

MARKUSCHEVICH BASES AND DUALITY THEORY

BY

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Abstract. Several duality theorems concerning Schauder bases in locally convex spaces have analogues in the theory of Markushevich bases. For example, a locally convex space with a Markushevich basis is semireflexive iff the basis is shrinking and boundedly complete.

The strong existence Theorem III.1 for Markushevich bases allows us to show that a separable Banach space is isomorphic to a conjugate space iff it admits a boundedly complete Markushevich basis, and that a separable Banach space has the metric approximation property iff it admits a Markushevich basis which is a generalized summation basis in the sense of Kadec.

I. Introduction. In recent years a number of papers have discussed applications of Schauder bases to the duality theory of locally convex spaces. (For example, see [2], [5], [9], [10], and [11].) However, the lack of a good existence theorem for Schauder bases severely limits the applicability of these results. In this paper we discuss a generalization of Schauder bases (called Markushevich bases or M -bases) for which there are good existence theorems. In fact, Markushevich [8] showed that every separable Banach space admits a M -basis and Theorem III.1 gives a better existence theorem for general linear topological spaces.

In §II we introduce the concepts of shrinking and boundedly complete Markushevich bases. The main results of this section are that a locally convex space with a M -basis is semireflexive iff the M -basis is shrinking and boundedly complete (Theorem II.6) and that a Banach space which admits a boundedly complete M -basis is canonically isomorphic to the adjoint of the coefficient space of the M -basis (Theorem II.5). Of course, these theorems have analogues in Schauder basis theory (see [5], [9], and [1]).

Theorem III.1 shows that every strongly separable, strongly closed, total subspace of the adjoint of a separable linear topological space X is the coefficient space of some countable M -basis for X . (This result is perhaps implicit in the results of [4], but we include a proof for completeness.) This theorem has several interesting applications. For example, a separable locally convex space has a strongly separable adjoint iff the space admits a countable shrinking M -basis (Corollary III.3). A separable Banach space is isomorphic to a conjugate Banach space iff the space admits a boundedly complete M -basis (Theorem III.4).

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In §IV we discuss generalized summation bases, a type of Markushevich bases introduced by Kadec [6]. We use Theorem III.1 to show that a separable Banach space admits a generalized summation basis iff the space has the metric approximation property.

We use the notation and terminology of [7]. X always represents a Hausdorff linear topological space and X^* represents the set of continuous linear functionals on X . We assume that X^* is endowed with the topology of uniform convergence on $w(X, X^*)$ bounded subsets of X , and call this topology on X^* the strong topology. When X^* is total over X , the natural embedding of X into X^{**} is denoted by " $\hat{}$ ".

If f is a function on a set Z and Y is a subset of Z , $f|_Y$ denotes the restriction of f to Y . The linear span of a subset Y of a linear space is denoted by $\text{sp}(Y)$. The domain, range, and null space of a linear operator T are denoted, respectively, by $D(T)$, $R(T)$, and $\ker(T)$.

II. Applications of M -bases to duality theory. Let (X, T) be a linear topological space. Recall that a biorthogonal collection $\{x_i, f_i\}_{i \in I}$ in (X, X^*) is a Markushevich basis (M -basis) for X iff $\{x_i\}_{i \in I}$ is fundamental in (X, T) and $\{f_i\}_{i \in I}$ is total over X . The strong closure of $\text{sp}(\{f_i\}_{i \in I})$ in X^* is called the coefficient space of the M -basis $\{x_i, f_i\}$.

DEFINITION II.1. Let $\{x_i, f_i\}_{i \in I}$ be a M -basis for X . $\{x_i, f_i\}$ is said to be shrinking iff $\{f_i\}_{i \in I}$ is strongly fundamental in X^* . (Equivalently, $\{x_i, f_i\}$ is shrinking iff $\{f_i, \hat{x}_i\}$ is a M -basis for X^* when X^* is endowed with the strong topology.)

