ON PROPERTIES OF SUBSPACES OF $l_p$, $0 < p < 1$

BY

W. J. STILES

Abstract. The material presented in this paper deals with some questions concerning projections, quotient spaces, and linear dimension in $l_p$ spaces, and also includes a remark about weak Schauder bases in $l_p$ spaces and an example of an infinite-dimensional closed subspace of $l_p$ which is not isomorphic to $l_p$.

1. Introduction. For $p > 0$, let $l_p$ be the set of all real sequences, $(a_n)$, such that $\sum |a_n|^p < \infty$. It is well known that $l_p$ is a complete linear metric space with paranorm given by $\sum |a_n|^p$ when $0 < p < 1$, and that $l_p$ is a Banach space with norm given by

$$\left( \sum |a_n|^p \right)^{1/p} \quad \text{when } p \geq 1.$$

A great deal is known about the structure of $l_p$ spaces for $p \geq 1$. Perhaps not quite so much is known about these spaces for $0 < p < 1$. We plan to discuss here properties of subspaces of the latter spaces and show that in most cases these properties are quite different from those of the normed spaces.

The paper will be divided into five sections. §1 will contain some comments and definitions which might be helpful to the reader as well as a summary of results. §2 will deal mainly with the concept of complemented subspaces of $l_p$, $0 < p < 1$. We will show in this section that each $l_p$ space is isomorphic to all of its subspaces of finite codimension, that each $l_p$ space contains a subspace isometrically isomorphic to $l_p$ no infinite-dimensional subspace of which is complemented in $l_p$, and that if $l_p$ is isometrically isomorphic to one of its subspaces which has the Hahn-Banach extension property, then this subspace is complemented in $l_p$. §3 will contain an example of a subspace of $l_p$ which is not isomorphic to $l_p$. §4 will contain some results about subspaces of $l_p$ which are obtained as kernels of linear mappings. In particular, we will show that each $l_p$, $0 < p < 1$, contains a closed proper subspace such that any continuous linear functional on $l_p$ which vanishes on this subspace vanishes on all of $l_p$. We will also show that $l_p$ contains a closed subspace which is not contained in any proper complemented subspace of $l_p$. Finally, §5 will contain some results on linear dimension, complementing those known for $l_p$, $p \geq 1$.

Most of our terminology is standard; however, a few remarks probably are in order. We use the word norm to denote the $l_p$ paranorm when $0 < p < 1$, and we
use the standard notation for a norm to denote this paranorm—even though it is $p$-homogeneous ($\|ra\| = |r|^p \|a\|$), and not homogeneous in the usual sense. In cases of possible ambiguity, we will denote the $p$-norm by $\| \|_p$. We will say that a subspace, $X$, of $l_p$ has the Hahn-Banach extension property if each continuous linear functional on $X$ can be extended to a continuous linear functional on all of $l_p$. We will let $\{e_i\}$ denote the unit vector bases in $l_p$, i.e., $e_i = (0, \ldots, 0, 1, 0, \ldots)$, we will write $X \cong Y$ to denote that $X$ is linearly isomorphic to $Y$, and we will use the known properties of the space $(X_1 \oplus X_2 \oplus \cdots)_X$ given in [9]. Finally, we will say that two bases, $\{x_n\}$ and $\{y_i\}$, are equivalent bases if $\sum a_i y_i$ converges if and only if $\sum a_i x_i$ converges.

2. Complemented subspaces. It is known [9] that each infinite-dimensional closed subspace of $l_p$, $p \geq 1$, contains an infinite-dimensional subspace which is complemented in $l_p$, or in the terminology of Whitley [12], $l_p$, $p \geq 1$, is subprojective. That the situation is considerably different when $0 < p < 1$ will be shown in Theorem 2.3. We begin by proving a theorem which is basically not new (see [3]) but whose proof contains estimates which are essential to our later work.

**Theorem 2.1.** If $X$ is a closed infinite-dimensional subspace of $l_p$, $0 < p < 1$, then $X$ contains a subspace isomorphic to $l_p$.

