CORRECTIONS TO "ON SEQUENTIAL CONVERGENCE"

BY

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In my paper [2], Theorems 8.2 and 8.3 are false. I am indebted to Dennis Sentilles who sent me a counterexample to Theorem 8.2.

The wrong statements concern "LS-spaces." These are linear spaces $S$ on which there is a metric $\rho$ and a function $f$ such that $x_n \to x$ in $S$ if and only if both $\rho(x_n, x) \to 0$ and $f(x_n)$ remains bounded. Without restating the further conditions imposed on $\rho$ and $f$, we single out for the present two tractable subclasses of the class of $LS$-spaces:

(I) LF-spaces, i.e. strict inductive limits of Fréchet spaces ([1, [2, Theorem 7.7]).

(II) If $B$ is a separable Banach space, and $S$ is its dual $B^*$ with weak-star convergence of sequences, then $S$ is an LS-space [2, Theorem 7.1]. (Actually the same is true with "Fréchet" in place of "Banach": see [1, Théorème 5, p. 84].)

In case (I), Theorems 8.2 and 8.3 are true. The Banach-Steinhaus theorem for LF-spaces is well known [1, Théorème 2, p. 73]. But in case (II), if $B$ is infinite-dimensional, the elements of its unit ball give pointwise bounded continuous linear forms on $S$ which are not equicontinuous for the LS-topology. The proof of Theorem 8.2 in [2] errs in assuming that multiplication by a positive scalar is an open mapping in the relative topology of a (convex, symmetric, closed, metrizable) set.

The proof of Theorem 8.3 is invalid since it rests on Theorem 8.2. Disproving the statement of 8.3 requires some further work. Here is one counterexample.

Let $C$ be the space of continuous real functions on $[0, 1]$ with supremum norm. Then its dual $C^*$ is the space of finite signed measures on $[0, 1]$. $C^*$ with weak* sequential convergence is an LS-space. It has a countable dense set and is complete as an LS-space. The LS-topology $T$ on $C^*$ is the topology of uniform convergence on sequences $\{f_n\}$ in $C$ with $\|f_n\| \to 0$ [2, around Theorem 7.8]. By the Mackey-Arens theorem, the dual space of $(C^*, T)$ is $C$ (see also [1, Théorème 6, p. 85]).

A sequence in $C$ is weakly convergent if and only if it is uniformly bounded and converges pointwise. Thus the following fact contradicts Theorem 8.3:

**Proposition.** Bounded pointwise convergence in $C$ is not countably quasi-metric.

**Proof.** Suppose there is a metric $d$ and a countable set $\{G_n\}_{n=1}^\infty$ of functions on $C$ such that if $\|f_n\| \leq 1$, then $f_n \to 0$ pointwise if and only if both:

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623
(a) \( d(f_n, 0) \to 0 \) and
(b) for each \( m \), \( \sup_n G_m(f_n) < \infty \).

For some intervals \( I_m = [a_m, b_m] \), \( 0 = a_0 < a_1 < \cdots < a_m < \cdots < b_m < \cdots < b_1 < b_0 = 1 \), let \( \mathcal{C}_m = \{ f \in \mathcal{C} : \| f \| \leq 1, f(x) = 0 \) for all \( x \notin I_m \} \). We can define \( a_m \) and \( b_m \) inductively so that for each \( m \),
(c) \( \sup \{ G_m(f) : f \in \mathcal{C}_m \} < \infty \) and
(d) \( \sup \{ d(0, f) : f \in \mathcal{C}_m \} < 1/m \).

To do this, we can let \( a_{m+1} = (a_m + b_m)/2 \) and \( b_{m+1} = a_{m+1} + 1/r \) for \( r \) large enough (if \( g_r \in \mathcal{C}_r \), \( \| g_r \| \leq 1 \), and \( g_r = 0 \) outside \( [a_{m+1}, a_{m+1} + 1/r] \), then \( g_r \to 0 \) pointwise).

Now there is a \( c \) with \( a_m < c < b_m \) for all \( m \), and there are \( f_m \) in \( \mathcal{C}_m \) with \( f_m(c) = 1 \) for all \( m \). But (a) through (d) imply \( f_m(c) \to 0 \), a contradiction. Q.E.D.

Another correction: on p. 506 in [2], it is wrongly stated that topologies \( T(C) \) on distribution spaces \( \mathcal{D}' \) and \( \mathcal{S}' \) are not locally convex (see [3]).

REFERENCES