DEFINITION II.2. Let $\{x_i, f_i\}_{i \in I}$ be a M -basis for X . $\{x_i, f_i\}$ is said to be boundedly complete iff whenever $\{Y_d\}$ is a bounded net in X such that for each i in I , $\lim_d f_i(Y_d)$ exists, there is x in X such that for each $i \in I$, $f_i(x) = \lim_d f_i(Y_d)$.

Let $\{x_i, f_i\}_{i=1}^\infty$ be a Schauder basis for a locally convex space X . If $\{x_i, f_i\}$ is shrinking as a Schauder basis, then obviously it is shrinking as a M -basis. It is also easy to see that if $\{x_i, f_i\}$ is boundedly complete as a M -basis, then it is boundedly complete as a Schauder basis. The converses of these statements are true for uniformly bounded Schauder bases.

THEOREM II.3. Let $\{x_i, f_i\}_{i=1}^\infty$ be a uniformly bounded Schauder basis for a locally convex space X . (1) If $\{x_i, f_i\}$ is boundedly complete as a Schauder basis, then it is boundedly complete as a M -basis. (2) If $\{x_i, f_i\}$ is shrinking as a M -basis, then it is shrinking as a Schauder basis.

Proof. Let $\{S_n\}_{n=1}^\infty$ be the partial sum operators associated with $\{x_i, f_i\}$. (That is, $S_n(x) = \sum_{i=1}^n f_i(x)x_i$.) We are assuming that $\{S_n\}_{n=1}^\infty$ is uniformly bounded.

To prove (1), we let $\{Y_d : d \in D\}$ be as in Definition II.2. For each n , $\{S_n(Y_d) : d \in D\}$ is a Cauchy net in the finite dimensional Hausdorff space $(R(S_n), w(R(S_n), \{f_i\}_{i=1}^n))$, hence $\lim_d S_n(Y_d)$ exists. Since $\{S_n\}$ is uniformly bounded and $\{Y_d\}$ is bounded, $\{\lim_d S_n(Y_d)\}_{n=1}^\infty$ is bounded. Since $\{x_i, f_i\}$ is boundedly complete as a Schauder basis, $\{\lim_d S_n(Y_d)\}_{n=1}^\infty$ must converge to, say, x . Clearly $f_i(x)$

$=\lim_d f_i(Y_d)$, for all $i=1, 2, 3, \dots$, so $\{x_i, f_i\}$ is boundedly complete as a M -basis.

To prove (2), we note that the uniform boundedness of $\{S_n\}$ implies that $\{S_n^*\}_{n=1}^\infty$ is equicontinuous on X^* . Hence $A=\{x : \lim_n S_n^*(x)=x\}$ is closed in X^* . Since A obviously contains $\text{sp}(\{f_i\})$ and $\{f_i\}$ is fundamental in X^* , A must equal X^* . Then $\{f_i, \hat{x}_i\}_{i=1}^\infty$ is a Schauder basis for X^* , which is to say that $\{x_i, f_i\}$ is a shrinking Schauder basis for X .

The hypothesis in Theorem II.3 that $\{x_i, f_i\}_{i=1}^\infty$ is a uniformly bounded Schauder basis is necessary. Indeed, let Y be any separable, infinite dimensional Banach space and let $\{x_i, f_i\}_{i=1}^\infty$ be any M -basis for Y which is not a Schauder basis for Y . Let $X=\{x \in Y : \{\sum_{i=1}^n f_i(x)x_i\}_{n=1}^\infty \text{ converges weakly to } x\}$. Let X be endowed with the $w(X, Y^*)$ topology. It is easy to see that $\{x_i, f_i\}_{i=1}^\infty$ is boundedly complete as a Schauder basis for X . $\{x_i, f_i\}_{i=1}^\infty$ is not boundedly complete as a M -basis for X because X is a proper subspace of Y . Also, $\{x_i, f_i\}_{i=1}^\infty$ is obviously a shrinking M -basis for X . $\{x_i, f_i\}_{i=1}^\infty$ is not a shrinking Schauder basis for X because $\{f_i\}_{i=1}^\infty$ is not even a basic sequence in $Y^*(=X^*)$ and $s(Y^*, Y)=s(Y^*, X)$.

