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THE PSEUDO-CIRCLE IS UNIQUE(1)

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1. **Introduction.** A well-known and hitherto unsolved problem concerning circularly chainable continua is the question raised by R. H. Bing in 1951 in [3, p. 49] of whether or not each two planar non-snake-like hereditarily indecomposable circularly chainable continua, described in [3, p. 48], are homeomorphic. Such continua have subsequently been referred to in the literature as "pseudo-circles" and have been discussed by F. B. Jones at the Summer Institute on Set Theoretic Topology, University of Wisconsin, 1955, [7]. The purpose of this paper is to give an affirmative answer to this question of whether or not the pseudo-circle is topologically unique.

2. **Preliminaries.** Throughout this paper we shall use the terms and results developed by this author in [4], [5] and [6]. In general these terms and notations were suggested by those used by Bing in [2] and [3]. We shall also define a number of additional special terms related to the pseudo-circle and particular types of refinements.

A *pseudo-circle* is defined to be the intersection M of the sets of points of a sequence $Q_1, Q_2, Q_3, \ldots$ of circular chains such that, for each positive integer i,

(a) the diameter of each link of Q_i is less than $1/i$,

(b) Q_{i+1} is a refinement of Q_i and the cyclic r-pattern f_i of Q_{i+1} in Q_i has winding number 1,

(c) the closure of each link of Q_{i+1} is a subset of the link of Q_i to which it corresponds under f_i, and

(d) Q_{i+1} is crooked in Q_i.

The sequence of cirular chains $Q_1, Q_2, Q_3, \ldots$ will be referred to as a *defining sequence of circular chains associated with M.*

Let $P=P(a, b)$ and $Q=Q(c, d)$ be p-chains and let f be an r-pattern of P in Q. Then f will be defined to be a *left-normal r-pattern* if $f(a)=c$. If $f(a)=c$ and no link of P other than the link with subscript a corresponds to the link of Q with subscript c, then f is said to be *properly left-normal.* The terms "*right-normal*" and "*properly right-normal*" are then defined in a similar manner. If an r-pattern

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f is both left-normal and right-normal then f is said to be a *normal r-pattern.* In the case that f is an r-pattern which is both properly left-normal and properly right-normal, then f is defined to be a *properly normal r-pattern.*

Each of the preceding variations of the concept of "normal refinement" will be important in the discussion of r-patterns associated with refinements between circular chains. It will also be convenient to refer to refinements and their associated patterns by the same descriptive terms. Thus, an r-pattern f will be said to be *normal* or *crooked* according as f is a pattern of a p-chain P in a p-chain Q under which P is a normal refinement of Q or P is crooked in Q. Similar remarks apply to cyclic r-patterns.

Now, let $P=P(a, b)$ and $Q=Q(c, d)$ be p-chains and assume also that $a<b$ and $c<d$. If f is a normal r-pattern of P in Q and there are integers h and k, $a<h<k<b$, such that the restrictions of f to each of the sets $\{a, a+1, \ldots, h\}$, $\{h, h+1, \ldots, k\}$ and $\{k, k+1, \ldots, b\}$ is monotone (alternately nondecreasing and nonincreasing), and $f(h)>f(k)$, then f will be defined to be an *elementary bend.* In this situation P will be said to have an *elementary bend in* Q. It will sometimes be convenient to use the symbol $(c; f(h); f(k); d)$ to indicate this elementary bend of P in Q.

If a p-chain $P=P(a, b)$, $a<b$, is a normal refinement with r-pattern f in a p-chain $Q=Q(c, d)$, $c<d$, and there are integers $h_1, h_2, \ldots, h_n$ such that $a<h_1<h_2<\cdots<h_n<b$, and the restrictions of f to each of the sets $\{a, a+1, \ldots, h_1\}$, $\{h_1, h_1+1, \ldots, h_2\}, \ldots, \{h_n, h_n+1, \ldots, b\}$ are monotone and $c<f(h_1)$, $f(h_1)>f(h_2)$, $f(h_2)<f(h_3), \ldots, f(h_n)<d$, then f will be defined to be a *composite bend.* In this situation P will be said to have a *composite bend* in Q and the notation $(c; f(h_1); f(h_2); \ldots; f(h_n); d)$ will be used to indicate this relationship. It is observed that if $h_1, h_2, \ldots, h_n$ is a finite sequence of integers associated with a composite bend as in the definition of "composite bend" then n is an even integer.

Next, let f be a cyclic r-pattern with domain $\{0, 1, \ldots, n\}$ and range $\{0, 1, \ldots, m\}$, and let g be an r-pattern such that the domain of g is the set of all integers and, for each integer i,

$$|g(i)-f(j)| \bmod (m+1) = 0$$

whenever j is the least nonnegative integer such that

$$|i-j| \bmod (n+1) = 0.$$

Then g is defined to be the *universal linear representation* of f. The restriction of g to any finite collection of at least $n+1$ consecutive integers is defined to be a *linear representation* of f. In the case that $f(0)=0$, the restriction of g to the set $\{0, 1, \ldots, n\}$ is called the *canonical linear representation* of f. It is observed that the definitions of the last two terms are consistent with those given by this author in [5] and [6]. The new term "universal linear representation" is suggested by the similarity between the concept of linear representation and the concept of covering space. It will sometimes be convenient to indicate the universal linear representation of a

cyclic r-pattern f and the canonical linear representation of a cyclic r-pattern f by the notations u.l.r. $[f]$ and c.l.r. $[f]$, respectively.

The *length* of a p-chain P will be defined to be the number of links of P. If a p-chain P is a refinement of a p-chain Q, then the minimum of the lengths of the maximal sub-p-chains of P that are refinements of single-link sub-p-chains of Q will be defined to be the *rank* of P in Q. The *order* of a p-chain P in a p-chain Q is defined by replacing the word "minimum" by the word "maximum" in the preceding statement.

If $P=P(a, b)$ is a p-chain and x is an integer such that $P(a, x)$ is a sub-p-chain of P, then $P(a, x)$ is defined to be a *first section* of P. If $P=P(a, b)$ is a p-chain and x is an integer such that $P(x, b)$ is a sub-p-chain of P, then $P(x, b)$ is defined to be a *last section* of P.

The *length of a pattern* f will be defined to be the number of distinct elements in the range of f and will be denoted by $L(f)$.

All sets considered in this paper will be assumed to have the separable metric topology. It will also be assumed that all circular chains and circular p-chains considered in this paper have at least five links. This will allow us to avoid the formal awkwardness of considering situations where the cyclic structures or crookedness properties are trivial. Finally, we shall assume that the closures of nonadjacent links of a chain or circular chain do not intersect. It will be seen that these assumptions can be made without any loss in generality in the results of this paper.

3. **Crooked r-patterns.** In this section we shall establish relationships between elementary bends and crooked r-patterns which will be used subsequently in developing a combinatorial system for elementary bends.

THEOREM 3.1. *Let $P=P(0, n)$ and $Q=Q(0, m)$ be p-chains such that P is a principal normal refinement of Q with r-pattern f in Q, and P has an elementary bend $(0; f(h); f(k); m)$ in Q with the properties that $0<f(k)$ and $f(h)<m$. In addition, let $T=T(0, s)$ be a p-chain such that T is a normal refinement of Q, T is crooked in Q, and the rank of T in Q is not less than the order of P in Q. Then T is a normal refinement of P.*

Proof. Let g denote the r-pattern of T in Q. We choose d to be the least integer such that $f(d)=f(h)+1$, and choose v_1 to be the least integer such that $g(v_1)=f(h)+1$. Next we choose c to be the greatest integer such that $f(c)=f(k)-1$, and choose u_1 to be the greatest integer such that $u_1<v_1$ and $g(u_1)=f(k)-1$. Then, noting that the rank of T in Q is greater than or equal to the order of P in Q, there is an r-pattern r_1 of $T(0, u_1)$ in $P(0, h)$ such that $r_1(u_1)=c$. In addition, since T is crooked in Q and $(g(v_1)-g(u_1))>2$, the sub-p-chain $T(u_1, v_1)$ of T contains sub-p-chains $T(u_1, a_1)$, $T(a_1, b_1)$ and $T(b_1, v_1)$ such that $u_1<a_1<b_1<v_1$ and $T(u_1, a_1)$ is a normal refinement of $Q(f(k)-1, f(h))$, $T(a_1, b_1)$ is a normal refinement of $Q(f(h), f(k))$ and $T(b_1, v_1)$ is a normal refinement of $Q(f(k), f(h)+1)$. Hence, again noting that the rank of T in Q is greater than or equal to the order of P in Q, there is an

r-pattern r_2 of $T(u_1, v_1)$ in $P(c, d)$ such that $r_2(u_1)=c$ and $r_2(v_1)=d$. Therefore, $T(0, v_1)$ is a normal refinement of $P(0, d)$.

