ELEMENTS WITH TRIVIAL CENTRALIZER 
IN WREATH PRODUCTS

BY
WOLFGANG P. KAPPE AND DONALD B. PARKER(*)

Abstract. Groups with self-centralizing elements have been investigated in recent papers by Kappe, Konvisser and Seksenbaev. In particular, if \( G = A \wr B \) is a wreath product some necessary and some sufficient conditions have been given for the existence of self-centralizing elements and for \( G = \langle S_\alpha \rangle \), where \( S_\alpha \) is the set of self-centralizing elements. In this paper \( S_\alpha \) and the set \( R_\alpha \) of elements with trivial centralizer are determined both for restricted and unrestricted wreath products. Based on this the size of \( \langle S_\alpha \rangle \) and \( \langle R_\alpha \rangle \) is found in some cases, in particular if \( A \) and \( B \) are \( p \)-groups or if \( B \) is not periodic.

1. Introduction. An element \( x \) is said to have trivial centralizer in the group \( G \) if \( \langle x, y \rangle \) is cyclic for each \( y \in c_\alpha x \). An element \( x \in G \) is self-centralizing in \( G \) if \( c_\alpha x = \langle x \rangle \). Clearly self-centralizing elements have trivial centralizer but the converse is not true. The existence of a self-centralizing element \( x \) has a profound effect on the structure of the group. For example, if \( x \) is self-centralizing in \( G \) then the center of \( G \) is cyclic since \( Z_1 G \leq c_\alpha x = \langle x \rangle \), and there are other less obvious relations between some of the invariants of the group [1], [2], [3]. In many cases \( G \) is generated by the set \( S_\alpha \) of all self-centralizing elements or the set \( R_\alpha \) of all elements with trivial centralizer in \( G \) [2]. For the particular case of restricted wreath products Seksenbaev [5] has given some necessary and some sufficient conditions for \( \langle S_\alpha \rangle = G \) and \( \langle R_\alpha \rangle = G \), mainly for finite \( p \)-groups of odd order. In Theorem 1 of this paper we give a complete description of \( S_\alpha \) and \( R_\alpha \) for a wreath product \( G = A \wr B \) with nontrivial factors, and based on this we obtain the following results on the size of \( \langle S_\alpha \rangle \), \( \langle R_\alpha \rangle \) and a related group \( P_\alpha \) for the restricted wreath product \( G = A \wr B \) of \( A \neq 1 \) and \( B \neq 1 \).

**Theorem 2.** Define \( P_\alpha = \langle xy \mid x, y \in S_\alpha \rangle \) for any group \( H \). If \( A \) and \( B \) are \( p \)-groups, \( B \) cyclic and \( S_\alpha \neq \emptyset \) then

(a) \( G/\langle S_\alpha \rangle \cong A/A'P_\alpha \).

(b) \( |G:P_\alpha| = 2|A:A'P_\alpha| \) for \( p = 2 \).

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Suppose $B_i \neq 1$ are cyclic $p$-groups and let $A_1 = A$, $A_{i+1} = A_i \wr B_i$, $W = A_{k+1}$. Then

$$|W : P_w| = 2^k |A : A' P_A| \quad \text{for } p = 2 \quad \text{and} \quad W / \langle S_w \rangle \cong A / A' \langle S_A \rangle \quad \text{for } p \neq 2.$$  

**Theorem 3.** Let $B^*$ be the subgroup generated by the elements of infinite order in $B$ and let $A$ be the base group of $G = A \wr B$. If $B$ is not a torsion group then $\langle S_A \rangle = \overline{AB}^*$. 

**2. Definitions and notations.** Throughout this paper we will always assume that $A \neq 1$, $B \neq 1$ and $G = A \wr B$. Our notation for the standard wreath product essentially follows [4]. Let $F$ be the group of all functions on $B$ with values in $A$ and define $f^b \in F$ for $b \in B$ and $f \in F$ by $f^b(x) = f(xb^{-1})$ for all $x \in B$. The unrestricted wreath product $A \wr B$ is the semidirect product $F B$. The support of $f \in F$ is the set of all $x \in B$ with $f(x) \neq 1$. For a subgroup $H$ of $A$ we define

$$H = \{ f \in F \mid f(x) = 1 \text{ for all } x \in B \}$$

The restricted wreath product $A \wr B$ is the semidirect product $\overline{A}B$. The natural homomorphism of $A \wr B$ onto $A \wr B / \overline{A} \cong B$ is denoted by $\mu$. For each $a \in A$ we define a function $\gamma_a \in \overline{A}$ by $\gamma_a(1) = a$ and $\gamma_a(x) = 1$ for all $x \neq 1$ in $B$. The mapping $\gamma : a \mapsto \gamma_a$ is then an isomorphic embedding of $A$ in $\overline{A}$ and $A \wr B$ is generated by $A^\gamma$ and $B$.

For a given element $bf \in A \wr B$ with $b \in B$ and $f \in \overline{A}$ we define an element $h_x \in A$ for each $x \in B$ by

$$h_x = f(xb^i) \quad \text{if } b \text{ has infinite order},$$

$$h_x = f(xb)f(xb^2) \cdots f(xb^{|b|}) \quad \text{if } b \text{ has finite order}.$$ 

For any integer $i$ we have $(bf)^i = b^i f^{i+1} \cdots f$. In particular, if $b$ has finite order let $d = (bf)^{|b|} \in \overline{A}$. Then

$$d(x) = f(xb^{|b|-1}) \cdots f(x) = h_x \quad \text{for all } x \in B.$$ 

**3. Some preliminary results.** In the following lemma some useful information about elements commuting with a fixed element $bf \in G = A \wr B$ is collected.

**Lemma 1.** Let $b, c \in B$ and $g, f \in \overline{A}$. Denote by $T$ a left transversal of $\langle b \rangle$ in $B$.

(i) $[cg, bf] = 1$ if and only if $g = f^{-1} g^bf$ and $[c, b] = 1$.

(ii) If $b$ has infinite order then $[g, bf] = 1$ implies $g = 1$.

