(e - \epsilon A)$ let $L(\epsilon) q = (e - \epsilon A)^{-1} q$. By (I) $L(\epsilon)$ is nonexpansive on $R(e - \epsilon A)$, i.e., if $u, v \in R(e - \epsilon A)$ then

$$\| L(\epsilon) u - L(\epsilon) v \| \leq \| u - v \|$$

**Theorem.** Let $A$ satisfy conditions (I), (II), and (III). If $p \in C$ and

$$\gamma_p = \min \{ \text{dist} (p, \partial C) / \| Ap \|, \alpha \},$$

then there is a continuously differentiable function $g_p$ from $[0, \gamma_p)$ to $S$ such that $g_p(0) = p$ and if $0 \leq \epsilon < \gamma_p$, $g_p'(x) = Ag_p(x)$ and $g_p(x) = \int_0^x (e - \epsilon A)^{-1} p$. The theorem will be proved by means of a sequence of lemmas each of which is under the hypothesis of the theorem.

**Lemma 1.1.** If $q \in C$ and $0 \leq x, y < \alpha$, then $\| L(x) q - L(y) q \| \leq |x - y| \cdot \| Aq \|$

**Proof.** Using (2) we have that

$$\| L(x) q - L(y) q \| = \| L(x) q - L(x) (e - x A) L(y) q \|
= \| q - (e - x A) L(y) q \|
= \| q - [(x/y)(e - y A)] L(y) q + (1 - x/y) L(y) q \|
= |1 - x/y| \| q - L(y) q \|
\leq |1 - x/y| \| (e - y A) q - q \|
= |x - y| \| Aq \|.$$

**Lemma 1.2.** Let $q \in C$, let $0 < x < \gamma_q$, and let $\{ s_i \}_{i=0}^m$ be a chain from 0 to $x$. If $j$ is an integer in $[1, m]$ then

(3) $$\int_{i=1}^{i-1} L(s_i - s_{i-1}) q \in C,$$

(4) $$\int_{i=1}^{j} L(s_i - s_{i-1}) q - q \leq s_j \| Aq \|.$$

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and

$$(5) \quad \left\| A \prod_{i=1}^{j} L(s_i - s_{i-1})q \right\| \leq \|Aq\|.$$  

(Note that $\prod_{i=1}^{0} L(s_i - s_{i-1})$ denotes the identity map, i.e., $\prod_{i=1}^{0} L(s_i - s_{i-1})q = q$.)

**Proof.** The proof is by induction. For $j = 1$, $\prod_{i=1}^{1} L(s_i - s_{i-1})q = q \in C$, 

$$\left\| \prod_{i=1}^{1} L(s_i - s_{i-1})q-q \right\| \leq s_1 \cdot \|Aq\|$$

(by Lemma 1.1), and

$$A \prod_{i=1}^{1} L(s_i - s_{i-1})q = \|1/s_1[L(s_i - s_0)q - q]\| \leq \|Aq\|.$$  

Suppose that $j$ is an integer in $[1, m-1]$, $\prod_{i=1}^{j+1} L(s_i - s_{i-1})q \in C$, 

$$\left\| \prod_{i=1}^{j+1} L(s_i - s_{i-1})q-q \right\| \leq s_j \cdot \|Aq\|,$$

and $\|A \prod_{i=1}^{j} L(s_i - s_{i-1})q\| \leq \|Aq\|$. Then,

$$\prod_{i=1}^{j+1} L(s_i - s_{i-1})q \in C \subseteq D_{t_{j+1} - t_j}.$$

Further,

$$\left\| \prod_{i=1}^{j+1} L(s_i - s_{i-1})q-q \right\| = \left\| \sum_{i=1}^{j+1} \left[ \prod_{k=1}^{i} L(s_k - s_{k-1})q - \prod_{k=1}^{i+1} L(s_k - s_{k-1})q \right] \right\|$$