Results from Schauder basis theory suggest that "shrinking" and "boundedly complete" should be dual concepts. Under certain circumstances, this is the case.

THEOREM II.4. *Let $\{x_i, f_i\}_{i \in I}$ be a shrinking M -basis for a locally convex evaluable space X . Then $\{f_i, \hat{x}_i\}_{i \in I}$ is a boundedly complete M -basis for X^* .*

Proof. Let $\{Y_d\}$ be a bounded net in X^* such that for each i in I , $\lim_d \hat{x}_i(Y_d)$ exists. Note that $\{Y_d\}$ is equicontinuous on X because X is evaluable, hence $\{Y_d\}$ has a weak* cluster point, say, y . Clearly $\hat{x}_i(y)=\lim_d \hat{x}_i(Y_d)$, for all i in I , so $\{f_i, \hat{x}_i\}$ is boundedly complete.

THEOREM II.5. *Let $\{x_i, f_i\}_{i \in I}$ be a boundedly complete M -basis for a Banach space X . Let Y be the coefficient space of the basis $\{x_i, f_i\}$. Then the canonical embedding of X into Y^* is an isomorphism of X onto Y^* . Hence $\{f_i, \hat{x}_i|_Y\}_{i \in I}$ is a shrinking M -basis for Y .*

Proof. Note that the canonical embedding of X into Y^* is one-to-one, because Y is total over X . It is norm decreasing, hence continuous. We show that it is onto Y^* (and hence an isomorphism by the open mapping theorem).

Let G be in Y^* . Let D be the collection of finite subsets of I , and direct D by inclusion. By Helly's theorem (cf., e.g., [12, p. 103]), for each d in D there is Y_d in X such that for each i in d , $f_i(Y_d)=G(f_i)$, and $\|Y_d\| \leq \|G\| + 1$. Since $\{x_i, f_i\}$ is boundedly complete, there is x in X such that for each i in I , $\lim_d f_i(Y_d)=f_i(x)$. Clearly $f_i(x)=G(f_i)$, for all i in I . Since $\{f_i\}$ is fundamental in Y and both \hat{x} and G are continuous on Y , $f(x)=G(f)$, for all f in Y . Thus the canonical embedding of X into Y^* is onto.

Since $\{x_i\}$ is fundamental in X , $\{\hat{x}_i|_Y\}$ is fundamental in Y^* , and thus $\{f_i, \hat{x}_i|_Y\}$ is shrinking. This completes the proof.

Historically, a major reason for considering shrinking and boundedly complete

Schauder bases was to characterize reflexivity (see [5] and [9]). Similarly, semi-reflexivity is characterized by the existence of a boundedly complete, shrinking M -basis.

THEOREM II.6. *Let $\{x_i, f_i\}_{i \in I}$ be a M -basis for a locally convex space X . X is semireflexive iff $\{x_i, f_i\}$ is both shrinking and boundedly complete.*

Proof. Suppose first that X is semireflexive. $\{f_i\}$ is total over X , hence is weak* fundamental in X^* , hence is weakly fundamental in X^* , hence is fundamental in X^* . Thus $\{x_i, f_i\}$ is shrinking. Now let $\{Y_\alpha\}$ be a bounded net as in Definition II.2. Since X is semireflexive, $\{Y_\alpha\}$ has a weak cluster point, say, x . Clearly $f_i(x) = \lim_\alpha f_i(Y_\alpha)$, for all i in I , so that $\{x_i, f_i\}$ is boundedly complete.

