COMPACT IMBEDDING THEOREMS FOR QUASIBOUNDED DOMAINS

BY

ROBERT A. ADAMS(1)

1. Introduction. Let $G$ be an unbounded open set in Euclidean $n$-space $E_n$. In this paper we investigate (for a large class of such domains) the problem of determining for which values of $m$, $p$, $j$ and $r$ the Sobolev space imbedding

$$W_{0}^{m,p}(G) \rightarrow W_{0}^{j,r}(G)$$

is or is not compact. Provided $j < m$ continuous imbeddings of this type are known to exist for $p \leq r \leq np(n - mp + jp)^{-1}$ if $n > mp - jp$ or for $p \leq r < \infty$ if $n \leq mp - jp$ (the Sobolev Imbedding Theorem, e.g. [5, Lemma 5]). If $G$ were bounded Kondrashov's compactness theorem [9] would yield the complete continuity of these imbeddings except in the extreme case $r = np(n - mp + jp)^{-1}$. Such compactness theorems are useful for studying existence and spectral theory for partial differential operators on $G$.

In a sequence of recent papers the writer [1]--[4] and C. W. Clark [6], [7], [8] have studied such compactness problems for various unbounded domains. It is clear that the imbedding (1) cannot be compact if $G$ contains infinitely many disjoint congruent balls, for if a fixed $C^\infty$ function has support in one of these balls then the set of its translates with supports in the other balls is bounded in any space $W_{0}^{m,p}(G)$ but is not precompact in any such space. Thus a necessary condition for the compactness of imbedding (1) is that $G$ should be quasibounded, i.e. that $\text{dist} (x, \text{bdry } G) \rightarrow 0$ whenever $|x| \rightarrow \infty$, $x \in G$. In [1] the writer has shown that if $n > 1$ then quasiboundedness is not sufficient for compactness.

The dimension of the boundary of $G$ is a critical factor in determining whether or not (1) is compact. If $G$ is quasibounded and bounded by smooth "reasonably unbroken" $(n-1)$-dimensional manifolds then (1) is compact [3, Theorem 1] for any $m$ and $p$ and for the same values of $j$ and $r$ as in the case of bounded $G$. However if $G$ has discrete (0-dimensional) boundary then [2, Theorem 1] no such imbedding can be compact unless $mp > n$.

Our purpose in this paper is to study the compactness of imbedding (1) for quasibounded domains $G$ whose boundaries are comprised of smooth manifolds...
of various dimensions. Roughly speaking our results are as follows. If \( k \) is the smallest integer for which those boundary manifolds of \( G \) having dimension not less than \( n-k \) bound a quasibounded domain then no imbedding of type (1) can be compact when \( mp < k \). On the other hand, if, in addition, the boundary manifolds are "reasonably unbroken" and if \( mp > n+p-\eta k \) then (1) is compact for the same values of \( j \) and \( r \) as in the case of bounded \( G \). Our results thus interpolate between the extreme cases mentioned above. We consider first domains \( G \) with flat (planar) boundaries, establishing in §2 a necessary condition for the compactness of (1) for such \( G \), and in §3 a slightly stronger sufficient condition. If \( m=1 \) these conditions are equivalent for certain domains. In §4 similar results are obtained for nonflatly-bounded domains \( G \).

As usual, in this paper \( W^{m,p}_0(G) \) denotes, for \( p \geq 1 \) and \( m=0, 1, 2, \ldots \), the Sobolev space obtained by completing with respect to the norm

\[
\|u\|_{m,p,G} = \left\{ \sum_{f=0}^m |u|_{f,p,G}^p \right\}^{1/p}
\]

the space \( C^\infty_0(G) \) of all infinitely differentiable, complex functions having compact support in \( G \) where

\[
|u|_{f,p,G} = \sum_{|\alpha|=f} \int_G |D^\alpha u(x)|^p \, dx.
\]

\( \alpha \) denotes an \( n \)-tuple of nonnegative integers \( (\alpha_1, \ldots, \alpha_n) \); \( |\alpha| = \alpha_1 + \cdots + \alpha_n \); \( D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} \); \( D_j = \partial/\partial x_j \). Note that \( |u|_{0,p,G} = \|u\|_{0,p,G} \) is the norm of \( u \) in \( L^p(G) \). \( W^{m,p}_0(G) \) represents the completion with respect to the norm (2) of the space of all infinitely differentiable functions on \( G \) for which (2) is finite. Provided the boundary of \( G \) satisfies certain mild regularity conditions [5, Lemma 5] the Sobolev Imbedding Theorem referred to above, and also (provided \( G \) is bounded) the Kondra\u0103ov Compactness Theorem, remain valid for imbeddings of \( W^{m,p}_0(G) \). No compactness theorems of this sort are yet known if \( G \) is unbounded.

2. Flatly-bounded domains—noncompact imbeddings. Let \( H \) be a \( k \)-dimensional plane \( 0 \leq k \leq n-1 \) in \( E_n \) and let \( a \) be a point on \( H \). With respect to a new system of rectangular coordinates \( z \) in \( E_n \) having origin at \( a \) and obtained from the usual coordinates by an affine transformation, \( H \) has equations \( z_1 = z_2 = \cdots = z_{n-k} = 0 \), or more simply \( r = 0 \) where \( r = \sum_{i=k+1}^n z_i^2 \). The coordinate \( r \), together with \( n-k-1 \) angle coordinates collectively denoted \( \sigma \) and the coordinates \( z' = (z_{n-k+1}, \ldots, z_n) \) form a system of cylindrical polar coordinates in \( E_n \) with origin at \( a \) and cylindrical axis \( H \).

The \( k \)-tube \( T_\delta(H) \) of radius \( \delta \) and axis \( H \) is the set \( \{x \in E_n : \text{dist}(x, H) = r < \delta \} \). By a tube function for the tube \( T_\delta(H) \) we mean a \( C^\infty \) function \( \theta : E_n \to [0, 1] \) whose value at \( x \) depends only on \( r = \text{dist}(x, H) \) and which vanishes identically near \( H \) and is identically unity outside \( T_\delta(H) \).
Lemma 1. Let $H$ be a $k$-plane in $E_n$ and $a \in H$. Let $1 \leq p < \infty$. If $u(x) = v(r)$ where $r = \text{dist}(x, H)$ and $v \in C^{1,\alpha}((0, \infty))$ then for all $x \notin H$

$$|D^\alpha u(x)|^p \leq \text{const} \sum_{j=1}^{|\alpha|} |v^{(j)}(r)|^p r^{p|\alpha| - p}$$

where the constant depends only on $\alpha$, $p$ and $k$.