(note that $\prod_{k=j+2}^{j+1} L(s_k - s_{k-1})$ is the identity map)

$$\leq \sum_{i=1}^{j+1} \left\| L(s_i - s_{i-1})q-q \right\|$$

$$\leq s_{j+1} \cdot \|Aq\|.$$  

Moreover,

$$\left\| A \prod_{i=1}^{j+1} L(s_i - s_{i-1})q \right\| = \left\| \left( \frac{1}{s_{j+1} - s_j} \right) \prod_{i=1}^{j+1} L(s_i - s_{i-1})q - \prod_{i=1}^{j} L(s_i - s_{i-1})q \right\|$$

$$\leq \|Aq\||.$$  

**Lemma 1.3.** Let $q \in C$, let $0 < x < y_q$, and let $\{t_i\}_{i=0}^{n}$ be a chain from 0 to $x$. If $j$ is an integer in $[1, n]$ then

$$(6) \quad \prod_{i=j}^{n} L(t_i - t_{i-1})q - q = \sum_{i=j}^{n} (t_i - t_{i-1})A \prod_{k=f}^{i} L(t_k - t_{k-1})q.$$
Proof.
\[
\prod_{i=f}^{n} L(t_i-t_{i-1})q - q = \sum_{i=f}^{n} \left[ \prod_{k=j}^{i} L(t_k-t_{k-1})q - \prod_{k=j}^{i-1} L(t_k-t_{k-1})q \right] \\
= \sum_{i=j}^{n} (t_i-t_{i-1})A \prod_{k=j}^{i} L(t_k-t_{k-1})q \\
= \sum_{i=j}^{n} (t_i-t_{i-1})A \prod_{k=j}^{i} L(t_k-t_{k-1})q.
\]

Let \( p \in C \), let \( c > 0 \), and let \( m \) be a nonnegative integer. The number-sequence \( \{s_i\}_{i=0}^{m} \) is said to have property \( P_c \) provided that the following are true: (i) \( s_0 = 0 \), \( s_m < \gamma_p \) (ii) \( \{s_i\}_{i=0}^{m} \) is increasing, and (iii) if \( h \) is an integer in \([0, m-1] \), \( s_h \leq x \leq s_{h+1} \), \( \{t_i\}_{i=0}^{m} \) is a chain from \( s_h \) to \( x \), and \( j \) is an integer in \([0, n] \), then

\[
\left| A \prod_{k=j}^{n} L(t_k-t_{k-1}) - \prod_{i=j}^{n} L(s_i-s_{i-1})p \right| \leq c.
\]

**Lemma 1.4.** Let \( p \in C \), let \( c > 0 \), and let \( \{s_i\}_{i=0}^{m} \) have property \( P_c \). There is a number \( s_{m+1} \) such that \( s_m < s_{m+1} < \gamma_p \) and \( \{s_i\}_{i=0}^{m+1} \) has property \( P_c \).

**Proof.** Lemma 1.4 follows from Lemma 1.2 and the continuity of \( A \) at \( \prod_{i=1}^{m} L(s_i-s_{i-1})p \).

**Lemma 1.5.** Let \( p \in C \), let \( c > 0 \), and let \( \{s_i\}_{i=0}^{m} \) have property \( P_c \). Suppose that \( y \) is a number such that \( s_m < y < \gamma_p \) and if \( s_{m+1} \) is a number such that \( s_m < s_{m+1} < y \) then \( \{s_i\}_{i=0}^{m+1} \) has property \( P_c \). Then, if \( s_{m+1} = y \), \( \{s_i\}_{i=0}^{m+1} \) has property \( P_c \).

**Proof.** Let \( q = \prod_{i=1}^{n} L(s_i-s_{i-1})p \), let \( \{t_i\}_{i=0}^{n} \) be a chain from \( s_m \) to \( y \), and let \( d > 0 \). There is a positive number \( b \) such that if \( u \in C \) and \( \|u - \prod_{i=1}^{n} L(t_i-t_{i-1})q\| < b \) then

\[
\|Au - \prod_{i=1}^{n} L(t_i-t_{i-1})q\| < d.
\]

There is a positive number \( r \) such that \( t_{n-1} - r < t_n = y \) and \( t_n - r < b/\|Ap\| \). By Lemmas 1.1 and 1.2

\[
\left| L(r-t_{n-1}) \prod_{i=1}^{n-1} L(t_i-t_{i-1})q - \prod_{i=1}^{n} L(t_i-t_{i-1})q \right| \leq (t_n-r) \cdot \|Ap\| < b.
\]

Then, if \( j \) is an integer in \([0, n-1] \)

\[
\left| A \prod_{i=1}^{j} L(t_i-t_{i-1})q - A \prod_{i=1}^{n} L(t_i-t_{i-1})q \right| \\
\leq \left| A \prod_{i=1}^{j} L(t_i-t_{i-1})q - A \prod_{i=1}^{n-1} L(t_i-t_{i-1})q \right| \\
+ \left| A \prod_{i=1}^{n-1} L(t_i-t_{i-1})q - A \prod_{i=1}^{n} L(t_i-t_{i-1})q \right| \\
< c + d.
\]
Then, if \( j \) is an integer in \([0, n]\)
\[
\left\| A \sum_{i=1}^{j} L(t_i - t_{i-1})q - A \sum_{i=1}^{n} L(t_i - t_{i-1})q \right\| \leq c
\]
and so the lemma is established.