Now, if $g(i) \geqq f(k)$ for $v_1 \leqq i \leqq s$, we may choose an r-pattern r_3 of $T(v_1, s)$ in $P(k, n)$ such that $r_3(v_1)=d$ and $r_3(s)=n$. In this first case the proof is complete. Otherwise, there is a first integer v_2 greater than v_1 such that $g(v_2)=f(k)-1$. Let u_2 be the greatest integer such that $u_2<v_2$ and $g(u_2)=f(h)+1$. In this second case, it follows by an argument similar to that given in the preceding paragraph that there exists an r-pattern r_3 of $T(v_1, u_2)$ in a sub-p-chain of $P(k, n)$ such that $r_3(v_1)=r_3(u_2)=d$, and there exists an r-pattern r_4 of $T(u_2, v_2)$ in $P(d, c)$ such that $r_4(u_2)=d$ and $r_4(v_2)=c$. Next we choose v_3 to be the first integer greater than v_2 such that $g(v_3)=f(h)+1$ and we choose u_3 to be the greatest integer such that $u_3<v_2$ and $g(u_3)=f(k)-1$. The situation is again similar to that considered in the first paragraph of the proof and we conclude that there exist r-patterns r_5 and r_6 of $T(v_2, u_3)$ in a sub-p-chain of $P(0, h)$ and $T(u_3, v_3)$ in $P(c, d)$, respectively, such that $r_5(v_2)=r_5(u_3)=f(k)-1$, $r_6(u_3)=f(k)-1$ and $r_6(v_3)=d$.

Proceeding in this manner, in a finite number of steps we obtain a situation corresponding to the first case described in the preceding paragraph and the proof of the theorem is complete.

The theorem which follows shows that the restrictions "$0<f(k)$" and "$f(h)<m$" of Theorem 3.1 can be removed in the case that there exists an intermediate p-chain R of a certain type such that T is a refinement of R and R is a refinement of Q. The resulting theorem will be used directly in the next section.

THEOREM 3.2. *Let P, Q, R and T be p-chains such that P is a principal normal refinement of Q having an elementary bend in Q, R is a normal refinement of Q such that the rank of R in Q is at least two units greater than the order of P in Q, and T is a normal refinement of R such that T is crooked in R. Then T is a normal refinement of P.*

Proof. Let $P=P(0, n)$, $Q=Q(0, m)$ and $R=R(0, w)$, and let f and g be the r-patterns of P in Q and R in Q, respectively. In addition, let $(0; f(h); f(k); m)$ denote the elementary bend of P in Q and let t denote the order of P in Q. We shall assume, noting Theorem 3.1, that $f(k)=0$. Then, since the rank of R in Q is greater than t we may choose a first integer u such that each of the integers $g(u-t)$, $g(u-t+1), \ldots, g(u)$ is equal to $f(h)$. Similarly, we may choose a greatest integer v less than u such that each of the integers $g(v)$, $g(v+1), \ldots, g(v+t)$ is equal to $f(k)$.

Now, the rank of R in Q exceeds t by at least 2. Thus $g(u+1)=f(h)$ and $g(v-1)=f(k)$ so that $u<w$ and $v>0$. In addition, since $R(0, u)$ is a normal refinement of $Q(0, f(h))$ and the rank of $R(0, u)$ in $Q(0, f(h))$ is at least as great as the order of $P(0, h)$ in $Q(0, f(h))$, it follows that $R(0, u)$ is a normal refinement of $P(0, h)$. Similarly, $R(u-1, v)$ is a normal refinement of $P(h, k)$ and, since $f(k)=0$, $R(v+1, w)$ is a refinement of P such that the first and last links of $R(v+1, w)$ correspond to

the links of P with subscripts k and n, respectively. Then let S denote the p-chain sum

$$R(0, u)+R(u-1, v)+R(v+1, w)$$

and observe that S is a normal refinement of P and S is a principal normal refinement of R such that the order of S in R is equal to 1. Furthermore, if r denotes the r-pattern of S in R, there are integers i and j such that $0<r(j)$, $r(i)<w$ and S has the elementary bend $(0; f(i); r(j); w)$ in R. It follows, by Theorem 3.1, that T is a normal refinement of S. Therefore, from the transitivity of the relationship "normal refinement", T is a normal refinement of P.

4. **Uniformity of sequences of crooked r-patterns.** An important theorem in the paper of Bing on the pseudo-arc is [2, Theorem 6]. This theorem was re-proved by Lehner [8] who also made fundamental use of the result in [8]. In the terms used in the present paper this result can be stated in the following equivalent manner.

Let $Q=Q(0, m)$ be a chain, let f be an r-pattern with range $\{0, 1, \ldots, m\}$, and let $Q_1, Q_2, Q_3, \ldots$ be a sequence of chains such that

(1) $Q=Q_1$

and, for each positive integer i,

(2) the diameter of each link of Q_i is less than $1/i$,

(3) Q_{i+1} is a normal refinement of Q_i and the rank of Q_{i+1} in Q_i is at least equal to 2,

(4) Q_{i+1} is crooked in Q_i, and

(5) the closure of each link of Q_{i+1} is a subset of the link of Q_i to which it corresponds under the r-pattern of Q_{i+1} in Q_i. Then there is an integer k such that some consolidation of Q_k has the pattern f in Q.

In this investigation of the pseudo-circle we shall need a similar but stronger theorem. Specifically it will be proved that the integer k of the above theorem of Bing can be chosen uniformly in the sense that k depends only on f and not on the particular sequence $Q_1, Q_2, Q_3, \ldots$. A sequence of chains satisfying conditions (1)–(5), above, will be referred to as a *c-regular sequence of chains relative to f.*

THEOREM 4.1. *If Q is a chain and P is a p-chain which is a principal normal refinement of Q with r-pattern f in Q, then there exists an integer k such that, for any c-regular sequence of chains $Q_1, Q_2, Q_3, \ldots$ relative to f, the chain Q_k is a normal refinement of P.*

Proof. Let $P=P(0, n)$ and let $Q=Q(0, m)$. In addition, let t be the order of P in Q, and let $h_1, h_2, \ldots, h_w$ be integers such that $0<h_1<h_2<\cdots<h_w<n$, $0\leqq w\leqq m$, and $(0; f(h_1); f(h_2); \ldots; f(h_w); m)$ is a representation of the composite bend of P in Q.

We shall prove the theorem by induction on w. Specifically it will be shown that k can always be chosen to be equal to $(w+1)(t+3)$.

In the case that $w=0$, the r-pattern f is monotone and, noting condition (3) of

the definition of "c-regular sequence of chains relative to f", we need only choose $d=t+3$ to obtain the desired refinement Q_k of P. Next assume that the theorem has been established for all values of w less than or equal to some integer u, $0 \leqq u$. We note, from the definition of "composite bend", that u is an even integer and proceed to establish the induction step for the case that $w=u+2$.