Proof. Since $z_i = \sum_{j=1}^n c_j(x_j - a_j)$ and so $\partial/\partial x_i = \sum_{j=1}^n c_j \partial/\partial z_j$ we may assume with no loss of generality that $H$ is a coordinate plane and $z = x$. We show that there exist homogeneous polynomials $P_{\alpha,i}(x)$ of degree $|\alpha|$ (possibly the zero polynomial) such that for $r > 0$

$$|D^\alpha u(x)|^p = \sum_{j=1}^{|\alpha|} P_{\alpha,i}(x)v^{(j)}(r)r^{j - 2|\alpha|}.$$  

Since $|P_{\alpha,i}(x)| \leq \text{const } r^{|\alpha|}$ the conclusion of the lemma for $p = 1$ follows at once. The result for general $p$ then follows from the well-known inequality

$$\left| \sum_{j=1}^N A_j \right|^p \leq \text{const} \sum_{j=1}^N |A_j|^p$$

where the constant depends only on $p$ and $N$.

Note that $D^\alpha u(x) = 0$ unless $\alpha_{n-k+1} = \cdots = \alpha_n = 0$. If $1 \leq i \leq n-k$ then $D_i u(x) = v'(r)x_i/r$ which is of the required form. Assume (3) holds for all $\alpha$ with $|\alpha| \leq m$. If $|\beta| = m+1$ then $D^\beta = D_i D^\alpha$ for some $i$, $\alpha$ where $|\alpha| = m$. Applying the induction hypothesis and the chain rule we obtain

$$D^\beta u(x) = \sum_{j=1}^m \{ D_i P_{\alpha,i}(x)v^{(j)}(r)r^{j - 2m - 1} - (2m-j)P_{\alpha,i}(x)v^{(j)}(r)x_i r^{j - 2m - 2} \}$$

$$= \sum_{j=1}^{m+1} P_{\beta,i}(x)v^{(j)}(r)r^{j - 2(m+1)}$$

where $P_{\beta,i}$ is given by

$$P_{\beta,1}(x) = r^2 D_i P_{\alpha,1}(x) - (2m-1)x_i P_{\alpha,1}(x),$$

$$P_{\beta,j}(x) = r^2 D_i P_{\alpha,j}(x) - (2m-j)x_i P_{\alpha,j}(x) + x_i P_{\alpha,j-1}(x) \quad \text{if } 2 \leq j \leq m,$$

$$P_{\beta,m+1}(x) = x_i P_{\alpha,m}(x).$$

Clearly $P_{\alpha,i}(x)$ is a polynomial of the desired type and the proof is complete.

Lemma 2. Let $\lambda$ be a positive integer and let $r = s^\lambda$, $s > 0$. If $f \in C^\lambda((0, \infty))$ and $1 \leq p < \infty$ then

$$|(d/dr)^j f(r^{1/\lambda})|^p \leq \text{const} \sum_{i=1}^j \lambda^{-ip_2} s^{-j \lambda} |f^{(i)}(s)|^p$$

where the constant depends only on $j$ and $p$. 

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Proof. Again the case of general \( p \) follows from the special case \( p = 1 \) via (4). For \( p = 1 \) (5) is an immediate consequence of the formula

\[
(d/dr)^i = \lambda^{-i} \sum_{j=1}^{i} P_{j-i,i}(\lambda) s^{i-\lambda j}(d/ds)^j
\]

where \( P_{j-i,i} \) is a polynomial of degree \( i \) depending on \( j \). We prove (6) by induction on \( j \). Note that \( d/dr = \lambda^{-1}s^{1-\lambda}d/ds \) which is of the required form. Assuming (6) we have

\[
(d/dr)^{i+1} = \lambda^{-i} \sum_{j=1}^{i} P_{j-i,i}(\lambda) \lambda^{-1}s^{1-\lambda}d/ds(s^{i-\lambda j}(d/ds)^j)
\]

\[
= \lambda^{-(i+1)} \sum_{j=1}^{i} P_{j-i,i}(\lambda)(s^{i+1-\lambda j}(d/ds)^{j+1} + (i-\lambda j)s^{i-\lambda j}(d/ds)^j)
\]

\[
= \lambda^{-(i+1)} \sum_{j=1}^{i+1} P_{j+1-i,i+1}(\lambda)s^{i-\lambda(j+1)}(d/ds)^j
\]

where the polynomials \( P_{j+1-i,i+1} \) are given by

\[
P_{0,j+1}(\lambda) = P_0(\lambda),
\]

\[
P_{i,j+1}(\lambda) = P_i(\lambda) + (j + 1 - i - \lambda)P_{i-1,i}(\lambda)
\]

for \( 1 \leq i \leq j - 1, \)

\[
P_{j,j+1}(\lambda) = (1 - \lambda j)P_{j-1,j}(\lambda),
\]

which are of the desired form.

Lemma 3. Let \( T \) be a \( k \)-tube in \( E_n \) with axis \( H \) and radius \( \delta \leq 1 \). Let \( 1 \leq p < \infty \) and let \( \lambda \) be a positive integer. Then there exists a tube function \( \theta \) for \( T \) satisfying for \( |\alpha| > 0, \)

\[
|D^\alpha \theta(x)|^p \leq \text{const} \lambda^{-|\alpha|} s^p - \lambda p |\alpha|
\]

where \( s^\delta = r = \text{dist}(x, H) \) and the constant depends only on \( \alpha, n, p \) and \( k \) and not on \( \lambda \).

Proof. Let \( f: [0, \infty) \to [0, 1] \) be a fixed \( C^\infty \) function such that \( f(s) = 0 \) near \( s = 0 \) and \( f(s) = 1 \) for \( s^\delta \geq \delta \). Define \( \theta \) by \( \theta(x) = \alpha(r) = f(s) \). Clearly \( \theta \) is a tube function for \( T \). By Lemmas 1 and 2 we have

\[
|D^\alpha \theta(x)|^p \leq \text{const} \sum_{j=1}^{[\alpha]} r^{p_j - |\alpha|} |f^{(j)}(r)|^p
\]

\[
\leq \text{const} \sum_{j=1}^{[\alpha]} \sum_{i=1}^{[\alpha]} \lambda^{-1} r^p s^p - \lambda p |\alpha| |f^{(i)}(s)|^p
\]

\[
\leq \text{const} \lambda^{-|\alpha|} s^p - \lambda p |\alpha|.\]

The final inequality follows because whenever \( D^\alpha \theta(x) \neq 0, s^\delta \leq 1, \) and also \( |f^{(i)}(s)| \leq \text{const} \) for \( 1 \leq i \leq |\alpha| \).