**Proof.** Since $X$ is infinite dimensional, $X$ contains a sequence $\{b_n\}$ such that $\|b_n\| = 1$ and each $b_n$ is of the form

$$b_n = (0, \ldots, 0, b_{k_n}^{n}, b_{k_n+1}^{n}, \ldots),$$

where $k_n$ can be chosen arbitrarily large. Select $b_n$ such that

$$\sum_{k = k_n+1}^{\infty} |b_k^n|^p < \frac{1}{2^{n+1}},$$

and define the sequence $\{C_n\}$ such that

$$C_n = (0, \ldots, 0, b_{k_n}^{n}, \ldots, b_{k(n+1)-1}^{n}, 0, \ldots).$$

We note the $\{C_n\}$ is a basic sequence equivalent to the unit vector basis in $l_p$. Indeed, this follows immediately from the following:

$$\left\| \sum_{n=1}^{\infty} \lambda_n C_n \right\| = \sum_{n=1}^{\infty} |\lambda_n|^p \sum_{k=k_n}^{k(n+1)-1} |b_k^n|^p \leq \sum_{n=1}^{\infty} |\lambda_n|^p,$$

while

$$\left\| \sum_{n=1}^{\infty} \lambda_n C_n \right\| \geq \frac{1}{2} \sum_{n=1}^{\infty} |\lambda_n|^p.$$
We also note that \( \{b_n\} \) is a basis equivalent to \( \{C_n\} \). This follows from the following:

\[
\sum_{n=1}^{m} \lambda_n (b_n - C_n) = \sum_{n=1}^{m} \lambda_n (0, \ldots, 0, b_{n+1}^m, \ldots) \\
\leq \sum_{n=1}^{m} |\lambda_n|^p \frac{1}{2^{n+1}} \leq \frac{1}{2} \sum_{n=k}^{k+1} |\lambda_n|^p \sum_{k=n}^{\infty} |b_k|^p \\
= \frac{1}{2} \sum_{n=1}^{m} \lambda_n C_n.
\]

Hence \( \{b_n\} \) is a basis for a subspace of \( X \) which is isomorphic to \( l_p \).

**Corollary 2.2.** The space \( l_p, 0 < p < 1 \), contains no infinite-dimensional subspace isomorphic to a Banach space.

**Proof.** If \( B \) were a Banach space isomorphic to a subspace of \( l_p \), then by the previous theorem, \( l_p \) would be isomorphic to a subspace of \( B \). Since \( l_p \) contains no bounded convex neighbourhood, this is impossible.

**Theorem 2.3.** For each \( p, 0 < p < 1 \), \( l_p \) contains a subspace, \( Y \), isometrically isomorphic to \( l_p \) such that no infinite-dimensional subspace of \( Y \) is complemented in \( l_p \).

**Proof.** Let \( Y \) be the subspace of \( l_p \) whose basis elements, \( b_n \), are given by

\[
b_1 = (1, 0, \ldots) \\
b_2 = (0, 1/2^{1/p}, 1/2^{1/p}, 0, \ldots) \\
b_3 = (0, 0, 0, 1/3^{1/p}, 1/3^{1/p}, 0, \ldots) \\
\vdots
\]

We will show that no infinite-dimensional subspace of \( Y \) has the Hahn-Banach extension property, and this will prove the theorem.

Let \( z \) be an infinite-dimensional subspace of \( Y \). We note that the sequence

\[
(\pm 1, \pm 2^{1/p-1}, \pm 2^{1/p-1}, \pm 3^{1/p-1}, \pm 3^{1/p-1}, \pm 3^{1/p-1}, \ldots)
\]

represents a continuous linear functional on \( Y \) and hence on \( Z \) for any choice of signs. We denote this functional, when the signs are all positive, by \( f \), and for notational convenience, we let \( f \) be denoted by the sequence \( (f_1, f_2, \ldots) \), i.e., \( f_1 = 1, f_2 = 2^{1/p-1} \), etc. We choose a sequence, \( \{z_n\} \), of unit vectors in \( Z \) as follows.