First, since P is a normal refinement of Q, there are integers i and j, $i<j$, of the finite sequence $0, h_1, h_2, \ldots, h_w, n$ such that exactly two consecutive members h_v, h_{v+1} of the finite sequence $h_1, h_2, \ldots, h_w$ lie between i and j and one of the following two relationships is true:

$$f(i) \leqq f(h_{v+1}) < f(h_v) \leqq f(j), \qquad f(j) \leqq f(h_v) < f(h_{v+1}) \leqq f(i).$$

Since the arguments are similar in both of these cases, we shall assume that $f(i) \leqq f(h_{v+1}) < f(h_v) \leqq f(j)$. In this situation, we let

$$\begin{aligned} S &= P(0, i-1)+Q(f(i), f(j))+P(j+1, n) \quad \text{if } 0 < i, j < n, \\ &= Q(f(i), f(j))+P(j+1, n) \quad \text{if } i = 0, j < n, \\ &= Q(f(i), f(j)) \quad \text{if } i = 0 \text{ and } j = n. \end{aligned}$$

It follows that S is a principal normal refinement of Q and that the order of S in Q is at most equal to t. Furthermore if g denotes the r-pattern of S in Q then S has a composite bend in Q of the form

$$(0; f(z_1); g(z_2); \ldots; g(z_u); m).$$

Hence by the induction hypothesis, $Q_{(u+1)(t+3)}$ is a normal refinement of S. Now, P is a principal normal refinement of S such that the order of P in S is at most equal to t. Furthermore, P has an elementary bend in S. From condition (3), $Q_{(u+1)(t+3)+(t+2)}$ is a normal refinement of S such that the rank of $Q_{(u+1)(t+3)+(t+2)}$ in S exceeds the order of P in S by at least 2. In addition, by condition (4), $Q_{(u+1)(t+3)+(t+3)}$ is crooked in $Q_{(u+1)(t+3)+(t+2)}$. Therefore, by condition (3) and Theorem 3.2, Q_k is a normal refinement of P for $k=(u+3)(t+3)$. This completes the proof.

For the purpose of establishing the principal theorem of the next section which will also be the major preliminary theorem of this paper, it will be convenient to formulate a theorem corresponding to Theorem 4.1 entirely in terms of patterns.

THEOREM 4.2. *If f is a normal r-pattern, then there exists an integer k such that for any sequence of crooked normal r-patterns $f_1, f_2, f_3, \ldots$ with the properties that*

(1) *the ranges of f and f_1 are identical, and, for each positive integer i,*

(2) *the range of f_{i+1} is identical with the domain of f_i, and*

(3) *the rank of f_i is greater than or equal to 2, then there is a normal r-pattern r such that*

$$fr = f_1 f_2 \cdots f_k.$$

Furthermore, the integer k depends only on the order of f and the number of elements in the composite bend representation of f.

5. **Uniformity of sequences of crooked cyclic r-patterns.** The purpose of this section is to establish a result for sequences of crooked cyclic r-patterns analogous to that obtained in §4 for sequences of crooked r-patterns.

First we formulate a definition which is partially motivated by the properties of sequences of crooked cyclic r-patterns established by this author in [5] and [6].

Let f be a properly left-normal cyclic r-pattern with winding number 1, and let $f_1, f_2, f_3, \ldots$ be a sequence of crooked cyclic r-patterns such that

(1) the range of f_1 is identical with the range of f, and, for each positive integer i,

(2) the range of f_{i+1} is identical with the domain of f_i,

(3) f_i is properly left-normal,

(4) f_i has winding number 1,

(5) f_i has rank greater than or equal to 2, and

(6) the canonical linear representation g_i of f_i has the property that $L(g_i) \geqq 2L(f_i)$.

Then the sequence $f_1, f_2, f_3, \ldots$ is defined to be a *c-regular sequence of monocyclic r-patterns relative to f.*

We now present the theorem which will be the principal result used in the proof of the uniqueness of the pseudo-circle.

THEOREM 5.1. *Let f be a properly left-normal cyclic r-pattern with winding number* 1. *Then there is an integer k such that $f_1, f_2, f_3, \ldots$ is any c-regular sequence of monocyclic r-patterns relative to f then there is a cyclic r-pattern r with winding number* 1 *such that*

$$fr = f_1 f_2 \cdots f_k.$$

The proof of this theorem is somewhat involved and it will be convenient to first establish two lemmas. These lemmas will be used in the sections of the proof of Theorem 5.1 in which the r-patterns considered have finite domains and ranges. Thus, it will be convenient to formulate these lemmas in terms of p-chains and their associated refinements.

LEMMA 5.1.1. *Let P, Q and T be p-chains such that P is a properly normal principal refinement of Q with r-pattern u in Q, and T is a normal refinement of Q with r-pattern v in Q. In addition, let the triple of p-chains P, Q and T have the property that whenever P^*, Q^* and T^* are sub-p-chains of P, Q and T, respectively, such that P^* or the conjugate of P^* is a properly normal principal refinement of Q^* with the r-pattern u^* in Q^* which is the restriction of u to the set of subscripts of the links of P^*, and T^* or the conjugate of T^* is a properly normal refinement of Q^* with r-pattern v^* in Q^* which is the restriction of v to the set of subscripts of the links of T^*, then T^* or the conjugate of T^* is a normal refinement of P^* with r-pattern w^* in P^* such that $u^*w^* = v^*$. Then T is a normal refinement of P with r-pattern w in P such that $uw = v$.*

Proof. Let $P = P(0, a)$, $Q = Q(0, b)$ and $T = T(0, c)$. Next, let $T(x_1, x_2)$, $T(x_2, x_3)$, $\ldots$, $T(x_{z-1}, x_z)$, $x_1 = 0$, $x_z = c$, be a finite sequence of sub-p-chains of T such that

each member $T(x_i, x_{i-1})$, $i=1, 2, \ldots, z-1$, of the sequence can be expressed by a p-chain sum of the form

$$T(x_i, x_{i+1}) = T_{i1}+T_{i2}+T_{i3}$$

where each of the first and third terms T_{i1} and T_{i3} of the p-chain sum is either empty or is a refinement of a sub-p-chain of Q consisting of a single link of Q, and the second term T_{i2} of the p-chain sum has the property that either T_{i2} or the conjugate of T_{i2} is maximal with respect to being a properly normal refinement of either Q, a first section of Q, or a last section of Q. A sequence of integers $x_1, x_2, \ldots, x_z$ satisfying the foregoing requirements can be constructed by first choosing a preliminary increasing finite sequence of integers $x'_1, x'_2, \ldots, x'_s$ such that $x'_1=0$, $x'_s=c$, and the sequence $x'_1, x'_2, \ldots, x'_s$ is maximal with respect to the two properties that every member x'_i of the preliminary sequence $x'_1, x'_2, \ldots, x'_s$ corresponds under v to the subscript of an end link of Q and that between each two adjacent members x'_i and x'_{i+1} of the preliminary sequence $x'_1, x'_2, \ldots, x'_s$ there is an integer which corresponds under v to the subscript of a link of Q which is not an end link of Q. Then, for each pair of adjacent members x'_i, x'_{i+1} of the preliminary sequence $x'_1, x'_2, \ldots, x'_s$ such that $v(x'_i)=v(x'_{i+1})$, if any, we choose an integer x''_i between x'_i and x'_{i+1} for which the expression $|v(x'_i)-v(x''_i)|$ is a maximum. The resulting increasing sequence consisting of the members of the preliminary sequence $x'_1, x'_2, \ldots, x'_s$ together with the additional integers with the properties such as those described for x''_i is a sequence with the properties required in the description of $x_1, x_2, \ldots, x_z$.

Now we define a related sequence of p-chains $P(y_1, y_2), P(y_2, y_3), \ldots, P(y_{z-1}, y_z)$, each member of which is a sub-p-chain of P or the conjugate of P, in the following manner. If i is an integer, $1 \leqq i \leqq z$, such that $v(x_i)$ is the subscript of an end link of Q, then we define y_i to be the unique integer such that $u(y_i)=v(x_i)$. If i is an integer, $1 \leqq i \leqq z$ such that $v(x_i)$ is not the subscript of an end link of Q, then we define y_i to be the least of the subscripts of the links of P such that $u(y_i)=v(x_i)$ if $v(x_{i-1})=0$, and define y_i to be the greatest of the subscripts of the links of P such that $u(y_i)=v(x_i)$ if $v(x_{i-1})=b$.