(iii) Suppose $b$ has infinite order and $c \in cb$. There exists an element $g \in \overline{A}$ such that $[c^{-1} g, bf] = 1$ if and only if $h_t = h_i$ for all $t \in T$.

(iv) Assume $b$ has finite order. For any $a \in cAh_i$ and $t \in T$ define $g_t \in \overline{A}$ inductively by

$$g_t(0) = a,$$

$$g_t(tb^i) = g_t(tb^i - 1) f(0b) \quad \text{for } 0 < i < |b|,$$

$$g_t(x) = 1 \quad \text{for } x \notin t\langle b \rangle.$$
Then \([g, bf] = 1\).

(v) If \(h_t = 1\) for all \(t \in T\) then \(bf\) and \(b\) are conjugate.

(vi) Assume \(g, g^* \in \overline{A}\) commute with \(bf\), \(b\) has finite order and \(g(t) = g^*(t)^\delta\) for some integer \(\delta\). Then \(g(x) = g^*(x)^\delta\) for all \(x \in t\langle b \rangle\).

**Remark 1.** For later applications in §5 it should be noted that (i), (iv) and (v) provided \(b\) has finite order hold also in the unrestricted wreath product \(X = A \odot Wr B\).

**Proof.** (i) \(1 = [cg, bf] = [c, b] \mod A \) and \(B \cap \overline{A} = 1\) imply \([c, b] = 1\). Hence
\[
1 = [cg, bf] = g^{-1}c^{-1}f^{-1}b^{-1}cbf = g^{-1}c^{-1}f^{-1}cb^{-1}g = g^{-1}f^{-cg}bf.
\]
Conversely if \([c, b] = 1\) and \(g = f^{-cg}bf\) then \([cg, bf] = g^{-1}f^{-cg}bf = g^{-1}g = 1\).

(ii) If \([g, bf] = 1\) then by (i), \(g(x) = f^{-1}(xg(xb^{-1})f(x))\) and by iteration \(g(x)\) is conjugate to \(g(xb^i)\) for all \(x \in B\) and all integers \(i\). Since \(g\) has finite support there is some \(x_t\) in each coset \(t\langle b \rangle\) such that \(g(x_t) = 1\) hence \(g = 1\).

(iii) Suppose there is some \(g \in A\) such that \([c^{-1}g, bf] = 1\). From (i) we have \(g(x) = f^{-1}(xg(xb^{-1})f(x))\) for all \(x \in B\) hence by iteration for all \(t \in T\) and all integers \(j > 0\)
\[
g(tb^j) = f^{-1}(tb^{j+1}) \cdots f^{-1}(tb^{-1}c)g(tb^{-1})f(tb^{-j+1}) \cdots f(tb^j).
\]
Since \(g\) and \(f\) have finite support and \(b\) has infinite order there exists an integer \(N \geq 0\) such that \(1 = f(tb^j) = f(tb^j) = g(tb^j)\) for all \(t \in T\) and all \(j\) with \(|j| \geq N\). Thus if \(c \in c_{b}b\) we have
\[
1 = g(tb^j) = f^{-1}(tcb^j) \cdots f^{-1}(tcb^{-1}c)g(tb^{-1})f(tb^{-j+1}) \cdots f(tb^j)
= h_t^{-1}h_t \quad \text{for all } t \in T.
\]
Conversely assume \(h_t = h_t\) for all \(t \in T\) and let \(N \geq 0\) be an integer such that \(f(tb^j) = f(tb^j) = 1\) for all \(t \in T\) and all \(j\) with \(|j| \geq N\). Define a function \(g : B \to A\) inductively by
\[
g(tb^{j+1}) = 1 \quad \text{for all } t \in T \text{ and all } j \geq N,
\]
\[
g(tb^j) = f^{-1}(tb^{j+1})g(tb^j) \quad \text{for } i > -N.
\]
By construction we have \(g(x) = f^{-1}(xg(xb^{-1})f(x))\) for all \(x \in B\) and \([c, b] = 1\) by assumption, hence \([c^{-1}g, bf] = 1\) by (i). To show \(g \in \overline{A}\) observe that for \(j \geq N\) we have \(g(tb^j) = g(tb^j) = h_t^{-1}g(tb^{-1})h_t = h_t^{-1}h_t = 1\).

Further
\[
S = \{ t \mid t \in T \text{ and } f(z) \neq 1 \text{ for some } z \in t\langle b \rangle \}
\]
and
\[
S^* = \{ t \mid t \in T \text{ and } f(z) \neq 1 \text{ for some } z \in tc\langle b \rangle \}
\]
are finite sets since \(f(z) \neq 1\) for only finitely many \(z \in B\) and \(Tc\) is also a left transversal of \(\langle b \rangle\) in \(B\) if \([c, b] = 1\). Thus \(g(z) = 1\) unless \(z\) belongs to the finite set of elements of the form
\[
sb^j \quad (s \in S; \ |j| \leq N), \quad s^*b^{j} \quad (s^* \in S^*; \ |j| \leq N).
\]
This proves \(g \in \overline{A}\) and hence (iii).
(iv) By (i) we have to prove $g_t(x) = f^{-1}(x)g_t(xb^{-1})f(x)$ for all $x \in B$. This is immediate from the definition for $x \notin \langle b \rangle$ and $x = tb^i$ with $0 < i < |b|$. For $x = t = tb^{b_1}$ we have inductively

$$f^{-1}(tb^{b_1})g_t(tb^{b_1})f(tb^{b_1}) = \cdots = g_t(t)f(tb^{b_1})f^{-1}(tb^{b_1}) = g_t(t)^{hi} = g_t(t),$$

since $g_t(t) = a \in c_t h_t$. Thus $g_t(x) = f^{-1}(x)g_t(xb^{-1})f(x)$ for all $x \in B$.