\[
z_1 = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \ldots, \alpha_{n_2}) \\
z_2 = (0, 0, 0, \alpha_{n_3}, \alpha_{n_3} + n_3, \ldots, \alpha_{n_4}, \ldots, \alpha_{n_4} + n_4, \ldots) \\
z_k = (0, \ldots, 0, \alpha_{n_{2k-1}}, \ldots, \alpha_{n_{2k-1}} + n_{2k-1}, \ldots, \alpha_{2k}, \ldots, \alpha_{2k} + n_{2k}, \ldots)
\]
where we have chosen our notation to indicate that a "block" begins at \( n_{2k} \) and ends at \( n_{2k} + n'_{2k} \). Choose \( z \), arbitrarily and then choose \( n_\delta \) such that
\[
\sum_{j=1}^{n_\delta} |a_j| |f_j| - \sum_{j=n_\delta + n'_\delta + 1}^{\infty} |a_j| |f_j| \geq \frac{1}{2} \sum_{j=1}^{\infty} |a_j|.
\]
Having chosen \( z_{k-1} \) and \( n_{2k-2} \), choose \( z_k \) so that \( n_{2k-1} > n_{2k-2} + n'_{2k-2} \). Then choose \( n_{2k} \) so that
\[
\sum_{j=1}^{n_{2k} + n'_{2k}} |a_j| |f_j| - \sum_{j=n_{2k} + n'_{2k} + 1}^{\infty} |a_j| |f_j| \geq \frac{K^{1/p-1}}{2} \sum_{j=1}^{\infty} |a_j|.
\]
The above choice of \( n_{2k} \) is clearly possible because of the forms of the linear functionals involved and because of the fact that our choice of \( n_{2k-1} \) has required us to "skip" at least one block at each step of the process.

We now let "sign" denote the function such that \( \text{sign } r = r/|r| \) if \( r \neq 0 \) and \( \text{sign } 0 = 0 \), and we let \( g \) be the continuous linear functional on \( Z \) whose representation is given by
\[
(f_1 \text{ sign } a_1, \ldots, f_{n_3-1} \text{ sign } a_{n_3-1}, f_{n_3} \text{ sign } a_{n_3}, \ldots, f_{n_5-1} \text{ sign } a_{n_5-1}, f_{n_6} \text{ sign } a_{n_6}, \ldots).
\]
Suppose that \( g \) has a continuous extension to \( l_p \). Then there is a linear functional, \( h \), on \( l_p \) which agrees with \( g \) on the sequence \( \{z_n\} \). Let \( (m_1, m_2, \ldots) \) be a bounded sequence which is the representation of \( h \), and suppose that \( \sup |m_j| \leq M \). Then
\[
g(z_k) \geq \frac{K^{1/p-1}}{2} \sum_{j=1}^{\infty} |a_j| \geq \frac{K^{1/p-1}}{2M} |h(z_k)|,
\]
and since \( 0 < p < 1 \), this is clearly impossible.

**Proposition 2.4.** Suppose that \( X \) is a closed subspace of \( l_p \), \( 0 < p < 1 \), such that \( X \) contains no subspace which is both complemented and isomorphic to \( l_p \). Then given any \( \varepsilon > 0 \) there exists an integer \( N \) such that \( n \geq N \) and \( a = (0, \ldots, 0, a_n, a_{n+1}, \ldots) \in X \) with \( \|a\| \leq 1 \) implies \( \sum_{j=n}^{n_\delta} |a_j| \leq \varepsilon \).