Then, for $i=1, 2, \ldots, z-1$, the p-chain T_{i2} is a properly normal refinement of $Q(v(x_i), v(x_{i+1}))$. In addition, noting that P is a properly normal refinement of Q, the p-chain $P(y_i, y_{i+1})$ is also a properly normal refinement of $Q(v(x_i), v(x_{i+1}))$. Thus, by the condition described in the second sentence of the statement of the lemma, it follows that T_{i2} is a normal refinement of $P(y_i, y_{i+1})$. Hence $T(x_i, x_{i+1})$ is a normal refinement of $P(y_i, y_{i+1})$ with an r-pattern w_i in $P(y_i, y_{i+1})$ such that if v_i is the restriction of v to the set $(x_i, x_i+1, \ldots, x_{i+1})$ then $uw_i=v_i$. Therefore, T is a normal refinement of P with an r-pattern w in P such that $uw=v$.

LEMMA 5.1.2. *Let* $f=((0, f(0)=0), (1, f(1)), \ldots, (n, f(n)))$ *be a cyclic r-pattern with range* $(0, 1, \ldots, m)$ *such that f has rank greater than or equal to* 2, *f has winding number* 1, *and the canonical linear representation g of f is properly left-normal. In addition, let h denote the universal linear representation of f and assume that*

there is an integer t such that $m+1<t<L(g)$ and if h^* is any r-pattern which is a restriction of h of length t then h^* is crooked. Finally, let T and Q be p-chains such that g is an r-pattern of T in Q with respect to which each link of Q corresponds to at least one link of T. Then there is a last section T' of T such that T' is a normal refinement of the sub-p-chain $Q(t, m+1)$ of the conjugate of Q.

Proof. We note that $T=T(0, n)$ and $Q=Q(0, L(g)-1)$. Now, since f is a properly left-normal cyclic r-pattern with positive winding number 1 and h is the universal linear representation of f, we may define d to be the greatest negative integer such that $h(d)=-1$. Next, again using the property that h is the universal linear representation of a cyclic r-pattern with positive winding number, we may choose e to be the least integer greater than d such that $h(e)=t-m$. Then the restriction h^* of h to the set $(d, d+1, \ldots, e)$ has length less than t. Hence h^* is a crooked r-pattern. It follows that there exist integers y_1 and y_2 such that $d<y_1<y_2<e$ and $h^*(y_1)=t-(m+1)$, $h^*(y_2)=0$. In addition, since the canonical linear representation g of f is properly left-normal and f has positive winding number, the integer y_2 is less than or equal to zero so that y_1 is a negative integer.

Let y be the greatest negative integer such that $h(y)=t-(m+1)$ and note that $-(n+1)<y$. Then the restriction of h to the set $(y, y+1, \ldots, 0)$ is a normal r-pattern with range $(t-(m+1),\ t-(m+1)-1, \ldots, 0)$. In addition, f has rank greater than or equal to 2 and f is a properly left-normal cyclic r-pattern such that $f(0)=h(0)=0$. Thus $h(-1)=0$. Hence the restriction of h to the set $(y, y+1, \ldots, -1)$ is a normal r-pattern with range $(t-(m+1), t-(m+1)-1, \ldots, 0)$. Now, since h is the universal linear representation of f, h is periodic in the sense that for each pair of integers i, j

$$h(i+j(n+1)) = h(i)+j(m+1).$$

It follows that the restriction of h to the set $(y+n+1, y+n+2, \ldots, n)$ is a normal r-pattern with range $(t, t-1, \ldots, m+1)$. Therefore, the last section $T'=T(y+n+1, n)$ of T is a normal refinement of the sub-p-chain $Q(t, m+1)$ of the conjugate of Q.

Proof of Theorem 5.1. Let the sets $\{0, 1, \ldots, n\}$ and $\{0, 1, \ldots, m\}$ be the domain and range, respectively, of f, and let g and h denote the canonical linear representation of f and the universal linear representation of f, respectively. For each positive integer i, we define the domain, range, canonical linear representation, and universal linear representation, respectively, of f_i by replacing the symbols n, m, f, g and h of the preceding statement by n_i, m_i, f_i, g_i and h_i, respectively.

We now describe the determination of the integer k whose existence is asserted in the statement of the theorem. Note first, by Theorem 2.1 of [6] and properties (4) and (6) of the definition of "c-regular sequence of monocyclic r-patterns relative to f", above, that there is an integer t depending only on f such that, for each nonnegative integer i,

$$L(\text{c.l.r. } [f_1 f_2 \cdots f_{t+i}]) > 3L(g).$$

In particular we may choose t to be equal to the integer $3L(g)$. Now, consider the sequence $h, h_1, h_2, h_3, \ldots$ of universal linear representations of $f, f_1, f_2, f_3, \ldots$, respectively. Let $h', h_1', h_2', h_3', \ldots$ be a related sequence of normal r-patterns having the following properties:

(a) h' is a restriction of h such that $L(h') \leqq 2L(g)$,

(b) h_1' is a restriction of h_1 such that the range of h_1' is identical with the range of h', and

(c) for each positive integer i, h_i' is a restriction of h_i such that h_i' has rank greater than or equal to 2 and so that the range of h_{i+1}' is identical with the domain of h_i'. It will now be shown that, for each positive integer i, h_{t+i}' is a crooked r-pattern. Since f_{t+i} is a crooked cyclic r-pattern it will be sufficient to prove that $L(h_{t+i}') \leqq L(f_{t+i})$. Suppose on the contrary that $L(h_{t+i}') > L(f_{t+i})$. Then, since the number of elements in the domains of h_{t+i-1}' and f_{t+i-1}, respectively, are $L(h_{t+i}')$ and $L(f_{t+i})$, it follows that the domain of the composite normal r-pattern $h_1'h_2' \cdots h_{t+i-1}'$ contains more elements than does the domain of the composite cyclic r-pattern $f_1f_2 \cdots f_{t+i-1}$. In addition it is easily verified that $h_1h_2 \cdots h_{t+i-1}$ is the universal linear representation of $f_1f_2 \cdots f_{t+i-1}$. Hence, $h_1'h_2' \cdots h_{t+i-1}'$ is a linear representation of $f_1f_2 \cdots f_{t+i-1}$. Now, a linear representation of a cyclic r-pattern with winding number 1 has length greater than or equal to the length of the canonical linear representation of that cyclic r-pattern minus the length of the cyclic r-pattern. Therefore,

$$\begin{aligned} L(h_1'h_2' \cdots h_{t+i-1}') &\geqq L(\text{c.l.r.}\,[f_1f_2 \cdots f_{t+i-1}]) - (m+1) \\ &> 2L(g). \end{aligned}$$

But these relationships are contrary to the hypothesized relationships

$$L(h_1'h_2' \cdots h_{t+i-1}') = L(h_1') = L(h') \leqq 2L(g).$$

From the preceding construction and result it follows that we may apply Theorem 4.2 to obtain the conclusion that there is an integer k' such that there is a normal r-pattern r' with the property that

$$h'r' = h_1'h_2' \cdots h_{k'}'.$$

We shall refer to this relationship subsequently in this proof as the "piecewise-commutativity property". In addition, since k' depends only on the order of h' and the number of elements in the composite bend representation of h', and since h is the universal linear representation of f, there is a finite collection $\{k'\}$ of integers such as k' such that every sequence of normal r-patterns satisfying conditions (a), (b) and (c) is associated in the above described manner with some member of $\{k'\}$. We define the integer k to be the maximum of the finite collection of integers consisting of $\{k'\}$ and the integer t.

It remains to show that there is a cyclic r-pattern r with winding number 1 such that

$$fr = f_1f_2 \cdots f_k.$$

This last section of the proof is somewhat involved and to facilitate the presentation of this portion of the proof a preliminary outline of the method and steps to be followed will be made. The terminology and properties of p-chains will be used in this final section of the proof since the argument will primarily involve patterns whose domains and ranges are finite.