In the following lemma we consider several \((n-k)\)-tubes in \( E_n \) simultaneously. Hence all the related quantities \( \theta, r, \sigma, z', s, H \) carry subscripts ranging from 1 to \( N \).
Lemma 4. Let $S$ be a bounded open set in $E_n$. Let $H_1, \ldots, H_N$ be a finite collection of $(n-k)$-planes which intersect $\overline{S}$. Let $m$ be a positive integer and let $\epsilon > 0$. If either $p > 1$ and $mp \leq k$ or $p = 1$ and $m < k$ then there exists a function $\phi \in C^\infty(E_n)$ with the properties:

(i) $\phi(x) = 0$ for $x$ near $\bigcup_{i=1}^N H_i$,
(ii) $0 \leq \phi(x) \leq 1$ for all $x$,
(iii) $\phi(x) = 1$ for $x$ in $E_n - \bigcup_{i=1}^N T_\delta(H_i)$, $\delta > 0$,
(iv) $\|D^\alpha\phi\|_{0,p,\delta} \leq \epsilon$ for $0 < |\alpha| \leq m$.

Proof. We first consider the case that no two of the planes $H_i$ intersect in $\overline{S}$. It is then possible to choose $\delta \leq 1$ small enough so that if $T_i = T_\delta(H_i)$ then $T_i \cap T_j \cap S$ is empty if $i \neq j$. By Lemma 3 there exist tube functions $\theta_i$ for $T_i$ satisfying

$$|D^\alpha \theta_i(x)|^p \leq \text{const} \lambda^{-p} s_i^{-\lambda p |\alpha|}$$

where $s_i^\delta = r_i = \text{dist} (x, H_i)$ and the constant is independent of $\lambda$ and $\alpha$ for $0 < |\alpha| \leq m$. Let $\phi(x) = \theta_1(x)\theta_2(x) \cdot \cdot \cdot \theta_N(x)$. Clearly $\phi$ satisfies (i)-(iii). Note that $D^\alpha \phi = 0$ outside $\bigcup_{i=1}^N T_i$ and that $D^\alpha \phi(x) = D^\alpha \theta_i(x)$ in $T_i$. We have

$$\|D^\alpha \phi\|_{0,p,\delta} = \sum_{i=1}^N \int_{T_i} |D^\alpha \theta_i(x)|^p \, dx$$

$$\leq \text{const} \lambda^{-p} \sum_{i=1}^N \int_{T_i} s_i^{-p - \lambda p |\alpha|} r_i^{k-1} \, dr_i \, d\alpha_i \, dz_i$$

$$\leq \text{const} \lambda^{-p} \int_0^1 s_i^{-p - \lambda p |\alpha| + \lambda k - 1} \, ds_i.$$ 

The final constant depends on $\alpha$, $p$, $n$, $k$, $N$ and diam $S$ but not on $\lambda$. If $|\alpha| \leq m$ and $mp \leq k$ then $p - \lambda p |\alpha| + \lambda k > 0$ and so

$$\|D^\alpha \phi\|_{0,p,\delta} \leq \text{const} \lambda^{-p} (p + \lambda k - \lambda p |\alpha|)^{-1}.$$ 

The expression on the right can be made arbitrarily small for sufficiently large $\lambda$ provided either $p > 1$ or $m < k$. This establishes (iv).

The case of intersecting $H_i$ remains to be considered. Again pick $\delta < 1$ small enough so that $T_i \cap T_j \cap S$ is empty whenever $H_i \cap H_j \cap S$ is empty. Define $\theta_i$ and $\phi$ as above. The general Leibniz formula states

$$D^\alpha \phi(x) = \sum_{\beta_1 + \cdot \cdot \cdot + \beta_N = \alpha}^\alpha \beta_1^\alpha \cdot \cdot \cdot \beta_N^\alpha (D^\beta_1 \theta_1)(x) \cdot \cdot \cdot (D^\beta_N \theta_N)(x).$$

For estimates of $|D^\alpha \phi(x)|$ we may drop from terms in the Leibniz expression any factor $D^\beta_i \theta_i(x)$ for which $\beta_i = 0$ because $|\theta_i(x)| \leq 1$. For simplicity consider a term $D^\beta_1 \theta_1(x) \cdot \cdot \cdot D^\beta_N \theta_N(x)$ where no $\beta_i$ is zero. Decompose $S$ into the union of $N$ sub-regions $S_j$ such that in $S_j$ we have $s_i \leq s_j$ for $i \neq j$. We now obtain via Lemma 3 in the manner above, noting that whenever $D^\beta_i \theta_i(x) \neq 0$ then $s_i < \delta \leq 1$
\[
\int_{S} |D^{\delta_{1}}\theta_{1}(x) \cdots D^{\delta_{n}}\theta_{n}(x)|^{p} \, dx = \sum_{j=1}^{N} \int_{S_{j}} |D^{\delta_{1}}\theta_{1}(x) \cdots D^{\delta_{n}}\theta_{n}(x)|^{p} \, dx \\
\leq \text{const} \lambda^{-Np} \sum_{j=1}^{N} \int_{S_{j}} s_{j}^{\gamma_{1}} - \gamma_{1}^{p} \cdots s_{j}^{\gamma_{n}} - \gamma_{n}^{p} \, dx \\
\leq \text{const} \lambda^{-Np} \sum_{j=1}^{N} \int_{S_{j}} s_{j}^{\gamma_{1}} - \gamma_{1}^{p} \cdots s_{j}^{\gamma_{n}} - \gamma_{n}^{p} |a| \, dx \\
\leq \text{const} \lambda^{-Np} \int_{0}^{1} s^{\gamma_{1}} - \gamma_{1}^{p} |a| + \gamma_{2} - 1 \, ds \\
\leq \text{const} \lambda^{-Np}(Np + \gamma_{2} - \gamma_{1}^{p}|a|)^{-1}.
\]

Similar estimates can be found for all terms in (7) and (iv) follows once more by taking \( \lambda \) sufficiently large.