To go the other way, suppose that $\{x_i, f_i\}$ is shrinking and boundedly complete. To show that X is semireflexive, it is sufficient to show that bounded, weakly Cauchy nets in X are weakly convergent. Let $\{Y_\alpha\}$ be a bounded, weakly Cauchy net in X . Clearly $\lim_\alpha f_i(Y_\alpha)$ exists for each i in I , hence by the boundedly complete assumption there is x in X such that for all i in I , $f_i(x) = \lim_\alpha f_i(Y_\alpha)$. We need to show that $\{Y_\alpha\}$ weakly converges to x . Now $\{Y_\alpha\}$ is bounded, so $\{\hat{Y}_\alpha\}$ is equicontinuous on X^* . $\{\hat{Y}_\alpha\}$ converges to \hat{x} pointwise on the subset $\{f_i\}$ of X^* . By the shrinking assumption, $\{f_i\}$ is fundamental in X^* , hence $\{\hat{Y}_\alpha\}$ converges to \hat{x} pointwise on X^* . That is, $\{Y_\alpha\}$ converges weakly to x . This completes the proof.

III. Existence theorem for countable Markushevich bases. Unfortunately, there is no general existence theorem for Schauder bases. In contrast to this situation, Theorem III.1 provides a very fine existence theorem for M -bases.

THEOREM III.1. *Let X be separable and let Y be a closed, separable, total subspace of X^* . Then X admits a M -basis $\{y_i, g_i\}_{i=1}^\infty$ whose coefficient space is Y .*

Proof. Let $\{x_i\}_{i=1}^\infty$ be a fundamental subset of X and let $\{f_i\}_{i=1}^\infty$ be a fundamental subset of Y . Note that $\{f_i\}_{i=1}^\infty$ is total over X . Hence we can assume, without loss of generality, that $f_1(x_1) \neq 0$.

Let $y_1 = x_1, g_1 = f_1/f_1(x_1), k(1) = 1$.

Suppose that $\{y_i, g_i\}_{i=1}^{k(n)}$ have been defined so that

- (1) $\{y_i, g_i\}_{i=1}^{k(n)}$ is biorthogonal;
- (2) $\text{sp}(\{y_i\}_{i=1}^{k(n)}) \supset \text{sp}(\{x_i\}_{i=1}^n)$;
- (3) $Y \supset \text{sp}(\{g_i\}_{i=1}^{k(n)}) \supset \text{sp}(\{f_i\}_{i=1}^n)$.

A. If x_{n+1} is in $\text{sp}(\{y_i\}_{i=1}^{k(n)})$, we let $k = k(n)$ and proceed to B. If x_{n+1} is not in $\text{sp}(\{y_i\}_{i=1}^{k(n)})$, we let $k = k(n) + 1$, and let $y_k = x_{n+1} - \sum_{i=1}^{k(n)} g_i(x_{n+1})y_i$. (1) implies that $g_i(y_k) = 0$, for each $i \leq k(n)$. By using the Hahn-Banach theorem in the space $(X, w(X, Y))$, we can find g_k in Y such that $g_k(y_k) = 1$ and $g_k(y_i) = 0$ for each $i \leq k(n)$. Note that $\{y_i, g_i\}_{i=1}^k$ is biorthogonal, $\text{sp}(\{y_i\}_{i=1}^k) \supset \text{sp}(\{x_i\}_{i=1}^{n+1})$, and $Y \supset \text{sp}(\{g_i\}_{i=1}^k)$. Now proceed to B.

B. If f_{n+1} is in $\text{sp}(\{g_i\}_{i=1}^k)$, let $k(n+1) = k$. If f_{n+1} is not in $\text{sp}(\{g_i\}_{i=1}^k)$, let $k(n+1)$

$=k+1$, and let $g_{k(n+1)} = f_{n+1} - \sum_{i=1}^k f_{n+1}(y_i)g_i$. By using the Hahn-Banach theorem in the space $(Y, w(Y, X))$, we can find $y_{k(n+1)}$ in X such that $g_{k(n+1)}(y_{k(n+1)}) = 1$ and $g_i(y_{k(n+1)}) = 0$ for all $i \leq k$.