First let v denote the properly left-normal composite r-pattern $h_1h_2\cdots h_{k-1}g_k$ and let $(0, 1, \ldots, c)$ and $(0, 1, \ldots, a)$ denote the domain and range, respectively, of v. We choose $T=T(0, c)$ and $Q=Q(0, a)$ to be p-chains such that T has the r-pattern v in Q. Next let b denote the greatest integer such that the restriction u of the universal linear representation h of f to the set $(0, 1, \ldots, b)$ has the range $(0, 1, \ldots, a)$. Then we choose $P=P(0, b)$ to be a p-chain such that P is a principal refinement of Q under the properly left-normal r-pattern u of P in Q. It is observed that since $k \geqq t$,

$$L(\text{c.l.r.}(f_1f_2\cdots f_k)) > 3L(g)$$

and since in addition

$$L(v) = L(\text{c.l.r.}(f_1f_2\cdots f_k))$$

we obtain the relationship $L(v)>L(g)$. Thus, noting that $L(u)=L(v)$, it follows that b is greater than n and therefore u is a linear representation of f.

We now describe the method that will be followed in this last portion of the proof and simultaneously develop relationships which will be an intrinsic part of the proof. First we note from the proof of Lemma 3.3.1 of the paper [5] of this author that an inverse operation to the linearization of a cyclic r-pattern can be performed in certain circumstances. Essentially, an r-pattern can be "projected" onto a cyclic r-pattern provided that the end elements of the domain of the r-pattern project onto cyclically identical or adjacent elements of the domain of the cyclic r-pattern and provided that a similar statement can be made regarding the ranges of the r-pattern and cyclic r-pattern. For the purposes of the present theorem the method of proof will be to show that there exists an r-pattern w of T in P such that w satisfies the commutativity relationship $uw=v$ and the end element relationships $w(0)=0$ and $w(c)=n+1$. Then as in the proof of Fearnley [5, Lemma 3.3.1], since $|w(c)-w(0)| \bmod n \leqq 1$, it will follow that the r-pattern w determines a cyclic r-pattern r with domain $(0, 1, \ldots, c)$ and range $(0, 1, \ldots, n)$ such that r is a linear representation of w. In addition, since $w(0)=0$ and $w(c)=n+1$ and w is a linear representation of r it will follow from the Corollary to Theorem 3.1 of [5] that the winding number of r is equal to 1. Finally, from the properties that u, v, and w are linear representations of f, $f_1f_2\cdots f_k$, and r, respectively, together with the fact that $uw=v$, we will be able to conclude by an argument similar to that given in the last portion of the proof of Fearnley [5, Lemma 3.3.1] that the remaining requirement that $fr=f_1f_2\cdots f_k$ is also satisfied.

The existence of an r-pattern w of T in P satisfying the requirements described above will be established by first proving by an induction on certain sub-p-chains of

T that there is an r-pattern w of T in P such that $uw=v$, and then showing that the r-pattern w of T in P developed by the constructions of this particular induction argument has the required additional properties that $w(0)=0$ and $w(c)=n+1$. In the presentation of the induction argument it will be understood that whenever a sub-p-chain T^* of T is referred to as being a refinement of a sub-p-chain P^* of P and w^* is the r-pattern of T^* in P^* then w^* has the property that uw^* is equal to the restriction of v to the set of subscripts of the links of T^*.

First, let c_1 be the least integer such that $v(c_1)=L(g)$. Such an integer c_1 must exist since v and g are left-normal r-patterns, $v(0)=h(0)=0$, and $L(v)>L(g)$. Next, let b_1 be the least integer such that $u(b_1)=L(g)$. The existence of b_1 is established by reasoning similar to that given for the existence of c_1. Now, since u and v are properly left-normal r-patterns, each of the sub-p-chains $P(1, b_1-1)$ of P and $T(1, c_1-1)$ of T are normal refinements of the sub-p-chain $Q(1, L(g)-1)$ of Q. Furthermore, if h' denotes the restriction of u to the set $(1, 2, \ldots, b_1-1)$, there is a sequence of r-patterns h', h'_1, h'_2, h'_3, ... having the properties (a), (b) and (c) described in the previous section of the proof. Thus, by the piecewise-commutativity property, $T(1, c_1-1)$ is a normal refinement of $P(1, b_1-1)$. In addition $u(0)=v(0)=0$ and $u(b_1)=v(c_1)=L(g)$. Therefore, $T_1=T(0, c_1)$ is a refinement of $P(0, b_1)$.

Before proceeding to present the general induction hypothesis we observe that Lemma 5.1.1 together with the argument given in the preceding paragraph establishes that if P^*, Q^* and T^* are sub-p-chains of P, Q and T, respectively, having the properties that P^* or the conjugate of P^* is a properly normal refinement of Q^*, T^* or the conjugate of T^* is a normal refinement of Q^*, and Q^* has at most $2L(g)$ links, then T^* or the conjugate of T^* is a normal refinement of P^*. This result may be considered as an extension of the p-chain formulation of the piecewise-commutativity property and will be used repeatedly in establishing the induction step. The result will be referred to as the "extended p-chain piecewise-commutativity property".

We are now ready to state the induction hypothesis for the construction of the r-pattern w of T in P. Assume for some positive integer j that a finite sequence of integers $c_1, c_2, \ldots, c_j$ has been chosen such that

(A) $0<c_1<c_2<\cdots<c_j<c$,

(B) the sub-p-chain $T_j=T(0, c_j)$ of T is a refinement of (a sub-p-chain of) P, and

(C) there is a positive integer i_j and a last section T'_j of T_j such that either T'_j or the conjugate of T'_j is a properly normal refinement of

$$Q(i_j(m+1), (i_j-1)(m+1)+L(g)).$$

We observe that the integer c_1 satisfies the requirements of the induction hypothesis since condition (B) was established for c_1 in the first step of the induction, and conditions (A) and (C) with $i_1=1$ can be readily verified. It will be shown next

that there is an integer c_{j+1} such that either c_{j+1} satisfies the conditions corresponding to (A), (B) and (C) of the induction hypothesis, or $c_{j+1}=c$ and c_{j+1} satisfies the condition corresponding to (B) of the induction hypothesis. Since there are only a finite number of integers that can satisfy condition (A) of the induction hypothesis, it will follow that at some stage there is a value of j such that $c_{j+1}=c$ and c_{j+1} satisfies the condition corresponding to (B) of the induction hypothesis, so that then the induction proof that there is an r-pattern w of T in P will be complete.

First note that $m+1<L(g)+m+1<L(v)$ and that r-patterns of length $L(g)+m+1$ that are restrictions of $h_1h_2\cdots h_k$ are crooked. Hence we can apply Lemma 5.1.2 to the cyclic r-pattern $f_1f_2\cdots f_k$ and its associated canonical linear representation v which is an r-pattern of T in Q to obtain the result that there is a last section T' of T such that T' is a normal refinement of the sub-p-chain $Q(L(g)+m+1, m+1)$ of the conjugate of Q. In addition, by the induction hypothesis, there is an integer i_j such that either $v(c_j)=i_j(m+1)$ or $v(c_j)=(i_j-1)(m+1)+L(g)$. From these results it follows that exactly one of the following three relationships is true:

(1) The integer i_j is equal to 1 and the link of T with subscript c_j is a link of a last section $T(x, c)$ of T such that $T(x, c)$ is a normal refinement of $Q(L(g)+m+1, m+1)$ and for each integer i with the property that $x<i\leqq c$, $v(i)$ satisfies the relationship $m+1\leqq v(i)<L(g)+m+1$.