**Definition.** Let \( G \) be an open set in \( E_{n} \). \( G \) is called a regular domain if \( \text{bdry} \, G = \bigcup_{k=0}^{0} G_{k} \) where \( G_{k} \) is the union of a locally finite collection of smooth manifolds of dimension \( k \) in \( E_{n} \). A regular domain \( G \) whose boundary manifolds are all segments of planes of various dimensions will be called a regular flatly-bounded domain. An unbounded regular domain \( G \) is called 0-quasibounded if it is quasi-bounded, i.e. if there exist at most finitely many disjoint congruent balls \( Q \) in \( G \), having any specified positive radius, which do not intersect the boundary of \( G \). \( G \) is called \( k \)-quasibounded \( (1 \leq k \leq n-1) \) if there exist at most finitely many disjoint congruent balls \( Q \) in \( G \), having any specified positive radius, such that \( Q \cap \text{bdry} \, G = \bigcup_{k=1}^{k} G_{k} \); i.e. if \( \text{dist} \,(x, \bigcup_{k=1}^{k} G_{k}) \rightarrow 0 \) as \( |x| \rightarrow \infty, \, x \in G \). For \( 1 \leq k \leq n-1 \) the condition of \( k \)-quasiboundedness is stronger than that of \((k-1)\)-quasiboundedness.

**Theorem 1.** Let \( G \) be a regular, quasibounded, flatly-bounded domain in \( E_{n} \). Let \( k \) be the smallest integer \( (1 \leq k \leq n) \) for which \( G \) is \((n-k)\)-quasibounded. If either \( mp \leq k \) and \( p > 1 \) or \( m < k \) and \( p = 1 \) then no imbedding of the form

\[
W_{0}^{m,p}(G) \rightarrow W_{0}^{k,n}(G)
\]

can be compact.

**Proof.** For \( 2 \leq k \leq n \) since \( G \) is not \((n-k+1)\)-quasibounded there is a sequence of congruent open balls \( \{Q_{i}\}_{i=1}^{\infty} \) in \( G \) such that \( Q_{i} \cap \text{bdry} \, G \) is contained in the union of finitely many \((n-k)\)-planes. If \( k = 1 \) balls \( Q_{i} \) with this property exist trivially since \( G \) is regular and flatly-bounded. Let \( Q \) denote any one of these balls and let \( H_{1}, \ldots, H_{n} \) be the corresponding \((n-k)\)-planes. Let \( \varphi \in C_{0}^{\infty}(Q) \) be a function for which

\[
\|\varphi\|_{0,r,Q} = 2C > 0, \quad \|\varphi\|_{m,p,Q} = K < \infty.
\]

There exists a constant \( M \) such that for all \( x \in E_{n} \) and for all \( \alpha \) with \( 0 \leq |\alpha| \leq m \), \(|D^{\alpha}\varphi(x)| \leq M \). Choose \( \delta_{0} > 0 \) small enough so that the sum of the volumes of the intersections with \( Q \) of the \((n-k)\)-tubes \( T_{\delta_{0}}(H_{i}), \, i = 1, \ldots, N \), does not exceed \((C/M)^{\gamma}\). Let

\[
e = K \left[ M \sum_{|\alpha| \leq m} \sum_{\beta < \alpha} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) \right]^{-1}.
\]
By Lemma 4 there exists for some $\delta \leq \delta_0$ a function $\psi \in C_C^\infty(E_n - \bigcup_{i=1}^n H_i)$ satisfying $0 \leq \psi(x) \leq 1$, $\psi(x) = 1$ outside $\bigcup_{i=1}^n T_{\delta}(H_i)$ and $\|D^\alpha \psi\|_{0,p,q} \leq \varepsilon$ for $0 < |\alpha| \leq m$. Let $\gamma = \varphi \cdot \psi = \varphi - \varphi(1 - \psi)$. Clearly $\gamma \in C_C^\infty(Q \cap G)$. Putting $T = T_{\delta}(H_i)$ we have

$$\|\gamma\|_{0,p,q} \geq \|\varphi\|_{0,p,q} - \|\varphi\|_{0,p,q\cap(T_1 \cup \cdots \cup T_n)} \geq 2C - M[\text{vol } Q \cap (T_1 \cup \cdots \cup T_n)]^{1/r} \geq C.$$

Moreover, for $0 < |\alpha| \leq m$

$$D^\alpha \gamma = \psi D^\alpha \varphi + \sum_{\beta < \alpha} \binom{\alpha}{\beta} D^\beta \varphi D^{\alpha - \beta} \psi,$$

and

$$\|D^\alpha \gamma\|_{0,p,q} \leq \|D^\alpha \varphi\|_{0,p,q} + \sum_{\beta < \alpha} \binom{\alpha}{\beta} M \varepsilon \leq 2K.$$

Thus if $K_i = \sum_{|\alpha| \leq m} 1$ we have $\|\gamma\|_{m,p,q} \leq 2KK_1$.

Now let $\varphi_i$, $i = 1, 2, \ldots$ be a translate of $\varphi$ with support in $Q_1$ and let $\gamma_i$ be constructed from $\varphi_i$ as $\gamma$ from $\varphi$ above, so that

$$\|\gamma_i\|_{0,p,q} \geq C, \quad \|\gamma_i\|_{m,p,q} \leq 2KK_1.$$

The sequence $(\gamma_i)$, though bounded in $W_0^{m,p}(G)$ has no subsequence converging in $L'(G)$. In fact if $i \neq j$ then $\|\gamma_i - \gamma_j\|_{0,p,q} \geq 2^{1/r}$. Thus the embedding

$$W_0^{m,p}(G) \to W_0^{0,n}(G) = L'(G)$$

if it exists cannot be compact. Neither can the embedding $W_0^{m,p}(G) \to W_0^{0,n}(G)$ for if this latter imbedding were compact then so would be the composition

$$W_0^{m,p}(G) \to W_0^{0,n}(G) \to L'(G).$$

3. Flatly-bounded domains—compact imbeddings. Let $H$ be a $k$-plane in $E_n$ and let $a \in H$. We denote by $T_{\delta,0}(H, a)$ the tube segment of radius $\delta$ and length $2\rho$ having axis $H$ and centre $a$. Thus, if $P$ is the orthogonal projection operator on $H$ then

$$T_{\delta,0}(H, a) = \{x \in E_n : \text{dist } (x, H) < \delta, \text{dist } (Px, a) < \rho\}.$$
Integrating over $\Sigma$ and $Z$ the domains of the variables $\sigma$ and $z'$ respectively in $T$ we obtain

$$\|y\|_{1,1,T} = \int_\Sigma d\sigma \int_Z dz' \int_0^\delta |y(\sigma, z')|^p d\sigma \int_0^\delta |y(t, \sigma, z')|^p t^{k-1} dt \int_0^\delta t^{-(k-1)(p-1)} dt \left(\int_0^\delta t^{-k-1} dt\right)^{p-1} \leq \left(\frac{p-1}{p-k}\right)^{p-1} \delta^{p-1} \int_0^\delta |y(t, \sigma, z')|^p t^{k-1} dt.$$