**Proof.** Suppose that there exists some \( \varepsilon > 0 \) such that for any \( N \) we can find a vector \( a = (0, \ldots, 0, a_n, a_{n+1}, \ldots) \in X \) with \( \|a\|_p = 1 \), \( \|a\|_1 \geq \varepsilon \) and \( n \geq N \). We construct a sequence \( \{a_n\} \) of \( p \)-unit vectors in \( X \) of the form
\[
a_1 = (a_1^1, a_2^2, \ldots, a_{n_\delta}^2, \ldots)
\]
\[
a_2 = (0, a_1^3, a_2^3, a_{n_\delta+1}^3, \ldots, a_{n_4}^3, \ldots)
\]
\[
\ldots \ldots \ldots \ldots \ldots
\]
\[
a_k = (0, \ldots, 0, a_{n_{2k-1}}^k, \ldots, a_{n_{2k}}^k, \ldots)
\]
\[
\ldots \ldots \ldots \ldots \ldots
\]
in the following manner. Choose \( a_1 \) arbitrarily. Then choose \( n_2 \) such that
\[
\sum_{j=1}^{n_2} |a_j| \leq \frac{\varepsilon}{2} \quad \text{and} \quad \sum_{j=n_2 + 1}^{\infty} |a_j|^p \leq \frac{1}{2} \left( \frac{\varepsilon}{2} \right)^p.
\]
Having chosen \( a_{k-1} \) and \( n_{2k-2} \), choose \( a_k \) such that \( n_{2k-1} > n_{2k-2} \). Then select \( n_{2k} \) such that
\[
\sum_{j=n_{2k-1}}^{n_{2k}} |a_j^p| \geq \frac{\varepsilon}{2^k} \quad \text{and} \quad \sum_{j=n_{2k}+1}^{\infty} |a_j^p| < \frac{1}{2^{k+1}} \left( \frac{\varepsilon}{2} \right)^p.
\]
Since \( \varepsilon \leq 1 \), the last inequality implies that
\[
\sum_{j=n_{2k}+1}^{\infty} |a_j^p| < \frac{1}{2^{k+1}}.
\]
Hence, according to the calculations contained in the proof of Theorem 2.1, the sequence \( \{b_n\} \) given by
\[
b_1 = (a_1, \ldots, a_{n_{2k-1}}, 0, \ldots)
\]
\[
b_2 = (0, \ldots, 0, a_{n_{2k+1}}, a_{n_{2k+2}}, 0, \ldots)
\]
\[\vdots\]
\[
b_k = (0, \ldots, 0, a_{n_{2k-1}}, \ldots, a_{n_{2k}}, 0, \ldots)
\]
is a basic sequence equivalent to the unit vector basis in \( l_p \), and, furthermore, the sequence \( \{a_n\} \) is a basic sequence in \( X \) equivalent to the unit vector basis in \( l_p \). If \( Y \) denotes the subspace spanned by \( \{b_k\} \), it is easy to see that the mapping \( P \) defined by
\[
P(x) = \sum_{j=1}^{\infty} f_j(x) b_j,
\]
where \( f_j \) is the linear functional corresponding to the sequence
\[
(0, \ldots, 0, m_{n_{2k-1}}, \ldots, m_{n_{2k}}, 0, \ldots),
\]
where each \( m_j \) may be chosen so that \( \sup_i |m_j| \leq 2/\varepsilon \) and \( f_k(b_k) = 1 \), is a continuous projection of \( l_p \) onto \( Y \).

Let \( A \) be the linear mapping defined by
\[
A(x) = x - P(x) + \sum_{j=1}^{\infty} f_j(P(x)) a_j.
\]
\( A \) is a well-defined continuous mapping since \( \{a_j\} \) and \( \{b_j\} \) are equivalent bases. We will show that \( A \) is a one-to-one mapping of \( l_p \) onto itself, and this will imply that \( A \) is bicontinuous by the open mapping theorem. A minor calculation shows that
\[
A(x) = x + \sum_{k=1}^{\infty} f_k(x)(0, \ldots, 0, a_{n_{2k+1}}, \ldots),
\]
and from this it is easy to deduce that \( A \) is one-to-one. To see that \( A \) maps onto \( l_p \), let \( y \) be any arbitrary element such that \( \|y\|_p = 1 \). We will determine an \( x \) in \( l_p \).
such that $Ax = y$. Let $x_j = y_j$ for $j = 1, \ldots, n_2$. Then, let

$$x^{n_2} = (x_1, x_2, \ldots, x_{n_2}, 0, \ldots),$$

and let $x_j = y_j - f_1(x^{n_2})a_j^1$ for $j = n_2 + 1, \ldots, n_4$. Having chosen $x_1, x_2, \ldots, x_{n_2}$, let

$$x^{n_2k} = (0, \ldots, 0, x_{n_2k-2+1}, \ldots, x_{n_2k}, 0, \ldots),$$

and let

$$x_j = y_j - \sum_{r=1}^{k} f_r(x^{n_r})a_j^r$$

for $j = n_2k + 1, \ldots, n_2k + 2$. We will show that $x = (x_1, x_2, \ldots)$ is in $l_p$ and this will complete the proof that $A$ is an isomorphism.

Note that $||x^{n_2}||_p \leq 1$. Hence $||f_1(x^{n_2})|| \leq (2/e)||x^{n_2}||_1 \leq 2/e$. This implies that

$$||x^{n_2}||_p \leq 1 + (2/e)^p \|(0, \ldots, 0, a_{n_2+1}, \ldots, a_{n_4}, 0, \ldots)\|_p$$

$$\leq 1 + (2/e)^p(2)^{p/2} \leq 1 + \frac{1}{2} < 2.$$ 

Hence $||x^{n_2}||_1 \leq 2^{1/p}$, and this implies that $|f_2(x^{n_2})| \leq (2/e)2^{1/p}$. Assuming that $|f_2(x^{n_2})| \leq (2/e)2^{1/p}$ for $j = 2, \ldots, k$, we see that