(v) If $b = 1$ then $1 = h_i = f(t)$ for all $t \in T$, hence $f = 1$ and $1 = bf = f$. Hence we may assume $b \neq 1$ and if $b$ has finite order define $k \in A$ inductively for all $t \in T$ by

$$k(t) = 1, \quad k(tb^i) = f^{-1}(tb^i)k(tb^{i-1})$$

for $0 < i < |b|$. Further for $t = tb^{b_1}$ we have

$$k^{-1}(tb^{b_1})f(t)k(t) = k^{-1}(tb^{b_1})f(t)k(tb^{b_1}) = f^{-1}(tb^{b_1})k(t).$$

Hence $k^{-1}(tb^i)k(t) = h_i = 1$ and $k^{-1}fk = 1$.

If $b$ has infinite order there is some $N$ such that $f(tb^i) = 1$ for all $t \in T$ and all integers $|i| \geq N$. Defining $k \in A$ by $k(tb^i) = 1$ for $i \geq N$ and

$$k(tb^i) = f^{-1}(tb^i)k(tb^{i-1})$$

for $i > -N$ we will have $k(tb^i) = f^{-1}(tb^i) \cdots f^{-1}(tb^{-N})$, hence for all $i > N$

$$k(tb^i) = f^{-1}(tb^{i+1}) = (f(tb^{-N}) \cdots f(tb^i))^{-1} = h_i^{-1} = 1.$$ 

This proves $k \in A$ and $k(tb^i) = f^{-1}(tb^i)k(tb^{i-1})$ for all $i$, and hence $(k^{-1}fk)(tb^i) = k^{-1}(tb^{i-1})f(tb^i)k(tb^i) = 1$ for all integers $i$ and all $t \in T$.

We have now $k^{-1}fk = 1$ in both cases, and

$$b = (k^{-1}fk)b = b^{-1}k^{-1}bfkb = (bf)^{kb}$$

shows that $b$ and $bf$ are conjugate in $G$.

(vi) From Lemma 1(i) we get

$$g(x) = f^{-1}(x)g(xb^{-1})f(x) \quad \text{and} \quad g^*(x) = f^{-1}(x)g^*(xb^{-1})f(x)$$

and inductively that for each $x \in \langle b \rangle$ there is some $a \in A$ such that $g(x) = g(t)^a$, $g^*(x) = g^*(t)^a$. Hence $g(x) = g(t)^a = g^*(t)^{ba} = (g^*(t)^a)^{b^a} = g^*(x)^a$.

**Lemma 2.** (a) A finite abelian group $H = \langle a_i, \ldots, a_k, w \rangle$ is cyclic if $\langle a_i, w \rangle$ is cyclic for $i = 1, \ldots, k$ and $(|a_i|, |a_j|) = 1$ for all $i \neq j$.

(b) The finite abelian group $H = \langle u, v \rangle$ is cyclic provided one of the following conditions is satisfied:

(i) $H$ has a subgroup $W$ such that $H/W$ is cyclic and $(|u|, |W|) = 1$.

(ii) There is some integer $\alpha \neq 0$ such that $\langle u^\alpha, v^\alpha \rangle$ is cyclic and $(|H/\langle u \rangle|, \alpha) = 1$.

(c) The abelian group $H = \langle u, v \rangle$ with $u \neq 1, v \neq 1$ is cyclic if and only if there exist integers $\alpha, \gamma$ with $(\alpha, \gamma) = 1$ and $u^\alpha = v^\gamma$. 

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Proof. (a) follows immediately from the main theorem for finite abelian groups. We note also the consequence of the main theorem: if \( m \) and \( n \) are integers with \( (m, n) = 1 \) then \( G \) is cyclic if and only if \( G^n = \langle g^n \mid g \in G \rangle \) and \( G^n = \langle g^n \mid g \in G \rangle \) are cyclic. Apply with \( m = |W|, n = |u| \) for (i), with \( m = |H/\langle u \rangle|, n = \alpha \) for (ii) and with \( m = \alpha, n = \gamma \) for (c). For the converse in (c) let \( H = \langle w \rangle, u = w^\sigma, v = w^\tau \) and \( \omega \) the least common multiple of \( \sigma \) and \( \tau \). Then \( \omega = \sigma\lambda = \tau\mu \) with \( (\lambda, \mu) = 1 \) and \( u^\lambda = w^\omega = v^\omega \).

Lemma 3. Let \( g, f \in \overline{A}, 1 \neq b \in B \) and \( g \neq 1 \). If \( \langle g, bf \rangle \) is cyclic then \( b \) has finite order and there are integers \( \alpha, \beta \) with \( (\alpha, |b|\beta) = 1 \) and \( g^\alpha = d^\beta \), where \( d = (bf)^{|b|} \).

Proof. By Lemma 2(c) there are integers \( \alpha, \gamma \) with \( (\alpha, \gamma) = 1 \) and \( g^\alpha = (bf)^\gamma \). Since \( g \in \overline{A} \) and \( \langle b \rangle \cap \overline{A} = 1 \) this implies that \( b \) has finite order and \( |b| \) divides \( \gamma \), hence \( \gamma = |b|\beta \).

4. Determination of \( S_G \) and \( R_G \). We now state and prove

Theorem 1. Let \( f \in \overline{A} \) and \( b \in B \) such that \( bf \neq 1 \) and let \( T \) be a left transversal of \( \langle b \rangle \) in \( B \).

(a) The element \( bf \) has trivial centralizer in \( G = A \wr B \) if and only if one of the following conditions is satisfied:

(1) \( b \) has infinite order.

(2) For each element \( c \in c_Bb \) satisfying \( h_{tc} = h_t \) for all \( t \in T \) the subgroup \( \langle c, b \rangle \) is cyclic.

(b) The element \( bf \) is self-centralizing in \( G \) if and only if one of the following conditions is satisfied:

(4) \( b \) has infinite order.

(5) \( c \in \langle b \rangle \) if and only if \( c \in c_Bb \) and \( h_{tc} = h_t \) for all \( t \in T \).

(5.1) \( b \) has finite order.

(5.2) \( h_t \) is self-centralizing in \( A \) for all \( t \in T \).