For the special case $p=k=1$ we have

$$\|y\|_{1,1,T} = \int_\Sigma d\sigma \int_Z dz' \int_0^\delta |y(\sigma, z')| d\sigma \int_0^\delta |y(t, \sigma, z')| t^{k-1} dt \leq \int_\Sigma d\sigma \int_Z dz' \int_0^\delta |y(t, \sigma, z')| t^{k-1} dt.$$ 

**Corollary.** Under the conditions of the lemma, if $1 \leq q \leq p$ then there exists a constant depending only on $p$, $q$, $k$, $n$ such that for all $\delta > 0$

$$\|y\|_{0,q,T} \leq \text{const} \delta^{1+n/q-n/p} |y|_{1,1,T}$$

for all $y \in C^\infty(\mathbb{R}^n - H \cap T)$ where $T=T_{0,0}(H, a)$.

**Proof.** By Hölder’s inequality and since $\text{vol } T=\text{const} \delta^n$

$$\|y\|_{0,q,T} \leq \|y\|_{0,q,T}[\text{vol } T]^{1/q-1/p} \leq \text{const} \delta^{1+n/q-n/p} |y|_{1,1,T}.$$

**Definition.** Let $G$ be an unbounded, regular, flatly-bounded domain in $\mathbb{R}^n$. We shall say that $G$ has the $k$-tube property if for every sufficiently large positive number $R$ there exists a positive number $\delta=\delta(R)$ with the properties:

(i) $\delta(R) \to 0$ as $R \to \infty$,

(ii) for each $x \in G_R = \{y \in G : |y| > R\}$ there exists a $k$-plane $H$ and a point $a \in H$ such that $x \in T_{0,0}(H, a)$ and $H \cap T_{2\delta,0}(H, a) \subset \text{bdry } G$.

It is clear that if $G$ has the $k$-tube property then $G$ is $k$-quasibounded. Of course the converse is not true as the planar segments comprising the boundary of $G$ may have too many gaps to satisfy condition (ii). For domains $G$ whose boundaries consist only of whole planes the $k$-tube property is equivalent to $k$-quasibounded-
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ness. Other examples of domains with the k-tube property are not difficult to
construct—for example Clark’s “spiny urchin” [7] has the 1-tube property in \(E_a\).

**Lemma 6 (A variant on Poincaré’s Inequality).** Let \(G\) be an unbounded,
regular, flatly-bounded domain in \(E_a\) having the \((n-k)\)-tube property for some \(k\)
\((1 \leq k \leq n)\). If \(1 \leq r \leq p\) where either \(p > k\) or \(p = k = 1\) then there exists a constant
depending only on \(n, k, p\) and \(r\) such that for all \(u \in W^{1,p}_0(G)\) and all sufficiently large \(R\)

\[
\|u\|_{0,r,G_R} \leq \text{const} \left[ \delta(R) \right]^{1+n/r-n/p} \|u\|_{1,r,G}.
\]

**Proof.** Fix \(R\) large enough so that \(\delta = \delta(R)\) exists. If \(\alpha\) is an \(n\)-tuple of integers
(not necessarily nonnegative) let \(Q_\alpha = \{x \in E_n : \alpha n^{-1/2} \delta \leq x_1 \leq (\alpha_1 + 1)n^{-1/2} \delta\}.\) Then
\(E_n = \bigcup_\alpha Q_\alpha.\) If \(x \in G_R\) then \(x \in Q_\alpha\) for some \(\alpha\) and there exists an \((n-k)\)-plane \(H\)
and a point \(a \in H\) such that \(x \in T_0(\alpha, a) = T\) and \(H \cap T' \subset \text{bdry} \ G\) where \(T' = T_{a_2, a_2}(H, a)\). Clearly \(Q_\alpha \subset T'.\) For any \(\gamma \in C^\infty_0(G)\) since \(\gamma\) vanishes near \(H \cap T'\) we have by the corollary of Lemma 5

\[
\|\gamma\|_{0,r,G_R} \leq \text{const} (2\delta)^{1+n/r-n/p} \|\gamma\|_{1,r,T'}
\]

where \(Q'_a\) is the union of all the cubes \(Q_\alpha\) which intersect \(T'\). There is a number \(N\)
depending only on \(n\) such that any \(N+1\) of the sets \(Q'_a\) have empty intersection. Summing the above inequality over all \(\alpha\) for which \(Q_\alpha\) meets \(G_R\) we obtain

\[
\|\gamma\|_{0,r,G_R} \leq \text{const} N \cdot \delta^{1+n/r-n/p} \|\gamma\|_{1,r,G}
\]

and this inequality extends by completion from \(C^\infty_0(G)\) to \(W^{1,p}_0(G)\).

**Theorem 2.** Let \(G\) be an unbounded, regular, flatly-bounded domain in \(E_a\) having
the \((n-k)\)-tube property \((1 \leq k \leq n)\). If either \(p > k\) or \(p = k = 1\) then the imbedding
\(W^{1,p}_0(G) \rightarrow L^p(G)\) exists and is compact for \(m = 0, 1, 2, \ldots\) and \(1 \leq r < \infty\) if \(p \geq n\) or for \(1 \leq r < np(n-p)^{-1}\) if \(p < n\).

**Proof.** First consider the case \(1 \leq r \leq p, m = 0.\) To prove that the imbedding
\(W^{1,p}_0(G) \rightarrow L^p(G)\) is compact we use the following compactness criterion for sets in
\(L^p(G)\): a sequence \(\{u_i\}_{i=1}^\infty\) which is bounded in \(L^p(G)\) is precompact in \(L^p(G)\) provided

(a) for every bounded \(G' \subset G\) the sequence \(\{u_i|G'\}\) is precompact in \(L^p(G')\), and

(b) for each \(\varepsilon > 0\) there exists \(R > 0\) such that for all \(i, \|u_i\|_{0,r,GR} < \varepsilon\).