$$||x^{n_2k}||_p \leq 1 + \sum_{r=1}^{k} |f_r(x^{n_2r})|^p \|(0, \ldots, 0, a_{n_{2r+1}}, \ldots, a_{n_{2r+2}}, 0, \ldots)\|_p$$

$$\leq 1 + \left(\frac{2}{e}\right)^p \left(\frac{2}{e}\right)^p \frac{1}{2} + \sum_{r=2}^{k} \left(\frac{2}{e}\right)^{2+1} \frac{1}{2} \left(\frac{2}{e}\right)^p \leq 2,$$

and this implies that $||x^{n_2k+2}||_1 \leq 2^{1/p}$ which in turn implies that $|f_{k+1}(x^{n_2k+2})| \leq (2/e)2^{1/p}$. Thus, this last inequality holds for all $k \geq 1$. Using this fact, one can show very easily that $||x||_p \leq 2$.

Since the isomorphism, $A$, maps the space spanned by $\{b_k\}$ onto the space spanned by $\{a_k\}$, the mapping $Q$ given by $Q = APA^{-1}$ is a continuous projection of $l_p$ onto $X$.

**Lemma 2.5.** If $X$ is a complemented subspace of $l_p$, $0 < p < 1$, and $X$ contains a subspace $Y$ which is both complemented in $X$ and isomorphic to $l_p$, then $X$ is isomorphic to $l_p$.

**Proof.** See Proposition 4 of [9].

**Theorem 2.6.** For each $p$, $0 < p < 1$, $l_p$ is isomorphic to all of its subspaces of finite codimension.

**Proof.** It suffices to show that $l_p$ is isomorphic to all of its hyperplanes. Let $X$ be a hyperplane in $l_p$. Since $X$ is complemented in $l_p$, if $X$ is not isomorphic to $l_p$, $X$ contains no complemented subspace isomorphic to $l_p$ by Lemma 2.5. Thus Proposition 2.4 applies. Suppose that $l_p = Rx \oplus X$ where $Rx$ is the space spanned
by the vector \( x = (x_1, x_2, \ldots) \). Then there are vectors of the form \((0, \ldots, 0, a_n, 0, \ldots)\) in \( x + X \) for each \( n \) with \( a_n \neq 0 \). Thus, we can find two vectors of the form

\[
y = (-x_1, \ldots, -x_{n-1}, -x_n + a_n, -x_{n+1}, \ldots)
\]

and

\[
z = (-x_1, \ldots, -x_n, -x_{n+1} + a_n, -x_{n+2}, \ldots)
\]

in \( X \) for each choice of \( n \). Since \( y - z = (0, \ldots, 0, a_n, -a_n + 1, 0, \ldots) \in X \), we can find \( p \)-unit vectors in \( X \) with \( l_1 \) norms greater than \( 2^{1-1/p} \) in direct contradiction to Proposition 2.4.

**Lemma 2.7.** If \( 0 < p < 2 \) and \( \xi \) and \( \eta \) are real numbers, then

\[
|\xi + \eta|^p + |\xi - \eta|^p \leq 2(|\xi|^p + |\eta|^p)
\]

and equality holds only when \( \xi \) or \( \eta \) is zero.

**Proof.** See [11].

**Theorem 2.8.** If a subspace \( X \) of \( l_p, 0 < p < 1 \), is isometrically isomorphic to \( l_p \) and has the H-B-extension property, then \( X \) is complemented in \( l_p \).

**Proof.** Let \( T \) be the isometry, and let \( Te_i = f_i \). It is easy to see from Lemma 2.6 that \( f_i \) and \( f_j \) have disjoint supports when \( i \neq j \). Since \( \{f_i\} \) is a basis for \( X \) equivalent to \( \{e_i\} \), one can define a continuous linear functional \( h \) on \( X \) such that \( h(f_i) = 1 \). If \( \|f_i\|_1 \to 0 \) for some subsequence, \( h \) cannot be extended to \( l_p \). Hence \( \|f_i\|_1 \geq \epsilon \) for all \( i \) for some \( \epsilon > 0 \). We can now define a projection of \( l_p \) onto \( X \) as we did in the proof of Proposition 2.4.