(5.3) If \( B \neq \langle b \rangle \), then \( h_t \) has finite order for all \( t \in T \) and \( (|h_t|, |y|) = 1 \) for all \( s \neq t \) in \( T \).
Remark 2. It should be noted that in cases (3) and (5) the group $B$ is actually finite. Indeed, since $f$ has finite support only finitely many $h_t = f(t) b \cdots f(t) b^{(b)}$ are nontrivial, hence (3.2) or (5.2) imply that $T$ and hence $B$ is finite.

Proof. (1) Suppose $b f$ has trivial centralizer, $b$ has infinite order and $c \in c_{gb}$ satisfies $h_t = h_t$ for all $t \in T$. From Lemma 1(iii) follows the existence of some $g \in A$ such that $[c^{-1} g, b f] = 1$ and $b f \in R_0$ implies $\langle c^{-1} g, b f \rangle$ is cyclic. Let $H = \langle c, b \rangle$. Then $H^* = \langle c^{-1} g, b f \rangle^*$ is cyclic and $H^* = H A / A \cong H / H \cap A$. But $H \cap A \subseteq B \cap A = 1$ so $H \cong H^*$ is cyclic.

Conversely suppose $b$ and $f$ satisfy conditions (1.1) and (1.2) and $w \in G$ commutes with $b f$, where $w = c^{-1} k$ for some $c \in B$ and $k \in A$. Lemma 1(iii) gives $c \in c_{gb}$ and $h_t = h_t$ for all $t \in T$ from Lemma 1(iii), hence $\langle c, b \rangle$ is cyclic by (1.2). Let $K = \langle c^{-1} k, b f \rangle$. Then $K^* = \langle c, b \rangle^*$ is cyclic and $K^* = K A / A \cong K / K \cap A$. But $[g, b f] = 1$ for all $g \in K \cap A$ since $K$ is abelian and so from Lemma 1(ii) we have $K \cap A = 1$. Hence $K \cong K^*$ is cyclic.

(2) Suppose $b$ has finite order and $h_t = 1$ for all $t \in T$. By assumption $bf \neq 1$. Then $b \neq 1$ implies $f(t) = h_t = 1$ for all $t \in T$, hence $f(1) = 1 = b f$. To prove (2.3) observe that $b$ is conjugate to $b f \in R_B$ by Lemma 1(v) and hence $b \in B \cap R_A \subseteq R_B$.

Finally for (2.4) let $a \in A$ and let $x \in T$ define $k_t \in A$ by

$$k_t(x) = a \quad \text{for } x \in t \langle b \rangle, \quad k_t(x) = 1 \quad \text{for } x \notin t \langle b \rangle.$$ 

Then $[k_t, b] = 1$ by construction and $b \in R_A$ implies $\langle k_t, b \rangle$ cyclic. Hence $k_t$ and $a$ have finite order since $b \neq 1$ has finite order. If $r = (|a|, |b|)$ then $r = (|k_t|, |b|)$ and there are subgroups of order $r$ in $\langle k_t, b \rangle \cap A$, $\langle b \rangle$ and $\langle b \rangle$. But there is only one subgroup $H$ of order $r$ in $\langle k_t, b \rangle$ since $\langle k_t, b \rangle$ is cyclic hence $H \subseteq H = \langle k_t \rangle \cap \langle b \rangle \subseteq A \cap B = 1$. This proves $r = (|a|, |b|)$ and thus (2.4).

Conversely assume conditions (2) are satisfied. Since $b$ and $bf$ are conjugate by (2.2) and Lemma 1(v) it is sufficient to show $b \in R_A$. Let $g \in A$ and $c \in B$ be such that $[c g, b] = 1$. Then $[g, b] = 1 = [c, b]$ by Lemma 1(i) and $\langle c, b \rangle$ is a finite cyclic group by (2.3) and (2.1). Hence if $K = \langle c g, b \rangle$ then $K / K \cap A \cong K A / A = K^* = \langle c, b \rangle^*$ is finite cyclic, $K \cap A$ is finite abelian by (2.4) and $|K \cap A| = 1$. If $|A| = 1$. Thus $K$ is finite and Lemma 2(i) shows $K$ cyclic.

(3) Assume $b$ has finite order and $h_t \neq 1$ for some $t \in T$. Suppose $h_t = 1$ for some $s \in T$. Then $s = t$ and $h_t \in c_{h_t} = A$. Define $g_s \in A$ as in Lemma 1(iv) with $g_s(s) = h_t$. Then $\langle g_s, b f \rangle$ is abelian, hence cyclic and $g_s^* = d \theta$ with $(a, \beta) = 1$ by Lemma 3. This gives $h_t^* = 1$ for the argument $s$ and $1 = h_t^*$. Hence cyclic since $(a, \beta) = 1$ and $h_t \neq 1$.

To prove (3.2) let $a \in c_{h_t} h_t$ and define $g_t \in A$ as in Lemma 1(iv) such that $g_t(t) = a$. Then $[b f, g_t] = 1$ and $\langle b f, g_t \rangle$ is cyclic since $b f \in R_A$. Hence also the subgroup $H = \langle d, g_t \rangle$ is cyclic, $H = \langle h_t \rangle$ with $h_t \in A$, $d = h_t$ and $g_t = h_t^*$. So $h_t = d(t) = h(t)^t$, $a = g_t(t) = h(t)^t$ which proves that $\langle h_t, a \rangle$ is cyclic for all $a \in c_{h_t} h_t$.

To prove (3.3) we may assume $b \neq 1$ since the condition is vacuous for $b = 1$. 

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Defining $g_t \in \overline{A}$ as in Lemma 1(iv) for $1 \neq a \in c_A h_t$ with $g_t(t) = a$ we have $[g_t, bf] = 1$ and hence $\langle g_t, bf \rangle$ is cyclic. From Lemma 3 we obtain $g_t^a = d^a$ with $(a, b|\beta|) = 1$. For the argument $t$ this gives $a^t \in \langle h_t \rangle$, hence $c_A h_t/\langle h_t \rangle$ is periodic and $(a, b|\beta|) = 1$ proves (3.3).