Lemma 6 and condition \(i\) of the \((n-k)\)-tube property assures us that \((b)\) is satisfied for any sequence \(\{u_i\}\) bounded in \(W^{1,p}_0(G)\). To establish \((a)\) let \(G' \subset G\) be a bounded subset of \(G\). Then for some \(R, G' \subset K_R = \{x \in E_n : |x| \leq R\}.\) Let \(W^{1,p}(G, R)\) denote the completion with respect to the norm \(\| \cdot \|_{1,p,G,GR}\) of the space \(C^\infty_0(G)\). The imbedding \(W^{1,p}(K_R) \rightarrow L(K_R)\) is known to be compact (Kondrakov’s Theorem) and since an element of \(W^{1,p}(G, R)\) can be extended to be zero outside its support so as to belong to \(W^{1,p}(K_R)\) it follows that \(W^{1,p}(G, R)\) is compactly imbedded in...
$L'(G \cap K_R)$. But $\{u_i|K_R\}$ is bounded in $W^{1,p}(G, R)$ and hence precompact in $L'(G \cap K_R)$ whence $\{u_i|G'\}$ is precompact in $L'(G')$ as required.

By Sobolev's Imbedding Theorem $W^{1,p}_0(G)$ is continuously imbedded in $L^q(G)$ for any $q$ satisfying $p \leq q < \infty$ if $p \geq n$ or $p \leq q \leq np(n-p)^{-1}$ if $p<n$. Select such a $q$ and a sequence $\{u_i\}$ bounded in $W^{1,p}_0(G)$ so that, say, $\|u_i\|_{0,q,G} \leq C$. We may assume, passing to a subsequence if necessary, that $\{u_i\}$ converges in $L^q(G)$. By Hölder's Inequality if $p \leq r < q$

$$\|u_i - u_j\|_{0,r,G} \leq \|u_i - u_j\|_{0,p,G} \|u_i - u_j\|_{0,q,G} \leq (2C)^1-\lambda \|u_i - u_j\|_{0,p,G}$$

where $\lambda = p(q-r)r^{-1}(q-p)^{-1} > 0$. Hence $\{u_i\}$ converges in $L^q(G)$ and so the imbedding $W^{1,p}_0(G) \rightarrow L^q(G)$ is compact for $1 \leq r < \infty$ if $p \geq n$ and for $1 \leq r < np(n-p)^{-1}$ if $p<n$.

Finally, if $\{u_i\}$ is bounded in $W^{m+\eta,p}_0(G)$ then for any $\alpha$ with $0 \leq |\alpha| \leq m$, $\{D^\alpha u_i\}$ is bounded in $W^{1,p}_0(G)$ and so has a subsequence convergent to an element $v_\alpha$ of $L^q(G)$. In particular (for a suitable subsequence) $u_i \rightarrow v_0$ in $L^q(G)$ and so in the sense of distributions. Since $D^\alpha u_i \rightarrow v_\alpha$ in $L^q(G)$ and $D^\alpha u_i \rightarrow D^\alpha v_0$ in the sense of distributions it follows that $v_\alpha = D^\alpha v_0$ and $u_i \rightarrow v_0$ in $W^{m+\eta}_0(G)$. This completes the proof.

This theorem affords for imbeddings of the sort $W^{1,p}_0(G) \rightarrow L^q(G)$ on domains $G$ for which $(n-k)$-quasiboundedness is equivalent to the $(n-k)$-tube property, a complete converse to Theorem 1. For imbeddings $W^{0,p}_0(G) \rightarrow L^q(G)$, $m \geq 2$, we do not fare quite so well.

**Theorem 3.** Let $G$ be an unbounded, regular, flatly-bounded domain in $E_n$ having the $(n-k)$-tube property $(1 \leq k \leq n)$. Then the imbedding

$$W^{m,p}_0(G) \rightarrow W^{j,q}_0(G), \quad 0 \leq j < m,$$

is compact in any of the following cases:

(i) $m=p=k=1$,

(ii) $mp > n+p-np/k$, $p \leq r < p^*$,

(iii) $mp > n + (j+1)p - np/k$, $1 \leq r < p^*$,

where $p^* = np(n-mp+jp)^{-1}$ if $n > mp - jp$ and $p^* = \infty$ if $n \leq mp - np$.

**Proof.** The case $m=1$ has already been proved. If $m \geq 2$ the imbedding $W^{m,p}_0(G) \rightarrow W^{j,q}_0(G)$ is continuous for $p \leq q \leq np(n-mp+p)^{-1}$ if $n > mp - p$ and $p \leq r < \infty$ if $n \leq mp - p$. By Theorem 2 the imbedding $W^{j,q}_0(G) \rightarrow L^q(G)$ is compact provided $q > k$. Since $mp > np(n-mp+p)^{-1}$ is equivalent to $np(n-mp+p)^{-1} > k$ such $q > k$ can always be chosen and so the composed imbedding $W^{m,p}_0(G) \rightarrow L^q(G)$ is compact.

By a standard interpolation theorem for Sobolev spaces [5, Lemma 6] there exists a constant $K$ such that for $0 \leq j < m$, $p \leq q < p^*$ we have

$$\|u\|_{j,t,G} \leq K \|u\|_{m,p,G} \|u\|_{0,p,G}^{\lambda}$$
for all \( u \in W_0^{0,p}(G) \) where \( \lambda = (nr+jrp-np)(mrp)^{-1} \). Note that \( 0 \leq \lambda < 1 \) for all relevant values of \( j, m, n, r \) and \( p \). If \( \{u_i\}_{i=1}^\infty \) is a bounded sequence in \( W_0^{0,p}(G) \) then it has a subsequence again denoted \( \{u_i\} \) which is convergent in \( L^p(G) \). Since

\[
\|u_i - u_k\|_{j,r,G} \leq K \|u_i - u_k\|_{\varrho_{p,G}} \|u_i - u_k\|^1_{0,p,G}^{1-\lambda},
\]

it follows that \( \{u_i\} \) is a Cauchy sequence and hence convergent in \( W_0^{1,p}(G) \) proving case (ii).

If \( mp > n + (j+1)p - np/k \) (and in particular if \( j=0 \) in case (ii)) we have by Sobolev’s theorem and Theorem 2

\[
W_0^{0,p}(G) \rightarrow W_0^{1,n,p}(G),
\]

the second imbedding being compact since \( np(n-mp+jp+p)^{-1} > k \).