**Lemma 1.6.** Let \( p \in C \), let \( c > 0 \), and suppose that \( \{s_i\}_{i=0}^{\infty} \) is an infinite increasing number-sequence such that \( \lim \{s_i\}_{i=0}^{\infty} < \gamma_p \) and if \( n \) is a nonnegative integer \( \{s_i\}_{i=0}^{n} \) has property \( P_c \). Then there is a positive integer \( m \) and a sequence \( \{r_i\}_{i=0}^{m+1} \) such that if \( i \) is an integer in \([0, m]\) \( s_i = r_i, r_{m+1} = \lim \{s_i\}_{i=0}^{\infty}, \) and \( \{r_i\}_{i=0}^{m+1} \) has property \( P_c \).

**Proof.** Let \( q_0 = p \) and if \( n \) is a positive integer let \( q_n = L(s_n - s_{n-1})q_{n-1} \). If \( n \) is a positive integer then \( \|q_n - q_{n-1}\| = \|L(s_n - s_{n-1})q_{n-1} - q_{n-1}\| \leq (s_n - s_{n-1})\|Ap\| \). Let \( s = \lim \{s_i\}_{i=0}^{\infty} \), let \( q = \lim \{q_i\}_{i=0}^{\infty} \), and note that \( q \in C \) since \( \|q_n - p\| < s \cdot \|Ap\| \) and so \( \|q - p\| < \text{dist}(p, \partial C) \). There is a positive number \( b \) such that if \( u \in C \) and \( \|u - q\| < b \) then \( \|Au - Aq\| < c/2 \). Let \( m \) be a positive integer such that \( \|q - q_m\| < b/2 \) and \( s - s_m < b/2 \|Ap\| \). Let \( 0 < x \leq s - s_m \), let \( \{t_i\}_{i=0}^{\infty} \) be a chain from 0 to \( x \), and let \( j \) be an integer in \([0, n]\). By Lemma 1.2
\[
\left\| \sum_{i=1}^{j} L(t_i - t_{i-1})q_{m} - q_{m} \right\| \leq t_j \cdot \|Ap\| < b/2
\]
and so
\[
\left\| A \sum_{i=1}^{j} L(t_i - t_{i-1})q_{m} - Aq \right\| < c/2.
\]
Then, if \( j \) is an integer in \([0, n]\)
\[
\left\| A \sum_{i=1}^{j} L(t_i - t_{i-1})q_{m} - A \sum_{i=1}^{n} L(t_i - t_{i-1})q_m \right\|
\leq \left\| A \sum_{i=1}^{j} L(t_i - t_{i-1})q_{m} - Aq \right\| + \left\| Aq - A \sum_{i=1}^{n} L(t_i - t_{i-1})q_m \right\|
\leq c
\]
and so the lemma is established.

**Lemma 1.7.** Let \( p \in C \), let \( c > 0 \), and let \( 0 < x < \gamma_p \). There is a chain \( \{s_i\}_{i=0}^{\infty} \) from 0 to \( x \) such that \( \{s_i\}_{i=0}^{n} \) has property \( P_c \).

**Proof.** By Lemma 1.4 there is an infinite increasing number-sequence \( \{s_i\}_{i=0}^{\infty} \) such that \( \lim \{s_i\}_{i=0}^{\infty} < \gamma_p \) and if \( n \) is a nonnegative integer \( \{s_i\}_{i=0}^{n} \) has property \( P_c \). Let \( M \) denote the set of all such sequences. If \( s = \{s_i\}_{i=0}^{\infty} \) is in \( M \) let \( z(s) \) denote the limit of \( s \). If each of \( s \) and \( t \) belongs to \( M \) define \( s \leq t \) only in case \( s \) is \( t \) or if \( n \) is the greatest nonnegative integer such that if \( i \) is an integer in \([0, n]\) \( s_i = t_i \), then \( z(s) \leq t_{n+1} \). Then, \( \leq \) is a partial ordering of \( M \).