(2) There is a first integer e_j greater than c_j with the properties that $v(e_j)=i_j(m+1)+L(g)$ and for each integer i such that $c_i\leqq i\leqq e_i$, $v(i)$ satisfies the relationship

$$(i_j-1)(m+1) < v(i) \leqq i_j(m+1)+L(g).$$

(3) There is a first integer e_j greater than c_j with the properties that $v(e_j)=(i_j-1)(m+1)$ and for each integer i such that $c_i\leqq i\leqq e_i$, $v(i)$ satisfies the relationship

$$(i_j-1)(m+1) \leqq v(i) < i_j(m+1)+L(g).$$

The existence of the integer c_{j+1} required for the induction step will be established first for the case in which (1) is true, and then for the four additional cases determined by whether $v(c_j)=i_j(m+1)$ or $v(c_j)=(i_j-1)(m+1)+L(g)$ and whether (2) or (3) is true.

Case 1. The relationship described in (1) is true. In this case let y be the first integer greater than or equal to x such that $v(y)=2(m+1)$ and note that then $T(x, y)$ is a properly normal refinement of the sub-p-chain $Q(L(g)+m+1, 2(m+1))$ of the conjugate of Q. Now, by part (C) of the induction hypothesis, either $v(c_j)=m+1$ and there is a last section $T_j'=T(c_j', c_j)$ of $T_j=T(0, c_j)$ such that T_j' is a properly normal refinement of $Q(L(g), m+1)$, or $v(c_j)=L(g)$ and there is a last section $T_j'=T(c_j', c_j)$ of $T_j=T(0, c_j)$ such that T_j' is a properly normal refinement of $Q(m+1, L(g))$. In both cases c_j either follows or is equal to a subscript of a link of $T(x, c)$ which corresponds under v to the link of Q with subscript $m+1$. Thus,

$c_j > y$. In addition, by part (B) of the induction hypothesis, $T_j = T(0, c_j)$ is a refinement of P. Hence $T(0, y)$ is a refinement of P under an r-pattern w_y which is the restriction to the set $(0, 1, \ldots, y)$ of the r-pattern of T_j in P.

Now let $P(b_x, b_y)$ be the sub-p-chain of either P or the conjugate of P defined by the relationships $b_x = w_y(x)$ and $b_y = w_y(y)$. Then, since uw_y is equal to the restriction of v to the set $(0, 1, \ldots, y)$, we obtain the relationships $u(b_x) = L(g) + m + 1$ and $u(b_y) = 2(m+1)$. Furthermore, from the facts that $T(x, y)$ is a properly normal refinement under v of $Q(L(g)+m+1, 2(m+1))$ and that w_y is an r-pattern, it follows that $P(b_x, b_y)$ is exactly the sub-p-chain of P or the conjugate of P which corresponds under w_y to the sub-p-chain $T(x, y)$ of T. In addition, we obtain the results that $T(x, y)$ is a properly normal refinement of $P(b_x, b_y)$ and that $P(b_x, b_y)$ is a properly normal refinement of $Q(L(g)+m+1, 2(m+1))$.

The next stage in establishing the induction step for Case 1 is to use the foregoing results together with the periodicity and proper left-normality of u to prove that $b_y = 2(n+1)$ and that b_x is equal to the least integer that corresponds under u to the link of Q with subscript $L(g)+m+1$. First, note that since $u(0) = 0$ and g is the canonical linear representation of the left-normal cyclic r-pattern f, the maximum value of the images under u of the set $(0, 1, \ldots, n)$ is the integer $L(g) - 1$. In addition, from the fact that u is a linear representation of f, u satisfies the periodicity relationship

$$u(s_1 + s_2(n+1)) = u(s_1) + s_2(m+1)$$

for all integers s_1 and s_2 for which this relationship is defined. Thus in order that $u(b_x) = L(g)+m+1$ it is necessary that $b_x \geqq 2(n+1)$. Correspondingly, from the foregoing relationships and the fact that f is properly left-normal, in order that $u(b_y) = 2(m+1)$ it must be the case that $b_y \leqq 2(n+1)$. Now, suppose that $b_y < 2(n+1)$. Then $P(b_y, b_x)$ is a sub-p-chain of P which contains a link of P with subscript $2(n+1)$. Since $u(2(n+1)) = 2(m+1)$, this supposition involves a contradiction to the fact that $P(b_x, b_y)$ is a properly normal refinement of $Q(L(g)+m+1, 2(m+1))$. Therefore $b_y = 2(n+1)$. From this result that $b_y = 2(n+1)$ and the relationships that $b_x \geqq 2(n+1)$ and $P(b_y, b_x)$ is a properly normal refinement of $Q(2(m+1), L(g)+m+1)$, we also obtain the conclusion that b_x is the least integer such that $u(b_x) = L(g)+m+1$.

To complete the argument for Case 1, we observe that since f is a properly left-normal cyclic r-pattern with domain $(0, 1, \ldots, n)$ and positive winding number, and u is a linear representation of f, then the restriction of u to the set $(n+1, n+2, \ldots, b)$ is a properly left-normal r-pattern. Thus, since b_x is an integer greater than or equal to $2(n+1)$ such that b_x is the least integer with the property that $u(b_x) = L(g)+m+1$, it follows that $P(b_x, n+1)$ is a properly normal refinement of $Q(L(g)+m+1, m+1)$. Now, from the conditions of Case 1 described in (1), $T(x, c)$ is a normal refinement of $Q(L(g)+m+1, m+1)$. Therefore, by the extended p-chain piecewise-commutativity property, $T(x, c)$ is a normal refinement of $P(b_x, n+1)$. Thus, since $T(0, x)$ is a refinement of P under the r-pattern which is

the restriction of w_y to the set $(0, 1, \ldots, x)$ and $w_y(x)=b_x$, we conclude that T is a refinement of P.

Case 2. The relationship described in (2) is true and $v(c_j)=(i_j-1)(m+1)+L(g)$. In this case, from part (C) of the induction hypothesis, there is an integer c_j' less than or equal to c_j and a last section $T_j'=T(c_j', c_j)$ of the sub-p-chain $T_j=T(0, c_j)$ of T such that T_j' is a properly normal refinement of $Q(i_j(m+1), (i_j-1)(m+1)+L(g))$. Next, let w_{c_j} denote the r-pattern of $T(0, c_j)$ in P whose existence is guaranteed by part (B) of the induction hypothesis, and let b_{c_j} and $b_{c_j'}$ be integers defined by the relationships $b_{c_j}=w_{c_j}(c_j)$ and $b_{c_j'}=w_{c_j}(c_j')$. Then, by an argument similar to that given in the second paragraph of the argument for Case 1, it follows that $T(c_j', c_j)$ is a properly normal refinement of $P(b_{c_j'}, b_{c_j})$ and that $P(b_{c_j'}, b_{c_j})$ is a properly normal refinement of $Q(i_j(m+1), (i_j-1)(m+1)+L(g))$. In addition, by an argument similar to that given in the third paragraph of the argument for Case 1, we obtain the further results that $b_{c_j'}=i_j(n+1)$ and that b_{c_j} is an integer greater than or equal to $i_j(n+1)$ which is the least integer with the property that