**Remark.** It is easy to see that if \( X \oplus Y = l_p, 0 < p < 1 \), then \( X^c \oplus Y^c = l_1 \) where \( X^c, Y^c \) are the closures, in \( l_1 \), of \( X \) and \( Y \). It also follows that \( X^* \) and \( Y^* \) are isomorphic to \( m \) since \( X^* \) and \( Y^* \) are isomorphic to complemented subspaces of \( m \) which must be isomorphic to \( m \) [7]. It is not true, however, that the conjugate of every closed infinite-dimensional subspace of \( l_p \) is isomorphic to \( m \). This will be shown in the proof of Theorem 3.1. We note in closing this section that it is easy to show that every finite-dimensional subspace of \( l_p \) is complemented.

3. A subspace not isomorphic to \( l_p \). For each \( p, 1 \leq p < 2, l_p \) contains an infinite-dimensional subspace which is not isomorphic to \( l_p \) (see [5] and [9]). Whether this situation persists for \( p > 2 \) is still unknown; however, it does persist for \( 0 < p < 1 \). We show this in the following theorem.

**Theorem 3.1.** For each \( p, 0 < p < 1, l_p \) contains an infinite-dimensional subspace, \( X \), which is not isomorphic to \( l_p \).

**Proof.** Let \( \varphi_n(t) = \text{sign} \sin (2^n \pi t), n = 0, 1, \ldots, \) be the Rademacher functions for \( t \) in \([0, 1]\). By Khinchine’s inequality (see Paley [8]), given any \( p > 0 \), there exist constants \( B \) and \( C \) such that

\[
B(\sum |a_n|^2)^{p/2} \leq \int_0^1 \left| \sum a_n \varphi_n(t) \right|^p dt \leq C(\sum |a_n|^2)^{p/2}.
\]
For a given integer \( n \), divide the unit interval into \( 2^n \) equal intervals and then subdivide one of these intervals into infinitely many subintervals of length \( 2^{-(n+1)} \), \( 2^{-(n+2)} \), \ldots. We now embed \( l_p \) in \( L_p \) in the usual manner by constructing the appropriate scalar multiple of the characteristic function on each of the intervals. The embedding, \( T_n \), of \( l_2 \) into \( L_p \) given by

\[
T_n(a_1, \ldots, a_n) = \sum_{k=1}^{n} a_k \phi_k
\]

is also an embedding of \( l_2 \) into \( l_p \) because of the way we have embedded \( l_p \) in \( L_p \). Let \( M_n \) denote the image of \( l_2 \) in \( l_p \) under \( T_n \). Khinchine’s inequality yields

\[
\|T_n x\|_p \leq \|x\|_2 C \quad \text{and} \quad B \|x\|_2 \leq \|T_n x\|_p.
\]

Therefore

\[
\|T_n x\|_p \leq \|x\|_2^p C^p \quad \text{and} \quad B^p \|x\|_2 \leq \|T_n x\|_p^p.
\]

These inequalities imply that the mapping of

\[
R = (l_2 \oplus l_2 \oplus l_2 \oplus \cdots)_{l_2}
\]

onto

\[
(m_1 \oplus m_2 \oplus \cdots)_{l_p}
\]

is an isomorphism of \( R \) into \( (l_2 \oplus l_2 \oplus \cdots)_{l_p} \) and this last space is isomorphic to \( l_p \). If \( R \) is isomorphic to \( l_p \), then \( R^* \) must be isomorphic to \( m \), the space of all bounded sequences. However \( R^* \) is isomorphic to \( (l_2 \oplus l_2 \oplus \cdots)_{m} \) and it has been shown by Lindenstrauss in [5] that this space is not isomorphic to \( m \).

4. Subspaces which are kernels of mappings. It is well known that any separable Banach space is the image of \( l_1 \) under a continuous linear mapping. This statement has its analogue for \( 0 < p < 1 \), and the kernel of this mapping is a subspace of \( l_p \) which has some interesting properties. We say that a linear topological space is locally bounded if it has a bounded neighborhood of zero.

Theorem 4.1. Every separable locally bounded F-space is isomorphic to a quotient space of \( l_p \) for some \( p \) in \( (0, 1) \).