Remark. The condition \( mp > n+p - np/k \) implies \( mp > k \). The converse, however, is true only if \( k = n \) or \( k \leq p \). Thus even for domains \( G \) for which \( (n-k)\)-quasiboundedness is equivalent to the \( (n-k)\)-tube property imbeddings of \( W_0^{0,p}(G) \) corresponding to the cases \( 1 < p < k < np \leq n + p - np/k \) and \( 1 = p \leq k \leq m \) fail to be covered either by Theorem 1 or Theorem 3.

4. Extensions to nonflatly-bounded domains. Let \( G, G' \) be open sets in \( E_n \). A one-to-one transformation \( M \) from \( G \) onto \( G' \) is called an \( m\)-diffeomorphism of modulus \( C \) if all the components of \( M \) and \( M^{-1} \) have continuous partial derivatives of all orders up to and including \( m \), and these partials do not exceed \( C \) in modulus.

Lemma 7. Let \( G, G' \) be open in \( E_n \) and let \( M \) be an \( m\)-diffeomorphism of modulus \( C \) from \( G \) onto \( G' \). Let

\[
Au(y) = u(M^{-1}y), \quad y \in G'.
\]

Then \( A \) is a homeomorphism from \( W^{m,p}(G) \) [respectively \( W_0^{m,p}(G) \)] onto \( W^{m,p}(G') \) [resp. \( W_0^{m,p}(G') \)] and there exist constants \( C_1 \) and \( C_2 \) depending only on \( n, p \) and \( C \) and not on \( G \) or \( G' \) such that for all \( u \in W^{m,p}(G) \)

\[
C_1\|u\|_{m,p,G} \leq \|Au\|_{m,p,G'} \leq C_2\|u\|_{m,p,G}.
\]

Proof. \( A \) is a homeomorphism from \( L^p(G) \) onto \( L^p(G') \) for if \( \partial M/\partial x \) represents the Jacobian determinant of \( M \) then

\[
\left\{ \sup_{y \in G'} \left| \left. \frac{\partial M^{-1}(y)}{\partial y} \right|^{1/p} \right. \right\}^{-1} \|u\|_{0,p,G} \leq \|Au\|_{0,p,G'} \leq \sup_{x \in G} \left| \left. \frac{\partial M(x)}{\partial x} \right|^{1/p} \right. \|u\|_{0,p,G}.
\]

By induction and formal applications of the chain rule it is easily verified that in the sense of distributions on \( G' \)

\[
D^a(Au) = \sum_{|\alpha| \leq a} M_{\alpha} A(D^\alpha u)
\]
where $M_{ab}$ is a polynomial of degree $|\beta|$ in the derivatives of the components of $M^{-1}$ involving derivatives of orders not exceeding $|\alpha|$. It follows that

$$
\|Au\|_{m,p,G'}^p = \sum_{|\alpha| \leq m} \sum_{|\beta| \leq |\alpha|} M_{ab} A(D^\beta u)_{0,p,G'}^p \leq \text{const} \sum_{|\beta| \leq m} \|A(D^\beta u)\|_{0,p,G'}^p \leq \text{const} \|u\|_{m,p,G'}^p.
$$

The reverse inequality follows in a similar manner.

**Lemma 8.** Under the hypotheses of Lemma 7 $W^{m,p}(G)$ is compactly imbedded in $W^{1,r}(G)$ if and only if $W^{m,p}(G')$ is compactly imbedded in $W^{1,r}(G')$. A similar statement holds for the spaces $W_0^{m,p}(G)$.

**Proof.** Suppose the imbedding $W^{m,p}(G) \rightarrow W^{1,r}(G')$ is compact. Let $\{u_i\}_{i=1}^\infty$ be a bounded sequence in $W^{m,p}(G')$. Then $A^{-1}u_i$ is bounded in $W^{m,p}(G)$ and so has a subsequence converging in $W^{1,r}(G)$. The corresponding subsequence of $\{u_i\}$ is convergent in $W^{1,r}(G')$ whence the imbedding $W^{m,p}(G') \rightarrow W^{1,r}(G')$ is compact. The other cases are proved similarly.

Of course Lemma 8 can be used to obtain immediately the conclusions of Theorems 1–3 for any domain $G$ which is $m$-diffeomorphic to an unbounded, regular, flatly-bounded domain $G'$ satisfying the conditions of the particular theorem. As most quasibounded domains do not have this property we obtain generalizations of these theorems with localized hypotheses.

**Theorem 4.** Let $G$ be a regular, unbounded domain in $E_n$. Let $k$ be the largest integer ($1 \leq k \leq n$) for which for some constant $C$ there exist infinitely many mutually disjoint open sets $U$ in $\overline{G}$ each of which is $m$-diffeomorphic with modulus not greater than $C$ to the unit ball $B$ in $E_n$ in such a way that $U \cap \partial G$ is mapped into a subset of the union of finitely many $(n-k)$-planes. (In particular $G$ is not $(n-k+1)$-quasibounded.) If either $mp \leq k$, $p > 1$ or $m < k$, $p = 1$ then no imbedding of the form $W_0^{m,p}(G) \rightarrow W_0^{1,r}(G)$ can be compact.

**Proof.** Let $\{U_i\}_{i=1}^\infty$ be a sequence of mutually disjoint open sets in $\overline{G}$ for which there correspond $m$-diffeomorphisms $M_i : U_i \rightarrow B$ having modulus $\leq C$ and such that $M_i(U_i \cap \partial G) \subset B \cap P_i$ where $P_i$ is the union of finitely many $(n-k)$-planes. Let $\varphi \in C_0^\infty(B)$ be such that

$$
\|\varphi\|_{0,r,B} = C_1 > 0, \quad \|\varphi\|_{m,p,B} = K_1 < \infty.
$$

By the method used in the proof of Theorem 1 we can construct functions $\gamma_i \in C_0^\infty(B-P_i)$ such that

$$
\|\gamma_i\|_{0,r,B} \geq C_2 > 0, \quad \|\gamma_i\|_{m,p,B} \leq K_2 < \infty,
$$

the constants $C_2$ and $K_2$ being independent of $i$. Denoting by $A_i$ the operator for
which \( A_t \mu(y) = u(M_t^{-1}y) \), \( y \in B \) we have by Lemma 7 that there exist constants \( C_3 \) and \( K_3 \) again independent of \( i \) such that

\[
\|A_t^{-1} \gamma_i\|_{0,r,B} \geq C_3 > 0, \quad \|A_t^{-1} \gamma_i\|_{m,p,G} \leq K_3 < \infty
\]

and

\[
A_t^{-1} \gamma_i \in C_0^\infty(U_i \cap G).
\]

The noncompactness of the imbedding \( W_0^{m,p}(G) \rightarrow W_0^{j,q}(G) \) now follows as in Theorem 1.