Assume that there exists no member \( s \) of \( M \) such that \( z(s) > x \). Let \( L \) be a linearly ordered subset of \( M \) and let \( y \) be the smallest positive number such that if \( s \) is in
Let \( \{z(0)\}_{i=0}^{\infty}, \{z(1)\}_{i=0}^{\infty}, \ldots \) be an increasing sequence of points in \( L \) such that \( z(s(0)), z(s(1)), \ldots \) converges to \( y \). For each nonnegative integer \( i \) define 
\[
y_i = \sup_k s_i(k).
\]
Then, \( y_i \leq y_{i+1} \) and \( \lim_{i \to \infty} y_i = y \).

Suppose there is a positive integer \( n \) such that \( y_n = y \). Then there is a least positive integer \( n \) such that \( y_n = y \) and there must exist an integer \( k \) such that \( s_i(k) = s_j(j) \) for each integer \( i \) in \([0, n-1]\) and \( j \geq k \). In this case \( s_n(k), s_n(k+1), \ldots \) converges to \( y \) and so by Lemma 1.5 \( \{s_i\}_{i=0}^{\infty} \) is an upper bound for \( L \). If there is no positive integer \( n \) such that \( y_n = y \) then \( y_n < y \) for every \( n \), \( \{y_n\}_{n=0}^{\infty} \) is in \( M \), \( \{y_n\}_{n=0}^{\infty} \) is an upper bound for \( L \).

Thus, if \( L \) is a linearly ordered subset of \( M \), then \( L \) is bounded by a member of \( M \). By Zorn’s lemma there exists \( u \in M \) such that \( u \) is maximal. But then we have a contradiction since \( z(u) \leq x < y \) and by Lemma 1.6 there exists \( t \in M \) such that \( u < t \).

**Lemma 1.8.** Let \( p \in C \), let \( c > 0 \), and let \( 0 < x < \gamma_p \). There is a chain \( \{s_i\}_{i=0}^\infty \) from \( 0 \) to \( x \) such that if \( \{t_i\}_{i=0}^\infty \) is a refinement of \( \{s_i\}_{i=0}^\infty \) then

\[
\left| \sum_{i=1}^n L(t_i-t_{i-1})p - \sum_{i=1}^m L(s_i-s_{i-1})p \right| < c.
\]

**Proof.** Let \( \{s_i\}_{i=0}^\infty \) be a chain from \( 0 \) to \( x \) such that \( \{s_i\}_{i=0}^\infty \) has property \( P_c \). Let \( \{t_i\}_{i=0}^\infty \) be a refinement of \( \{s_i\}_{i=0}^\infty \), i.e., there is an increasing sequence \( u \) such that \( u_0 = 0, u_n = n \), and if \( i \) is an integer in \([0, m]\) \( s_i = t_u \). If \( i \) is an integer in \([1, m]\) let \( K_i = \sum_{i=1}^m L(t_j-t_{j-1}) \), let \( J_i = \sum_{i=1}^m L(s_j-s_{j-1}) \), let \( K_m+1 = e \), and let \( J_0 = e \). Then,

\[
\left| \sum_{i=1}^n L(t_i-t_{i-1})p - \sum_{i=1}^m L(s_i-s_{i-1})p \right| = \sum_{i=1}^m \left| K_i p - J_i p \right|
\]

\[
\leq \sum_{i=1}^m \|K_i J_{i-1} p - J_{i-1} p\|
\]

\[
\leq \sum_{i=1}^m \|K_i J_{i-1} p - L(s_i-s_{i-1})J_{i-1} p\|
\]

\[
\leq \sum_{i=1}^m \|e - (s_i-s_{i-1})A \| K_i J_{i-1} p - J_{i-1} p\|
\]