Proof. Aoki [1] and Rolewicz [10] have shown that a \( p \)-homogeneous norm, \( \|\cdot\|_p \), can be defined in a locally bounded F-space for every \( p \) satisfying \( 0 < p < \log C(X)^2 \) where \( C(X) \) is the modulus of concavity of the space \( X \). Let \( \{x_n\} \) be any countable collection of points in \( X \) which is dense in the unit sphere, \( S_X = \{x \in X : \|x\|_p = 1\} \), of \( X \), and define a mapping \( T \) of \( l_p \) into \( X \) as follows. Let \( \{e_i\} \) be the unit vector basis in \( l_p \), and let \( T e_1 = x_1 \). Extend \( T \) linearly to the span of \( \{e_i\} \). Since

\[
\left\| T \left( \sum_{k=1}^{n} \lambda_k e_k \right) \right\|_p \leq \sum_{k=1}^{n} |\lambda_k|^p \left\| x_k \right\|_p \leq \left\| \sum_{k=1}^{n} \lambda_k e_k \right\|,
\]

(1) After submitting this paper for publication, the author discovered that Theorem 4.1 was contained in J. H. Shapiro’s doctoral dissertation (University of Michigan, 1969).
T is continuous and can be extended continuously to all of $l_p$. Given any point $x$ in $S_X$, one can construct a series in $l_p$ which converges to a point whose image is $x$. Hence the mapping $T$ is onto $X$.

**Corollary 4.2.** For each $p$, $0 < p < 1$, and for each $q$, $q \geq p$, $l_q$ and $L_q$ are isomorphic to quotient spaces of $l_p$.

**Proof.** This follows immediately from the proof of the preceding theorem because of the fact that

$$
\left\| T\left( \sum_{k=1}^{n} \lambda_k e_k \right) \right\|_q \leq \sum_{k=1}^{n} |\lambda_k|^q \leq \left( \sum_{k=1}^{n} |\lambda_k|^p \right)^{q/p},
$$

if $p \leq q < 1$, and

$$
\left\| T\left( \sum_{k=1}^{n} \lambda_k e_k \right) \right\|_q \leq \sum_{k=1}^{n} |\lambda_k| \leq \left( \sum_{k=1}^{n} |\lambda_k|^p \right)^{1/p},
$$

if $q \leq 1$.

**Theorem 4.3.** For each $p$, $0 < p < 1$, $l_p$ contains a closed proper subspace, $X$, such that any continuous linear functional in $l_p$ which vanishes on $S$ vanishes on all of $l_p$.

**Proof.** Choose $X$ such that $l_p/X$ is isomorphic to $L_p$. Since $L_p$ contains no nonzero continuous linear functionals, there can be no nonzero linear functional in $l_p$ which vanishes on $X$.

**Corollary 4.4.** For each $p$, $0 < p < 1$, $l_p$ contains a closed proper subspace which is weakly dense in $l_p$.

**Corollary 4.5.** For each $p$, $0 < p < 1$, $l_p$ contains a weak Schauder basis which is not a basis.

**Proof.** Let $X$ be the subspace given in Corollary 4.3. It is easy to see that $X$ is a dense subspace of $l_1$. Hence, by a well-known theorem of Krein, Milman, and Rutman, $l_1$ has a basis $\{b_n\}$ where $b_n$ is in $X$ for each $n$. This last condition implies that $\{b_n\}$ cannot be a basis for $l_p$ while $\{b_n\}$ is clearly a weak Schauder basis for $l_p$.

**Theorem 4.6.** For each $p$, $0 < p < 1$, $l_p$ contains a closed subspace, $X$, which is not contained in any proper complemented subspace.

**Proof.** Let $X$ be the kernel of a continuous linear mapping of $l_p$ onto $L_p$, and suppose $X \subset Y$ where $Y$ is a subspace complemented in $l_p$. If $Y \oplus Y_0 = l_p$, then $Y/X \oplus (X + Y_0)/X \approx L_p$. But $(X + Y_0)/X \approx Y_0$ and $Y_0$, being a subspace of $l_p$, has nonzero continuous linear functionals. This means that $L_p$ also has nonzero continuous linear functionals which is not the case.

**Theorem 4.7.** For each $p$, $0 < p < 1$, $l_p$ contains a subspace, $X$, such that $X \oplus Y$ fails to be complete for all infinite-dimensional subspaces $Y$ of $l_p$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Proof. Let $X$ be the kernel of any continuous mapping of $l_p$ onto $l_1$. If $X \oplus Y$ is complete then $Y$ is isomorphic to a subspace of $l_1$, and this is impossible.