Finally for (3.4) assume $B \neq \langle b \rangle$ and let $1 \neq y \in c_A h_t$. Define $1 \neq g_t \in \overline{A}$ as in Lemma 1(iv) with $g_t(t) = y$. Then $\langle g_t, bf \rangle$ is abelian, hence cyclic, and $g_t^a = d^a$ with $(a, \beta|\beta|) = 1$ by Lemma 3. For the argument $s \neq t$ this gives $1 = h_s^b$, hence $h_s$ has finite order for all $s \in T$. For the argument $t$ we get $y^a = h_t^b$. Since $(a - \beta|\beta|) = 1$ this implies $(|h_s|, |h_t|) = 1$ for $y = h_t \in c_A h_t$. But $|y|$ divides $a|h_t|$ for all $y \in c_A h_t$, hence $(|h_t|, |y|)$ divides $(|h_t|, a|h_t|) = (\beta, a|\beta|) = 1$.

Conversely assume that conditions (3) are satisfied, let $cg \in c_G(bf)$ and suppose $c \notin \langle b \rangle$. Then also $[cg, d] = 1$ where $d = (bf)^{|b|}$, and Lemma 1(i) gives $d = g^{-1}d^g$ hence $d(x) = d(xc^{-1})^g(x)$ for all $x \in B$. Since $[d, bf] = 1$ we have also

$$d(x) = d(xc^{-1})^g(x)$$

for all $x \in B$, so $d(xc^{-1})$ is conjugate to $d(z)$ for each $z \in x\langle b \rangle$. Let $tc^{-1} = sb^\beta$ with $s \in T$ and an integer $\beta$. Then $h_t = d(s)$ and $h_t = d(t)$ are conjugate, so (3.4) implies $s = t$, hence $c \in \langle b \rangle$, say $c = b^i$ for some integer $i$. Since $cg(bf)^{-1} \circ \overline{A}$ and $\langle cg(bf)^{-1}, bf \rangle = \langle cg, bf \rangle$ we see that in order to prove that $bf$ has trivial centralizer in $G$ it suffices to show that $\langle g, bf \rangle$ is cyclic for each $g \in \overline{A}$ with $[g, bf] = 1$. Lemma 2(ii) with $u = bf$, $v = g$, $a = |b|$ and $d = u^a$ gives just that provided that we can show

(a) $\langle d, g \rangle$ is cyclic;

(b) $\langle bf, g \rangle/\langle bf \rangle$ is a $p'$-group for all primes $p$ dividing $|b|$.

Since $\langle bf \rangle \cap \overline{A} = \langle (bf)^{|b|} \rangle = \langle d \rangle$ then

$$\langle bf \rangle \cap \langle g \rangle = \langle bf \rangle \cup \langle \overline{A} \rangle \cap \langle g \rangle = \langle d \rangle \cap \langle g \rangle.$$  

Then

$$\langle bf, g \rangle/\langle bf \rangle \simeq \langle g \rangle/\langle bf \rangle \cap \langle g \rangle = \langle g \rangle/\langle d \rangle \cap \langle g \rangle$$

shows that (b) can be replaced by

(b*) $\langle g \rangle/\langle d \rangle \cap \langle g \rangle$ is a $p'$-group for each prime $p$ dividing $|b|$.

To show (a) assume first $B = \langle b \rangle$. Then $d(t) = h_t$ and (3.2) gives $\langle d(t), g(t) \rangle$ is cyclic, say $k_t = d(t)^a g(t)^b$, $d(t) = k_t^i$ and $g(t) = k_t^j$ with integers $\alpha, \beta, i, j$ and $k_t \in A$. Let $k = d^a g^b \in \overline{A}$ and observe that $d, g$ and $k$ commute with $bf$. From Lemma 1(vi) we get $d = k^l$ and $g = k^l$, so $\langle d, g \rangle = \langle k \rangle$ is cyclic.

For $B \neq \langle b \rangle$ the elements $h_t, g(s)$ have finite order for all $s \in T$ by (3.4). Since $g$ commutes with $bf$ we have from Lemma 1(i) that $g(s)$ and $g(x)$ are conjugate if $x \in s\langle b \rangle$. For each $t \in T$ define $d_t \in \overline{A}$ by $d_t(x_0) = d(x)$ for $x_0 \in t\langle b \rangle$, $d(x) = 1$ for $x \notin t\langle b \rangle$. Then $d_t$ commutes with $d_t, g$ and $bf$, only finitely many $d_t$ are nontrivial and $d$ is the product of all $d_t$. Let $\Pi$ be the set of all primes dividing $|h_t|$ and decompose $\langle g \rangle = \langle m \rangle \times \langle n \rangle$ with $m, n \in \overline{A}$ such that $\langle m \rangle$ is a II-group and $\langle n \rangle$ is a $\Pi'$-group. Since $(|g(s)|, |h_t|) = 1$ by (3.4) for $s \neq t$ and $|g(x)| = |g(s)|$ for $x \in s\langle b \rangle$
it follows that \( m(x) = 1 \) for \( x \notin \langle b \rangle \). By (3.2) the abelian group \( \langle m(t), d(t) \rangle \) \( \leq \langle g(t), h_i \rangle \) is cyclic, say \( \langle m(t), d(t) \rangle = \langle k_i \rangle \) with \( k_i = m(t)^a d(t)^b \). Let \( k = m^n d^n \), and observe that \( m \in \langle g \rangle \), \( d \), and \( k \) commute with \( bf \). Since \( m(x) = d(x) = k(x) = 1 \) for \( x \notin \langle b \rangle \) we get then from Lemma 1(vi) that \( \langle m, d \rangle = \langle k \rangle \) is a cyclic \( \Pi \)-group. Hence \( \langle d, g \rangle = \langle d, m \rangle \times \langle n \rangle \) is cyclic. Finally \( (|d_s|, |d_t|) = 1 \) for \( s \neq t \) in \( T \) by (3.4), so Lemma 2(a) can be applied to prove (a).