We have thus extended the sequence $\{y_i, g_i\}_{i=1}^{k(n)}$ to a sequence $\{y_i, g_i\}_{i=1}^{k(n+1)}$ such that $\{y_i, g_i\}_{i=1}^{k(n+1)}$ satisfies (1), (2), and (3) if $n+1$ is substituted for n . The sequence $\{y_i, g_i\}_{i=1}^{\infty}$ obviously satisfies the conclusion of the theorem.

REMARK III.2. Theorem III.1 shows that every separable linear topological space which admits a countable total family of continuous linear functionals must also admit a countable M -basis. Unfortunately, there are separable locally convex spaces which do not admit a countable total family of continuous linear functionals. (An example is the product of \aleph_1 copies of the real line with the product topology.) Of course, such spaces do not admit countable M -bases.

One instance of Theorem III.1 is of particular interest:

COROLLARY III.3. *Suppose that X is separable and X^* is total over X . Then X admits a countable shrinking M -basis iff X^* is strongly separable.*

THEOREM III.4. *Let X be a separable Banach space. X is isomorphic to a conjugate Banach space iff X admits a boundedly complete M -basis.*

Proof. If X is isomorphic to Y^* , then Y is separable, so that the conclusion follows from Corollary III.3 and Theorem II.4. Conversely, if X admits a boundedly complete M -basis, the conclusion follows from Theorem II.5.

Let X be a Banach space. Recall that a subspace Y of X^* is norming iff there is $k > 0$ such that for all x in X , $\|x\| \leq k \sup \{|f(x)| : f \in Y, \|f\| \leq 1\}$. Equivalently, Y is norming iff the canonical mapping of X into Y^* is an isomorphism of X into Y^* . Note that Theorem II.5 shows that the coefficient space of a boundedly complete M -basis is norming. However, there are M -bases whose coefficient spaces are not norming. This follows from Theorem III.1 and the well-known fact that there are closed total subspaces of c_0^* which are not norming. On the other hand, if X is separable it is easy to see that X^* contains a separable norming subspace. It thus follows from Theorem III.1 that every separable Banach space admits a M -basis whose coefficient space is norming.

For many applications, it is desirable to have a stronger existence theorem for M -bases than Theorem III.1. In particular, which nonseparable Banach spaces admit M -bases? (Dyer [3] has noted that for T uncountable, $m(T)$ has no M -bases.) Let X be a separable Banach space. Is there a M -basis $\{x_i, f_i\}$ for X such that $\{x_i\}$ is bounded in X , $\{f_i\}$ is bounded in X^* , and the coefficient space of $\{x_i, f_i\}$ is norming?

IV. Generalized summation bases in spaces with the metric approximation property.

In this section we assume that X is a separable Banach space, $\{x_i, f_i\}_{i=1}^{\infty}$ is a M -basis for X , and $\{S_n\}_{n=1}^{\infty}$ is the sequence of operators defined by $S_n(x) = \sum_{i=1}^n f_i(x)x_i$. I is the identity operator on X .

Following Kadec [6] we say that $\{x_i, f_i\}$ is a generalized summation basis (g.s.b.) for X iff there is a sequence $\{T_n\}_{n=1}^\infty$ of linear operators with $R(T_n) \subset D(T_n) = \text{sp}(\{x_i\}_{i=1}^n)$ such that the sequence $\{T_n S_n\}_{n=1}^\infty$ of linear operators on X converges pointwise to I . Kadec pointed out that not every countable M -basis is a g.s.b. Indeed, this follows from the comments at the end of §III and the easily verified fact that the coefficient space of a g.s.b. is norming (see [6] and [4]).

It is not known whether every separable Banach space admits a g.s.b. Note that the existence of a g.s.b. for X implies that X has the metric approximation property—i.e., that there is a sequence of continuous linear operators of finite range (but not necessarily of norm 1) on X which converges pointwise to I . In fact, Theorem IV.1 shows that the metric approximation property is equivalent to the existence of a g.s.b.