5. Linear dimension. If $X$ and $Y$ are linear topological spaces, then $\dim_l X \leq \dim_l Y$ if and only if $X$ is isomorphic to a subspace of $Y$. If this is the case, we will say that $X$ has linear dimension less than that of $Y$. If this is not the case, we will write $\dim_l X \not\leq \dim_l Y$. If neither $X$ can be embedded in $Y$ nor $Y$ embedded in $X$, we will say that they are incomparable. These ideas date back to Banach [2], and the problem of linear dimension for $l_1$ and $L_p$, $p \geq 1$, spaces has now been completely solved (see [6] and the references cited there). We will examine the case when $p > 0$.

If $f \in L_p$, let $S_p^e = \{t \in [0, 1] : |f(t)| \geq \epsilon \|f\|_p\}$ and say that $f \in M_p^e$ if and only if $|S_p^e| \geq \epsilon$ where $|S_p^e|$ denotes the Lebesgue measure of $S_p^e$. It has been shown by Kadec and Pełczyński in [4] that if $X$ is a subspace of $L_p$, $1 \leq p < \infty$, and $X \nsubseteq M_p^e$ for any $\epsilon > 0$, then $X$ contains a subspace isomorphic to $l_p$. We use these ideas in the following.

**Theorem 5.1.** Suppose that $p, q > 0$. If $p \neq q$, then $\dim_l l_p$ and $\dim_l l_q$ are incomparable; $\dim_l l_1 \leq \dim_l L_p$; $\dim_l L_p \leq \dim_l l_q$ implies $p = q = 2$; $\dim_l l_2 \not\leq \dim_l L_q$ if $p < q$; $\dim_l L_p \leq \dim_l L_q$ implies $p \geq q$; and $\dim_l L_p \leq \dim_l L_q$ if $q \leq p$ and $1 < p \leq 2$.

**Proof.** Suppose that $0 < p < q < 1$ and that $\dim_l l_p \not\leq \dim_l l_q$. Then $l_q$ contains a bounded basic sequence $\{f_n\}$ equivalent to the unit vector basis in $l_p$. Since the sequence $\{f_n\}$ is bounded, the series $\sum a_n f_n$ converges for all sequences $(a_n)$ in $l_q$. This is a contradiction which implies $\dim_l l_p \not\leq \dim_l l_q$. If $\dim_l l_q \not\leq \dim_l l_p$, then by Theorem 2.1, $l_q$ contains a subspace isomorphic to $l_p$ which was just shown to be impossible.

$\dim_l l_2 \leq \dim_l L_p$ follows immediately from Khinchine's inequality given in the proof of Theorem 3.1.

If $0 < p, q < 1$, then clearly $\dim_l L_p \not\leq \dim_l l_q$ since $L_p$ has no continuous linear functionals. The other cases are either well known or are obvious.

If $0 < p < q < 1$, then $\dim_l L_p \not\leq \dim_l L_q$ follows from the argument given in the first part of this proof. Since $l_p$ can be isometrically embedded in $L_p$, this implies that $\dim_l L_p \not\leq \dim_l L_q$ when $p < q$.

If $1 < p < 2$, select a number $r$ such that $1 < r < p$ and $q < r$. Then $L_p$ is isomorphic to a subspace $X$ of $L_r$ (see [6]), and since $l_1$ is not isomorphic to a subspace of $L_p$, $X \nsubseteq M_r^e$ for some $\epsilon > 0$. This implies that $X$ is a closed subspace of $L_q$ and so $\dim_l L_p \leq \dim_l L_q$.

We are unable to settle the remaining case which we leave as a problem: If $0 < p < 1$ and $q > p$ is $\dim_l L_q \leq \dim_l l_p$ or is $\dim_l l_q \leq \dim_l L_p$?

We note in conclusion that a stronger result than that contained in the last part of the preceding theorem can be obtained. This is done in the following.

**Theorem 5.2.** For each $p$, $1 < p \leq 2$, $L_p$ is isomorphic to a subspace of $M$, the space of all measurable functions on $[0, 1]$ with topology given by convergence in measure.
Proof. For a given $p$, we choose an $r$ as in the last part of the proof of the preceding theorem. Then if $L_p \simeq X$ where $X$ is a subspace of $L_r$, $X \subset M_\varepsilon^r$ for some $\varepsilon > 0$. The natural mapping of $X$ (as a subspace of $L_r$) into $M$ is continuous and one-to-one and the inverse of this mapping is also continuous since $X \subset M_\varepsilon^r$. This shows that $X$, as a subspace of $M$, is isomorphic to $L_p$.

REFERENCES


Florida State University,
Tallahassee, Florida 32306