To show (b*) assume first \( B = \langle b \rangle \) and \( p \) divides \( |b| \). Then by (3.3) there is some integer \( i \) such that \( (p, i) = 1 \) and \( g(t)^i \in \langle h_i \rangle \), say \( g(t)^i = d(t)^i \). But Lemma 1(vi) gives \( g(x)^i = d(x)^i \) for all \( x \in B \) so \( g^i \in \langle d \rangle \cap \langle g \rangle \).

If \( B \neq \langle b \rangle \), then \( c_a h_i \) is periodic for all \( t \in T \) by (3.4). Suppose there is some prime \( p \) dividing \( |B| \) and some \( k \in \langle g \rangle \) such that \( k^p \in \langle d \rangle \). In particular \( k(t) \in \langle h_i \rangle \) for all \( t \in T \) by (3.3), say \( k(t) = h_i^p \) with integers \( a_i \). Now \( (|h_s|, |h_t|) = 1 \) for \( s \neq t \) in \( T \) by (3.4), and the Chinese remainder theorem gives the existence of an integer \( a \) such that

\[
\alpha \equiv \alpha_t \mod |h_i| \quad \text{for all } t \in T.
\]

Finally \( d(t) = h_i \), so Lemma 1(vi) shows \( k(x)^i = d(x)^i \) for all \( x \in B \). Hence there is no element of order \( p \) in \( \langle g \rangle \cap \langle d \rangle \), which proves (b*).

(4) Assume now that \( b' \) is a self-centralizing element in \( G \) and \( b \) has infinite order. If \( c \in \langle b \rangle \) then \( c \in c_b g \) and clearly \( h_w = h_x \) for all \( x \in B \). Assume \( c \in c_b g \) satisfies \( h_w = h_x \) for all \( t \in T \). There exists \( g \in \bar{A} \) by Lemma 1(iii) such that \( [c^{-1}g, b'] = 1 \) hence \( c^{-1}g \in \langle b' \rangle \) since \( b' \) is self-centralizing. If \( c^{-1}g = (b')^j \) then in particular \( c^{-1} = b' \) so \( b \) is self-centralizing.

Conversely, assume (4.1) and (4.2) are satisfied and there are \( c \in B \) and \( g \in \bar{A} \) such that \( [c^{-1}g, b'] = 1 \). Then \( [c, b'] = 1 \) from Lemma 1(i) and Lemma 1(iii) together with (4.2) implies \( c = b^j \) with some integer \( j \). But \( k = (b')^{-t}(c^{-1}g) \in \bar{A} \) commutes with \( b' \), hence \( k = 1 \) by Lemma 1(ii) and so \( c^{-1}g \in \langle b' \rangle \).

(5) Suppose \( b \) has finite order and let \( a \in c_b h_i \). Define \( g_i \) as in Lemma 1(iv) with \( g_i(t) = a \). Then \( [g_i, b'] = 1 \) and by assumption \( g_i = (b')^j \) with some integer \( j \). But \( g_i \in \bar{A} \) implies that \( |b| \) divides \( i \) so \( i = |b|j \) and \( (b')^j = (b')^{|b|j} = d^j \). Hence \( a = g_i(t) = d(t)^j = h_i^j \) which proves (5.2).

Since a self-centralizing element has trivial centralizer we may apply the results of (2) and (3) to prove (5.3). We have \( c_b h_i = \langle h_i \rangle \) by (5.2) hence, \( h_i \neq 1 \) for all \( t \in T \) and condition (3.4) gives (5.3).

Conversely, assume (5) is satisfied, \( c \) commutes with \( b' \) and \( c \notin \langle b \rangle \). Then \( d = (b')^{|b|} \) commutes with \( c \) and \( b' \) so Lemma 1(i) implies

\[
d = f^{-1}d^bg \quad \text{and} \quad d = g^{-1}d^cg.
\]

In particular, if \( tc^{-1} = sb^\theta \) with \( s \in T \) and an integer \( \beta \), then \( h_i = d(t) \) is conjugate to \( d(s) = h_s \) and hence \( s = t \) by (5.3). But then \( c = b^{-\beta} \in \langle b \rangle \), a contradiction. Thus \( c \in \langle b \rangle \), say \( c = b^k \) and \( k = (b')^{-t}(cg) \in \bar{A} \) commutes with \( b' \) and \( d \). Since \( d(t) = h_i \) is self-centralizing by (5.2) there is an integer \( a_t \) for each \( t \in T \) such that \( k(t) = d(t)^{a_t} \).
From (5.3) we have \(|d(s)|, |d(t)| = 1\) for \(s \neq t\) in \(T\), so by the Chinese remainder theorem there is an integer \(\alpha\) with
\[
\alpha \equiv \alpha_t \mod |d(t)| \quad \text{for all} \quad t \in T.
\]
Then \(k(t) = d(t)^\alpha\) for all \(t \in T\) and Lemma 1(vi) implies \(k(x) = d(x)^\alpha\) for all \(x \in B\). Hence \(k \in \langle d \rangle \leq \langle b' \rangle\), and \(b'\) is self-centralizing.

**Corollary 1.** If \(B\) is a \(p\)-group and \(A\) is torsion-free or a \(p\)-group then \(1 \neq b'f \in R_G\) with \(b' \in B\) and \(f' \in A\) implies \(B = \langle b' \rangle\).

**Proof.** Since \(B\) is periodic we have only to consider conditions (2) and (3). Further under our hypothesis it is impossible to satisfy condition (2.4). Hence from (3.2) we get \(h_t \neq 1\) for all \(t \in T\). Thus \((|h_t|, |h_t|) \neq 1\) if \(A\) is a \(p\)-group, and hence \(B = \langle b' \rangle\) by (3.4). If \(A\) is torsion-free we observe that \(1 \neq h_t \in c_Ah_t\), hence \(1 \neq c_Ah_t \subseteq A\) is not periodic and \(B = \langle b' \rangle\) follows again from (3.4). For later applications we note the following.