As an analog of the compactness Theorems 2 and 3 we have

**Theorem 5.** Let \( G \) be an unbounded open set in \( E_n \) with the property that there exist constants \( C, R_0 \) and \( K \) such that for each \( R \geq R_0 \) there exist positive numbers \( d(R) \) and \( \delta(R) \) with the following properties:

(i) \( d(R) + \delta(R) \rightarrow 0 \) as \( R \rightarrow \infty \),

(ii) \( d(R)/\delta(R) \leq K, \ R \geq R_0 \),

(iii) for each \( x \in G_R = \{ x \in G : |x| > R \} \) the ball \( B_\delta(x) \) of radius \( \delta(R) \) and center \( x \) can be mapped by a \( 1 \)-diffeomorphism \( M \) of modulus \( \leq C \) onto a set \( S \) in \( E_n \) such that for some \( (n-k) \)-plane \( H \) \((1 \leq k \leq n)\) and some point \( a \in H \) we have \( S \subset T_{d(R),\delta(R)}(H, a) \) and \( H \cap T_{d(R),\delta(R)}(H, a) \subset M(bdy G \cap B_\delta(x)) \).

Then the imbedding \( W_0^{m,p}(G) \rightarrow W_0^{j,q}(G), 0 \leq j < m \) is compact in any of the following cases:

(a) \( m=p=k=1 \),

(b) \( mp > n+p-np/k, p \leq r < p^* \),

(c) \( mp > n+(j+1)p-np/k, 1 \leq r < p^* \),

where \( p^* = np(n-mp+jp)^{-1} \) if \( n > mp-jp \) and \( p^* = \infty \) if \( n \leq mp-jp \).

**Proof.** The conclusion is the same as that of Theorem 3 and the proof is identical if we reprove Lemma 6 (Poincaré's inequality) under the conditions of this theorem. Thus, let \( p > k \) or \( p = k = 1 \) and let \( 1 \leq r \leq p \). Fix \( R \geq R_0 \) and let \( d = d(R) \) and \( \delta = \delta(R) \). Define the cubes \( Q_\alpha \) as in the proof of Lemma 6. If \( x \in G_R \) then for some \( \alpha, x \in Q_\alpha \subset B_\delta(x) \). There exists a \( 1 \)-diffeomorphism \( M \) of \( B = B_\delta(x) \) onto \( S \subset E_n \) having modulus \( \leq C \) and there exists an \( (n-k) \)-plane \( H \) and a point \( a \in H \) such that \( S \subset T = T_{d,\delta}(H, a) \) and \( H \cap T \subset M(bdy G \cap \delta(x)) \). For any \( \gamma \in C_0^\infty(G) \) we have that \( A \gamma \) (defined by \( A \gamma(y) = \gamma(M^{-1}y), y \in S \)) vanishes near \( H \cap T \). Thus by the corollary of Lemma 5, Lemma 7 and the fact that \( d \leq K \delta \)

\[
\|\gamma\|_{0,r,B} \leq \|\gamma\|_{0,r,B} \leq \text{const} \|A \gamma\|_{0,r,S} \\
\leq \text{const} d^{1+n/r-n/p} \|A \gamma\|_{1,p,T} \\
\leq \text{const} \delta^{1+n/r-n/p} \|\gamma\|_{1,p,M^{-1}(T)} \\
\leq \text{const} \delta^{1+n/r-n/p} \|\gamma\|_{1,p,Q_\delta}
\]

where \( Q_\delta \) is the union of all the cubes \( Q_\alpha \) which intersect \( M^{-1}(T) \). Since the modulus of \( M \) is bounded, \( M^{-1} \) is Lipschitzian and there exists a constant \( \lambda \) such that \( M^{-1}(T) \subset B_{\lambda d}(x) \subset B_{\lambda \delta d}(x) \). Thus there is a constant \( N \) independent of \( R \) and \( x \)
such that any \( N + 1 \) of the sets \( Q' \) have empty intersection. Summing the above inequality over those \( \alpha \) for which \( Q_{\alpha} \) meets \( G_{\lambda} \) we obtain, as in Lemma 6, the required form of Poincaré's inequality.

5. **An application to differential operators.** Let \( L \) be a linear partial differential operator of order \( 2m \) in \( G \) given by

\[
Lu(x) = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha u(x)
\]

with coefficients \( a_\alpha \) infinitely differentiable, bounded, complex functions on \( G \). Suppose \( L \) is such that it satisfies the boundedness condition

\[
\left| \int_G L\varphi(x) \overline{\psi(x)} \, dx \right| \leq c_0 \| \varphi \|_{m,2,G} \| \psi \|_{m,2,G}
\]

for all \( \varphi, \psi \in C_0^\infty(G) \), and also Garding's inequality

\[
\text{Re} \int_G L\varphi(x) \overline{\psi(x)} \, dx \geq c_1 \| \varphi \|_{m,2,G}^2 - c_2 \| \psi \|_{0,2,G}^2
\]

for all \( \varphi \in C_0^\infty(G) \), where \( c_0, c_1 > 0 \) and \( c_2 \) are constants. The realization of \( L \) in \( L^2(G) \) corresponding to null Dirichlet boundary data is an operator \( T \) in \( L^2(G) \) defined by

\[
\text{Dom}(T) = W^{m,2}_0(G) \cap \{ f \in L^2(G) : Lf \in L^2(G) \}
\]

\[
Tf = Lf, \quad f \in \text{Dom}(T).
\]

**Theorem 6.** *If \( G \) is open in \( E_n \) and satisfies the conditions of either Theorem 3 or Theorem 5 with \( 2m > n + 2 - 2n|k| \) then \( T \) as defined above is a closed linear operator in \( L^2(G) \); the spectrum \( \sigma(T) \) is discrete and has no finite limit points; for \( \lambda \notin \sigma(T) \) the resolvent operator \( R_{\lambda}(T) = (\lambda I - T)^{-1} \) is completely continuous.*

The proof is identical to that of the standard theorem of this type. A sketch can be found in [6].

**References**


University of British Columbia, 
Vancouver, B.C., Canada