$$u(b_{c_j}) = (i_j-1)(m+1)+L(g).$$

Now, let x_1, x_2, x_3, x_4 be a nondecreasing sequence of four integers chosen in the following manner. First, let $x_1=c_j'$ and let $x_4=e_j$. Next, let x_3 be an integer such that $x_1 \leqq x_3 \leqq x_4$ and such that $v(x_3)$ is a minimum of the set $(v(x_1), v(x_1+1), \ldots, v(x_4))$. Finally, let x_2 be an integer such that $x_1 \leqq x_2 \leqq x_3$ and such that $v(x_2)$ is a maximum of the set $(v(x_1), v(x_1+1), \ldots, v(x_3))$. It is observed that since $v(x_1)=v(c_j')=i_j(m+1)$ then $v(x_3) \leqq i_j(m+1)$. In addition, from condition (C) of the induction hypothesis, $v(x_2) \geqq v(c_j)$ so that $v(x_2) \geqq (i_j-1)(m+1)+L(g)$. Next we choose a sequence y_1, y_2, y_3, y_4 of four integers, related to the sequence x_1, x_2, x_3, x_4, in the following manner. First let $y_1=b_{c_j'}$ and note that $u(y_1)=v(x_1)$. Next let y_2 be the least integer such that $u(y_2)=v(x_2)$ and note since $v(x_2) \geqq (i_j-1)(m+1)+L(g)$ and f is a properly left-normal cyclic r-pattern with winding number 1, with domain $(0, 1, \ldots, n)$, and with canonical linear representation g of length $L(g)$, that $i_j(n+1) \leqq y_2$. Hence $y_1 \leqq y_2$. The integer y_3 is now chosen to be the greatest integer such that $u(y_3)=v(x_3)$. Then, since f is a properly left-normal cyclic r-pattern with positive winding number and domain $(0, 1, \ldots, n)$ and u is a linear representation of f, the restriction of u to the set $(y_1, y_1+1, \ldots, b)$, $y_1=b_{c_j'}=i_j(n+1)$, is a properly left-normal r-pattern such that $u(y_1)=i_j(m+1)$. In addition, $v(x_3) \leqq i_j(m+1)$. Hence $y_3 \leqq y_1$ and thus $y_3 \leqq y_2$. Finally we choose y_4 to be the least integer such that $u(y_4)=v(x_4)$ and observe, since $v(x_4)=i_j(m+1)+L(g)$ and f is a properly left-normal cyclic r-pattern with winding number 1, domain $(0, 1, \ldots, n)$, and canonical linear representation g of length $L(g)$, that $(i_j+1)(n+1) \leqq y_4$. Hence, since $y_3 \leqq i_j(n+1)$, it follows that $y_3 < y_4$.

From the choice of the integers x_1, x_2, x_3 and x_4 we obtain the relationship that $T(x_1, x_2)$, $T(x_2, x_3)$ and $T(x_3, x_4)$ are normal refinements of $Q(i_j(m+1), v(x_2))$, $Q(v(x_2), v(x_3))$ and $Q(v(x_3), i_j(m+1)+L(g))$, respectively. Furthermore, by the

condition described in (2) that $(i_j-1)(m+1)<v(i)\leqq i_j(m+1)+L(g)$ whenever $c_j\leqq i\leqq e_j$ together with the fact that $T(c_j', c_j)$ is a normal refinement of $Q(i_j(m+1), (i_j-1)(m+1)+L(g))$ under the r-pattern v, the refinements of $T(x_1, x_2)$, $T(x_2, x_3)$ and $T(x_3, x_4)$ in Q have r-patterns of length at most $2L(g)$. In addition, from the choice of the integers y_1, y_2, y_3 and y_4, and the inequalities $y_1\leqq y$, $y_2\geqq y_3$, $y_3<y_4$ established above, it follows that $P(y_1, y_2)$, $P(y_2, y_3)$ and $P(y_3, y_4)$ are properly normal refinements of $Q(i_j(m+1), u(y_2))$, $Q(u(y_2), u(y_3))$ and $Q(u(y_3), i_j(m+1)+L(g))$, respectively. Therefore, by the extended p-chain piecewise-commutativity property together with the facts that $u(y_1)=v(x_1)$, $u(y_2)=v(x_2)$, $u(y_3)=v(x_3)$ and $u(y_4)=v(x_4)$, we obtain the results that $T(x_1, x_2)$ and $T(x_3, x_4)$ are normal refinements of $P(y_1, y_2)$, $P(y_2, y_3)$ and $P(y_3, y_4)$, respectively. Now, from part (B) of the induction hypothesis, the r-pattern w_{c_j} of $T(0, c_j')$ in P has the property that $w_{c_j}(c_j')=b_{c_j'}=y_1$. Thus, $T(0, e_j)$ is a refinement of P. Furthermore, since e_j is the least integer greater than c_j such that $v(e_j)=i_j(m+1)+L(g)$, there is a last section of $T(0, e_j)$ which is a properly normal refinement of $Q((i_j+1)(m+1), i_j(m+1)+L(g))$. Thus, if we define $c_{j+1}=e_j$, the integer c_{j+1} has the required properties corresponding to conditions (A), (B) and (C) of the induction hypothesis and the induction step for this case is complete.

The proofs of the induction step for the remaining cases involve techniques similar to those used in Cases 1 and 2 and these will be given in condensed form.

Case 3. The relationship described in (2) is true and $v(c_j)=i_j(m+1)$. In this case, from parts (B) and (C) of the induction hypothesis, the sub-p-chain $T(0, c_j)$ of T is a refinement of P under an r-pattern w_{c_j} and there is a last section $T(c_j', c_j)$ of $T(0, c_j)$ which is a properly normal refinement under w_{c_j} of the p-chain

$$Q((i_j-1)(m+1)+L(g),\ i_j(m+1)).$$

In addition, by a similar argument to that given in the first two paragraphs of Case 1, $T(c_j', c_j)$ is a properly normal refinement under w_{c_j} of $P(b_{c_j'}, b_{c_j})$, where $b_{c_j}=i_j(n+1)$ and $b_{c_j'}$ is an integer greater than or equal to b_{c_j} such that $b_{c_j'}$ is the least integer with the property that $u(b_{c_j'})=(i_j-1)(m+1)+L(g)$.

Now, from (2) and the fact that $v(c_j)=i_j(m+1)$, there is a least integer d_j greater than or equal to c_j such that $v(d_j)=(i_j-1)(m+1)+L(g)$. We may assume that $d_j>c_j$ since if $d_j=c_j$ the induction step for Case 3 follows from Case 2. Next, let d_j' be an integer such that $c_j'\leqq d_j'\leqq d_j$ and $v(d_j')$ is equal to the minimum of the set $(v(c_j'), v(c_j'+1), \ldots, v(d_j))$ and note that $v(d_j')\leqq v(c_j)=i_j(m+1)$. Finally, let y be the greatest integer such that $u(y)=v(d_j')$ and note that since $v(d_j')\leqq i_j(m+1)$ and f is a properly left-normal r-pattern with domain $(0, 1, \ldots, n)$ and winding number 1, it follows that $y\leqq i_j(n+1)=b_{c_j'}$. Then, $T(c_j' d_j')$ and $T(d_j', d_j)$ are normal refinements of $Q((i_j-1)(m+1)+L(g), v(d_j'))$ and $Q(v(d_j'), (i_j-1)(m+1)+L(g))$, respectively. In addition, $P(b_{c_j'}, y)$ is a properly normal refinement of $Q((i_j-1)(m+1), u(y)=v(d_j'))$ and a similar statement involving their respective conjugates is automatically true. Therefore, from the extended p-chain piecewise-commutativity property for

these refinements, it follows that $T(c_j', d_j')$ is a normal refinement of $P(b_{c_j'}, y)$ and that $T(d_j', d_j)$ is a normal refinement of $P(y, b_{c_j'})$. Thus, noting that $w_{c_j}(c_j') = (i_j-1)(m+1)+L(g)$, we obtain the result that $T(0, d_j)$ is a refinement of P. In addition $d_j > c_j$ and d_j is the least integer greater than d_j' such that

$$v(d_j) = (i_j-1)(m+1)+L(g).$$

We conclude that if we define $c_{j+1} = d_j$ then c_{j+1} satisfies the required properties corresponding to (A), (B) and (C) of the induction hypothesis.

In the presentation of the proofs of the induction step for the final two cases use will be made of a duality inherent in the periodicity and proper left-normality of the r-pattern u of P in Q.