**Corollary 2.** If \(A\) and \(B\) are \(p\)-groups then \(b'f \in S_G\) if and only if \(B = \langle b' \rangle\) and \(h_1 \in S_A\).

**Corollary 3.** If \(A\) and \(B\) are \(p\)-groups then
\begin{enumerate}
  \item \(\{1\} \cup S_G = R_G\).
  \item \(S_G \neq \emptyset\) if and only if \(B\) is cyclic and \(S_A \neq \emptyset\).
\end{enumerate}

**Proof.** (i) Clearly \(\{1\} \cup S_G \subseteq R_G\). So let \(1 \neq b'f \in R_G\). From Corollary 1 we have \(B = \langle b' \rangle\), hence \(b' \neq 1\) and from (3.3) we see \(c_Ah_t = \langle h_t \rangle\). Thus condition (5) is satisfied for \(b'f\) and hence \(b'f \in S_G\).

(ii) Suppose \(S_G \neq \emptyset\) and \(b'f \in S_G\). Then \(B = \langle b' \rangle\) and \(S_A \neq \emptyset\) since \(h_1 \in S_A\) by Corollary 2. Conversely assume \(B = \langle b' \rangle\), \(a \in S_A\) and let \(f = \gamma_a\). Then
\[
h_1 = f(b) \cdots f(b^{(b)}) = a \in S_A,
\]
and \(b'f \in S_G\) by Corollary 2, hence \(S_G \neq \emptyset\).

5. **The size of \(S_G\) and \(P_G\).** We introduce a new characteristic subgroup \(P_H\) which will be useful to compute \(S_G\) if \(G\) is an iterated wreath product.

**Definition.** \(P_H = \langle xy \mid x, y \in S_H \rangle\).

By definition \(P_H \leq \langle S_H \rangle\), and if \(H\) is a \(p\)-group, \(p \neq 2\) then \(\langle x \rangle = \langle x^2 \rangle\) shows \(P_H = \langle S_H \rangle\). The generalized quaternion groups are examples for \(P_H = \langle S_H \rangle\), while for the dihedral groups of \(2\)-power order \(P_H \neq \langle S_H \rangle\).

**Lemma 4.** Suppose \(A\) and \(B = \langle b' \rangle\) are \(p\)-groups and \(S_A \neq \emptyset\). Then
\begin{enumerate}
  \item \((G') \leq P_G\).
  \item \((A'P_A)' = A' \cap P_A = A' \cap \langle S_G \rangle\).
\end{enumerate}

**Proof.** (i) Let \(u, v, w \in A\), \(uw = 1\) and \(1 \neq c \in B = \langle b' \rangle\). If \(a \in S_A\) then \(\gamma_a \gamma_w^c b \in S_G\) by Corollary 2, hence
\[
\gamma_a \gamma_w^c = \gamma_a \gamma_w^c (\gamma_w^c)^{-1} = (\gamma_a \gamma_w^c b) (\gamma_w^c b)^{-1} \in P_G.
\]
But the elements $\gamma a^2$ generate $G'$ [4, Corollary 4.5, p. 350] hence $G' \subseteq P_G$.

(ii) If $a_1, a_2 \in S_A$ then $a_1 b \in S_A$ and $(a_2^{-1})^b b \in S_A$ by Corollary 2 thus $(a_1 a_2)^b = a_1 a_2 = a_1 b ((a_2^{-1})^b b)^{-1} \in P_G$ hence $P_A \subseteq P_G$. Since $(A')^G \subseteq G'$ and $G' \subseteq P_G$ by Lemma 4(i) we have $(A' P_A)^G \subseteq A' \cap P_G$.

Conversely let $h \in A' \cap P_G$. Then

$$h = (b_1, f_1) \cdots (b_r, f_r) \quad \text{where } r \text{ is even},$$

and $h \in A' \subseteq A$ we may rewrite

$$h = f^*_1 \cdots f^*_r,$$

with $f^*_1$ conjugate to $f_1$ under $B$. In particular $f^*_1(b_1) \cdots f^*_1(b_1^{b_1}) = f_1(b_1) \cdots f_1(b_1^{b_1}) = u_1 \mod A'$. Now $h(b_1) \cdots h(b_1^{b_1}) = \prod_{i=1}^r (f^*_1(b_1) \cdots f^*_1(b_1^{b_1})) = u_1 \cdots u_r \mod A'$, and so $h(b_1) \cdots h(b_1^{b_1}) \in A' P_A$ since $r$ is even. But $h \in A'$ implies $h(b)^i = 1$ for $0 < i < |b|$, so $h(1) = h(b_1^{b_1}) \in A' P_A$, or $h \in (A' P_A)^G$. This proves $(A' P_A)^G = A' \cap P_G$.

For $p \neq 2$ we have $P_G = \langle S_G \rangle$. To prove $A' \cap P_G = A' \cap \langle S_G \rangle$ we may hence assume $p = 2$. Suppose $h \in A' \cap \langle S_G \rangle$. By Corollary 2

$$h = (b_1, f_1) \cdots (b_r, f_r),$$

with $b_1 f_1 \in S_A$, $B = \langle b_1 \rangle = \cdots = \langle b_r \rangle$ and $f_i \in A$. Since $h \in A' \subseteq A$ and $h \equiv b_1 \cdots b_r \mod A$, we have $b_1 \cdots b_r \in \bar{A} \cap B = 1$. But $B = \langle b_1 \rangle$ and $p = 2$ imply that each $b_i$ is an odd power of $b$, hence $r$ is even because $b$ has even order. Thus $h \in P_G$, which proves

$$A' \cap \langle S_G \rangle = A' \cap P_G,$$

since trivially $P_G \subseteq \langle S_G \rangle$.

We can now prove Theorem 2 announced in the introduction.