\[
= \sum_{i=1}^m \left[ \sum_{j=1}^{u_i} L(t_j-t_{j-1})J_{i-1} p - J_{i-1} p \right] - (s_i-s_{i-1})AK_i J_{i-1} p.
\]
Proof of the theorem. Let \( p \in C \). If \( x = 0 \), then \( \prod 0 (e - d \Lambda)^{-1} p = p \). If \( 0 < x < \gamma_p \), then \( \prod 0 (e - d \Lambda)^{-1} p \) exists by virtue of Lemma 1.8. If \( 0 \leq x < \gamma_p \), define \( \gamma_p(x) = \prod 0 (e - d \Lambda)^{-1} p \). By Lemma 1.2 we see that \( \gamma_p \) is Lipschitz continuous on \( [0, \gamma_p] \) with Lipschitz constant \( \leq \| A p \|, \gamma_p(x) \in C \) for \( x \in [0, \gamma_p] \), and \( \| A \gamma_p(x) \| \leq \| A p \| \) for \( x \in [0, \gamma_p] \). For \( 0 \leq x < \gamma_p \), we have that \( \text{dist}(p, \partial C) \leq \text{dist}(\gamma_p(x), \partial C) + \| p - \gamma_p(x) \| \leq \text{dist}(\gamma_p(x), \partial C) + x \| A p \| \). Hence,

\[
\text{dist}(p, \partial C)/\| A p \| \leq \text{dist}(\gamma_p(x), \partial C)/\| A p \| + x
\]

and so \( \gamma_p - x \leq \gamma_p(x) \). Thus, if \( 0 \leq x < \gamma_p \) and \( 0 \leq y < \gamma_p - x \), one sees that \( \gamma_p(x)(y) = \gamma_p(x + y) \). To show that \( \gamma_p' = A \gamma_p \), let \( 0 \leq x < \gamma_p \) and let \( c > 0 \). By Lemma 1.2 there is a positive number \( z < \gamma_p - x \) such that if \( 0 < y < z \) and \( \{s_i\}_{i=0}^m \) is a chain from 0 to \( y \), then

\[
\left\| A \prod_{i=1}^m L(s_i - s_{i-1}) \gamma_p(x) - A \gamma_p(x) \right\| < c/2.
\]

Let \( 0 < y < z \). There is a chain \( \{t_i\}_{i=0}^m 0 \) from 0 to \( y \) such that

\[
\left| \prod_{i=1}^n L(t_i - t_{i-1}) \gamma_p(x) - \gamma_p(x)(y) \right| < c \cdot y/2.
\]

Then,

\[
\left\| \frac{1}{y} [\gamma_p(x + y) - \gamma_p(x)] - A \gamma_p(x) \right\| < c/2 + \frac{1}{y} \left| \left( \prod_{i=1}^n L(t_i - t_{i-1}) \gamma_p(x) - \gamma_p(x) \right) - y A \gamma_p(x) \right|
\]

\[
= \frac{1}{2} + \frac{1}{y} \sum_{i=1}^n (t_i - t_{i-1}) A \prod_{j=1}^i L(t_j - t_{j-1}) \gamma_p(x) - y A \gamma_p(x) \right|
\]

\[
\leq \frac{c}{2} + \frac{1}{y} \sum_{i=1}^n (t_i - t_{i-1}) \left| A \prod_{j=1}^i L(t_j - t_{j-1}) \gamma_p(x) - A \gamma_p(x) \right| < c
\]
and so \( g_p'(x) = Ag_p(x) \). Thus, \( g_p^{\prime +} = Ag_p \) on \([0, \gamma_p)\) and so \( g_p \) has a continuous right derivative on \([0, \gamma_p)\). Then \( g_p \) has a continuous derivative on \([0, \gamma_p)\) and so the theorem is proved.

**Corollary.** Let \( A \) be a mapping from the Banach space \( S \) to \( S \) such that the following are true:

(I') \( A \) is dissipative on \( S \), i.e., if \( u, v \in D_A \) and \( \epsilon \geq 0 \) then \( \| (e - \epsilon A)u - (e - \epsilon A)v \| \geq \| u - v \| \)

(II') \( R(e^{-\epsilon A}) = S \) for each \( \epsilon \geq 0 \)

(III') \( A \) is continuous on \( S \).

If \( p \in S \) then there is a continuously differentiable function \( g_p \) from \([0, \infty)\) to \( S \) such that \( g_p(0) = p \) and if \( x \geq 0 \) \( g_p'(x) = Ag_p(x) \) and \( g_p(x) = \int_0^x (e - dIA)^{-1}p \).

**Proof.** The proof follows immediately from the theorem if one observes that \( a = +\infty \) and \( \text{dist} (p, \partial S) = +\infty \).