THEOREM IV.1. *Let X be a separable Banach space which has the metric approximation property. Then X admits a generalized summation basis.*

Proof. Let $\{L_n\}_{n=1}^\infty$ be a sequence of linear operators of finite range on X which converges pointwise to I . Let $\{x_i, f_i\}_{i=1}^\infty$ be any M -basis for X such that the coefficient space of $\{x_i, f_i\}$ contains $\bigcup_{n=1}^\infty R(L_n^*)$. (By Theorem III.1, such a M -basis exists.)

Write $L_1(x) = \sum_{i=1}^p g_i(x)y_i$, where $\{y_i\}_{i=1}^p \subset R(L_1)$ and $\{g_i\}_{i=1}^p \subset R(L_1^*)$. If $\epsilon > 0$, there is a positive integer $n(1)$ such that for each $i \leq p$, there are \bar{x}_i in $\text{sp}(\{x_i\}_{i=1}^{n(1)})$ and \bar{f}_i in $\text{sp}(\{f_i\}_{i=1}^{n(1)})$ such that $\|\bar{x}_i - y_i\| < \epsilon$ and $\|\bar{f}_i - g_i\| < \epsilon$.

Let $T_{n(1)}(x) = \sum_{i=1}^p \bar{f}_i(x)\bar{x}_i$. Note that

$$\begin{aligned} \|T_{n(1)}(x) - L_1(x)\| &= \left\| \sum_{i=1}^p (\bar{f}_i(x) - g_i(x))\bar{x}_i + \sum_{i=1}^p g_i(x)(\bar{x}_i - y_i) \right\| \\ &\leq \left[\sum_{i=1}^p \|\bar{f}_i - g_i\| \|\bar{x}_i\| + \sum_{i=1}^p \|g_i\| \|\bar{x}_i - y_i\| \right] \|x\|. \end{aligned}$$

Thus if ϵ is sufficiently small, we can assure that $\|T_{n(1)} - L_1\| < 1$.

Note that $T_{n(1)} = T_{n(1)}S_{n(1)}$ and $R(T_{n(1)}) \subset \text{sp}(\{x_i\}_{i=1}^{n(1)})$.

A simple extension of this argument shows that there is an increasing sequence $\{n(i)\}_{i=1}^\infty$ of positive integers and a sequence $\{T_{n(i)}\}_{i=1}^\infty$ of linear operators on X such that for each i ,

$$\|T_{n(i)} - L_i\| < 1/i, \quad R(T_{n(i)}) \subset \text{sp}(\{x_j\}_{j=1}^{n(i)}), \quad \text{and} \quad T_{n(i)} = T_{n(i)}S_{n(i)}.$$

$\{T_{n(i)}\}_{i=1}^\infty$ converges pointwise to I because $\{\|T_{n(i)} - L_i\|\}_{i=1}^\infty$ converges to 0 and $\{L_i\}_{i=1}^\infty$ converges pointwise to I .

We complete the sequence $\{T_j\}_{j=1}^\infty$ by defining

$$\begin{aligned} T_j &= S_j, & \text{if } j < n(1), \\ &= T_{n(i)}, & \text{if } n(i) \leq j < n(i+1). \end{aligned}$$

Note that for each j , $T_j = T_j S_j$. We have seen that $\{T_j\}_{j=1}^\infty$ converges pointwise to I ,

so $\{T_j S_j\}_{j=1}^{\infty}$ converges pointwise to I . Clearly $R(T_j) \subset \text{sp}(\{x_i\}_{i=1}^j)$, so we have shown that $\{x_i, f_i\}_{i=1}^{\infty}$ is a g.s.b. for X .

Theorem II.6 and the proof of Theorem IV.1 yield a (slight) strengthening of Theorem 4 of [6].

COROLLARY IV.2. *Let X be a separable reflexive Banach space which has the metric approximation property. Then every M -basis for X is a g.s.b.*

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