Case 4. The relationship described in (3) is true and $v(c_j) = i_j(m+1)$. To prove the induction result for this case we first identify properties of the r-pattern u that will be used in establishing the desired duality characteristics of u. First let $T(c_j', c_j)$ be the last section of $T(0, c_j)$ such that $T(c_j', c_j)$ is a properly normal refinement of $Q((i_j-1)(m+1)+L(g), i_j(m+1))$ and let $P(b_{c_j'}, b_{c_j})$ be the p-chain such that $T(c_j', c_j)$ is a properly normal refinement of $P(b_{c_j'}, b_{c_j})$ under the r-pattern w_{c_j} of $T(0, c_j)$ in P. That these choices are possible follows by arguments similar to those given in the previous cases. Then the characteristics of u with respect to b_{c_j} and the characteristics of u with respect to $b_{c_j'}$ are "symmetric" in the following respects. The restriction of u to the set $(b_{c_j}, b_{c_j}+1, \ldots, b)$ is properly left-normal and the restriction of u to the set $(0, 1, \ldots, b_{c_j'})$ is properly right-normal. In addition, if x is an integer such that $u(x) \leqq u(b_{c_j})$ then $x \leqq b_{c_j}$ and if y is an integer such that $u(y) \geqq u(b_{c_j'})$ then $y \geqq b_{c_j'}$. In Case 2 the foregoing properties of u except with b_{c_j} and $b_{c_j'}$ replacing each other in the above statements are the only properties of u used. The properties of the r-pattern v of T in Q used in Case 2 in proving that $T(0, e_j)$ is a refinement of P are that the restriction of v to the set $(c_j', c_j'+1, \ldots, e_j)$ has length at most $2L(g)$, $v(c_j') \leqq v(c_j)$, and $v(e_j) > v(c_j)$. Thus in the present case, in which the restriction of v to the set $(c_j', c_j'+1, \ldots, c_j)$ is a normal r-pattern, the restriction of v to the set $(c_j', c_j'+1, \ldots, e_j)$ has length at most $2L(g)$, $v(c_j') \geqq v(c_j)$, and $v(e_j) < v(c_j)$, the natural dual of the argument for Case 2 establishes in Case 4 that $T(0, e_j)$ is a refinement of P. Therefore, if we define $c_{j+1} = e_j$ the requirements corresponding to conditions (A) and (B) of the induction hypothesis are satisfied. In addition, since $v(c_j) = i_j(m+1)$ and e_j is the least integer greater than c_j such that $v(e_j) = (i_j-1)(m+1)$, there is a last section T_{j+1}' of $T_{j+1} = T(0, c_{j+1})$ such that T_{j+1}' is a properly normal refinement of $Q((i_j-2)(m+1)+L(g), (i_j-1)(m+1))$. Hence the remaining requirement corresponding to condition (C) of the induction hypothesis is also satisfied.

Case 5. The situation described in (3) is true and $v(c_j) = (i_j-1)(m+1)+L(g)$. The relationship of Case 5 to Case 3 is similar to the relationship of Case 4 to Case 2.

The foregoing induction argument establishes that there is an r-pattern w of T in P such that $uw = v$. Now, none of the integers c_{j+1} chosen by the procedures

described in Cases 2, 3, 4 or 5 of this preceding induction proof is equal to c since for each such integer c_{j+1} the relationship $v(c_{j+1}) \neq v(c_{j+1}-1)$ is true, whereas it follows from the fact that v is the canonical linear representation of a properly left-normal cyclic r-pattern of rank greater than 2 that $v(c)=v(c-1)$. Thus the image of the integer c under the r-pattern w is defined by the procedure described in Case 1. Hence $w(c)=n+1$. Finally since $u(0)=v(0)=0$ and u is properly left-normal it follows that $w(0)=0$. Therefore, by the argument preceding the inductive construction of w, we conclude that there exists a cyclic r-pattern with winding number 1 such that $fr=f_1f_2\cdots f_k$. This completes the proof of the theorem.

6. **The pseudo-circle is unique.** In this section we shall establish the principal result of the paper, that any two pseudo-circles are topologically equivalent.

First a modified form of Theorem 5.1 is developed involving defining sequences of circular chains associated with pseudo-circles.

THEOREM 6.1. *Let $Q_1, Q_2, Q_3, \ldots$ be a defining sequence of circular chains associated with a pseudo-circle M, let i be a positive integer and let T be a principal refinement of Q_i such that T has winding number* 1 *in Q_i. Then there is an integer j greater than i such that Q_j is a refinement of T and Q_j has winding number* 1 *in T.*

Proof. Let k be a positive integer. Since Q_{i+k+1} is a refinement of Q_{i+k} such that Q_{i+k+1} has winding number 1 in Q_{i+k}, there is a cyclic r-pattern f_{i+k} of Q_{i+k+1} in Q_{i+k} such that $f_{i+k}(0)=0$ and f_{i+k} is properly left-normal. Similarly, there is a cyclic r-pattern g of T in Q_i such that $g(0)=0$ and g is properly left-normal. In addition, f_{i+k} can be chosen so that the closure of each link of Q_{i+k+1} is a compact subset of the link of Q_{i+k} to which it corresponds under f_{i+k}. Furthermore, the diameter of each link of Q_{i+k} is less than $(i+k)^{-1}$, $k=1, 2, 3, \ldots$. Thus, without loss in generality, we may assume that each of the cyclic r-patterns $f_i, f_{i+1}, f_{i+2}, \ldots$ has rank greater than or equal to 2.

Finally, by Theorem 3.1 of [6], we may assume that

$$L(\text{c.l.r.}\,[f_{i+k}]) \geqq 2L(f_{i+k}).$$

It follows by Theorem 5.1 that there is an integer j greater than i and a cyclic r-pattern r with winding number 1 such that

$$gr = f_if_{i+1}\cdots f_{j-1}.$$

Therefore, Q_j is a refinement of T and Q_j has winding number 1 in T.

The following theorem formulated in terms of the combinatorial category is an immediate consequence of Theorem 6.1.

THEOREM 6.2. *Let $Q_1, Q_2, Q_3, \ldots$ be a sequence of combinatorial simple closed curves such that the mesh of Q_k approaches zero as k increases without bound. In addition, let f_k be a simplicial mapping of Q_{k+1} onto Q_k, $k=1, 2, 3, \ldots$, such that the degree of f_k is* 1 *and f_k is a crooked simplicial mapping. Finally, for some positive integer i, let g be a simplicial mapping of degree* 1 *of a combinatorial simple closed*

curve T onto Q_i and let ε be a positive number. Then there is an integer j greater than i and a simplicial mapping r of degree 1 of Q_j onto T such that the function space distance betwen gr and $f_i f_{i+1} \cdots f_{j-1}$ is less than ε.

Finally we use the foregoing theorem and an inverse limit argument to prove that the psudo-circle is unique.

THEOREM 6.3. *If H and K are pseudo-circles then H and K are topologically equivalent.*

Proof. Since H is a circularly chainable continuum, H is homeomorphic with the inverse limit of an inverse system

$$C_1 \xleftarrow{f_1} C_2 \xleftarrow{f_2} C_3 \xleftarrow{f_3} \cdots$$

such that, for each positive integer i, C_i is a combinatorial simple closed curve and f_i is a simplicial mapping of C_{i+1} onto C_i. Similarly, the circularly chainable continuum K is homeomorphic with the inverse limit of an inverse system

$$D_1 \xleftarrow{g_1} D_2 \xleftarrow{g_2} D_3 \xleftarrow{g_3} \cdots$$

such that, for each positive integer i, D_i is a combinatorial simple closed curve and g_i is a simplicial mapping of D_{i+1} onto D_i. In addition, since H and K are pseudo-circles we may assume that each member of each of the sequences of bonding mappings $f_1, f_2, f_3, \ldots$ and $g_1, g_2, g_3, \ldots$ is a crooked simplicial mapping of degree 1.

Now, let ε be a positive number. By Theorem 6.2, if u is a mapping of winding number 1 of C_m onto D_n, $m, n > 0$, then there exists an integer j greater than n and a simplicial mapping r of degree 1 of D_j onto C_m such that the function space distance between ur and $g_n g_{n+1} \cdots g_{j-1}$ is less than ε. Furthermore, a similar statement can be made after substituting C for D and D for C. Therefore, by standard inverse limit relationships [1] or [9], we conclude that H and K are topologically equivalent.

In a subsequent paper this author uses a further development of the techniques of this present paper in establishing a topological classification of all hereditarily indecomposable circularly chainable continua.

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