It may be noted that a result of J. Dorroh [8] can be used to show that the solutions of \( g_p' = Ag_p \), \( g_p(0) = p \) in the corollary are unique. In [15] G. Minty has shown that if \( S \) is the Hilbert space then (I') and (III') imply (II'). More generally, it has been shown recently by T. Kato in [11] that (I') and (III') imply (II') in the case that \( S^* \) is uniformly convex. If \( S \) is a general Banach space F. Browder has shown in [4] that (I') and (III') imply (II') in the case that \( A \) is locally uniformly continuous.

By virtue of the corollary one may define for each \( x \geq 0 \) the transformation \( T(x) \) from \( S \) to \( S \) as follows: \( T(x)p = g_p(x) \) for each \( p \in S \). Then \( T \) is a strongly continuous semigroup of nonlinear nonexpansive transformations on \( S \), i.e.,

(i) \( T(x+y) = T(x)T(y) \) for \( x, y \geq 0 \),
(ii) \( T(0) = e \),
(iii) \( \| T(x)p - T(x)q \| \leq \| p - q \| \) for \( x \geq 0 \) and \( p, q \in S \) and
(iv) \( g_p(x) = T(x)p \) is continuous for \( p \) fixed and \( x \geq 0 \).

Further, \( A \) is the infinitesimal generator of \( T \), i.e., \( Ap = g_p'(0) \) for each \( p \in S \). In [2], [14], [17], [18], and [19] representations are given for nonlinear nonexpansive semigroups of transformations in terms of their infinitesimal generators using product integrals.

3. **Examples.** In conclusion we give some examples. In [6] a well-known example is given by J. Dieudonné of a continuous mapping \( A \) from a Banach space \( S \) to \( S \) for which there is no solution to the equation \( g' = Ag \) and \( g(0) = 0 \). This example is given in a Banach space which is not reflexive. Recently, J. Yorke [20] has given an example of a continuous mapping \( A \) from a Hilbert space to itself for which no solution exists to \( g' = Ag \), \( g(0) = 0 \).

In the examples below the mapping \( A \) satisfies conditions (I'), (II'), and (III') of the corollary.

**Example 1.** Let \( S = E_1 \) and let \( A \) be a continuous nonincreasing function from \( E_1 \) to \( E_1 \).
Example 2. Let $S = C_{[0,1]}$, i.e., $S$ is the Banach space of continuous real-valued functions on $[0, 1]$ with supremum norm. Let $F$ be a continuous increasing function from $E_1$ onto $E_1$ such that $F'$ is continuous and nonincreasing on $E_1$. Define the mapping $A$ on $C_{[0,1]}$ as follows:

$$ Af = F'[F^{-1}[f]] \quad \text{for each } f \in C_{[0,1]}.$$

The solutions $g_f$ of the corollary are then given by $g_f(x) = F[x + F^{-1}[f]]$ for $x \geq 0$.

In both Examples 1 and 2 $A$ may be neither linear nor Lipschitz continuous. In both, however, $A$ is locally uniformly continuous. In Example 3 the mapping $A$ is not locally uniformly continuous.

Example 3. Let $S = (c_0)$, i.e., $S$ is the Banach space of real-number sequences $x = (x_n)$ converging to 0 with $\|x\| = \sup_n |x_n|$. If each of $(a, b)$ and $(c, d)$ is a point in the plane define the function $F_{[(a, b), (c, d)]}$ from $[a, c]$ to $[b, d]$ by

$$ F_{[(a, b), (c, d)]}(x) = b + \left(\frac{d-b}{c-a}\right)(x-a) \quad \text{for } x \in [a, c].$$

For each positive integer $n$ define the function $A_n$ from $E_1$ to $E_1$ as follows:

$$ A_n(x) = \begin{cases} 1 & \text{if } x < -1 \\ 0 & \text{if } x \geq 0 \end{cases} $$

$$ = F_{([-1/k, 1/k], (-1/k+1/n)(1/k-1/(k+1), 1/(k+1))]}(x) \quad \text{if } x \in \left[\frac{-1}{k+1}, \frac{1}{k+1}\right]$$

$$ = \frac{1}{k+1} \quad \text{if } x \in \left[\frac{-1}{k+1}, \frac{1}{k+1}\right].$$

Define the mapping $A$ from $(c_0)$ to $(c_0)$ by $Ax = (A_n(x_n))$ for each $x = (x_n) \in (c_0)$. One sees that $A$ satisfies conditions (I'), (II'), and (III'), since for each positive integer $n$ $A_n$ is nonincreasing and continuous. Moreover, there is no neighborhood about 0 on which $A$ is uniformly continuous.

References


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