Proof. (a) We first observe that $G = A' \langle S_G \rangle$ since $\langle A', b \rangle = G$ and $\gamma a b \in S_G$ by Corollary 2 for $B = \langle b \rangle$ and $a \in S_A$. Hence $G/\langle S_G \rangle \cong A'/A' \cap \langle S_G \rangle$, and $A' \cap \langle S_G \rangle = (A' P_A)^G$ by Lemma 4(ii) implies

$$G/\langle S_G \rangle \cong A/A' P_A.$$

(b) By Corollary 2 an element $x \in S_G$ has the form $x = b^* f$ with $f \in \bar{A}$ and $B = \langle b^* \rangle$ where $b^*$ is an odd power of $b$. In particular $P_G \subseteq \bar{A} \langle b^2 \rangle$. But $\gamma a b \in S_G$ for $a \in S_A$ so $b^2 \in AP_G$. Hence $\bar{A} P_G \subseteq \bar{A} \langle b^2 \rangle$ and

$$|G: \bar{A} P_G| = |G: \bar{A} \langle b^2 \rangle| = 2.$$

Further $A' P_G$ is normal in $G$ since $G' \subseteq A' P_G$ by Lemma 4(i). But $\bar{A}$ is generated by conjugates of $A'$, hence $\bar{A} P_G = A' P_G$ and $\bar{A} P_G / P_G = A'/A' \cap P_G \cong A/A' P_A$ by Lemma 4(ii). This proves

$$|G: P_G| = |G: \bar{A} P_G| |\bar{A} P_G / P_G| = 2 |A: A' P_A|.$$
(c) For \( k = 1 \) this follows directly from (a) and (b). We proceed by induction on \( k \). Let \( V = A_{k+2} = W \wr B_{k+1} \). Then for \( p = 2 \), \( |V: P_V| = 2|W: W'P_w| \) by (b) and \( W' \subseteq P_w \) by Lemma 4(i). By induction \( |W: P_w| = 2^k |A: A'P_A| \) hence \( |V: P_v| = 2^{k+1} |A: A'P_A| \). For \( p \neq 2 \) we have \( V/\langle S_v \rangle \cong W/\langle W'P_w \rangle \) by (a), \( P_w = \langle S_w \rangle \) and \( W' \subseteq P_w \) by Lemma 4(i), hence \( V/\langle S_v \rangle \cong W/\langle S_w \rangle \) and by induction

\[ V/\langle S_v \rangle \cong W/\langle S_w \rangle \cong A/\langle A'P_A \rangle. \]

Proof of Theorem 3. Let \( b \) be an element of infinite order in \( B \), \( 1 \neq a \in A \) and choose \( T \) so that \( 1 \in T \). We show first that if \( f = \gamma_a \), then the element \( bf \) satisfies condition (4.2) of Theorem 1. Observe that

\[
\begin{align*}
    h_x &= 1 \text{ for } x \notin \langle b \rangle, \\
    h_x &= a \text{ for } x \in \langle b \rangle.
\end{align*}
\]

Hence \( h_{tc} = h_t \) implies for \( t = 1 \) that \( h_{tc} \neq 1 \) and so \( c \in \langle b \rangle \). This proves by Theorem 1(4) that \( b \gamma_a \in S_G \) and a similar argument shows \( b^2 \gamma_a \in S_G \) hence both \( b \) and \( \gamma_a \) are contained in \( \langle S_G \rangle \). But this implies \( B^* \subseteq \langle S_G \rangle \) and \( A' \subseteq \langle S_G \rangle \). Since \( A \) is generated by the conjugates of \( A' \) and \( \langle S_G \rangle \) is normal in \( G \), we have \( \overline{AB}^* \subseteq \langle S_G \rangle \).

On the other hand, if \( bf \in S_G \) then \( b \in B^* \) for otherwise \( b \) has finite order and Remark 2 implies \( B \) is finite which contradicts the assumption that \( B \) is not a torsion group. Thus \( S_G \subseteq \overline{AB}^* \), hence \( \langle S_G \rangle = \overline{AB}^* \).

6. Unrestricted wreath products. The characterization of self-centralizing elements and elements with trivial centralizer is much easier for the unrestricted wreath product as the following theorem shows. In particular if \( A \) and \( B \) are \( p \)-groups it suffices to consider the restricted wreath product.

Theorem 4. Suppose \( B \) has infinite order and \( X = A \Wr B \) is the unrestricted wreath product of \( A \) and \( B \).

(a) Let \( b \in B \) and \( f \in F \) such that \( bf \neq 1 \), and \( T \) a left transversal of \( \langle b \rangle \) in \( B \). Then \( bf \) has trivial centralizer in \( X \) if and only if the following conditions are satisfied:

(i) \( b \) has finite order.

(ii) \( h_t = 1 \) for all \( t \in T \).

(iii) \( b \neq 1 \) has trivial centralizer in \( B \).

(iv) \( A \) is periodic and elements in \( A \) have order prime to \( |b| \).

(b) \( X \) has no self-centralizing elements.

Proof. (a) Suppose \( bf \in R_X \). If \( b \) has infinite order, let \( 1 \neq a \in A \) and define \( 1 \neq g \in F \) by

\[ g(1) = a, \quad g(x) = g(xb^{-1}) f(x) \quad \text{for } x \in \langle b \rangle \]

and \( g(x) = 1 \) for \( x \notin \langle b \rangle \). Then by construction \( g^{bf} = g \), so that \( \langle g, bf \rangle \) must be cyclic. This contradicts Lemma 3 and this proves (i).

Let \( 1 \neq a \in c_xh_t \) and define \( g_t \) as in Lemma 1(iv) with \( g_t(t) = a \). Then \( \langle g_t, bf \rangle \) is
cyclic and by Lemma 3 \( g_i^\alpha = d^\beta \) with integers \( \alpha, \beta \) such that \( (\alpha, |b|\beta) = 1 \). In particular
\[
(*) \quad a^\alpha = h_i^\beta, \quad 1 = h_i^\beta \quad \text{for all } s \neq t \in T.
\]
Hence \( h_s \) has finite order, \( c_A h_t \) is periodic and there is some integer \( m > 0 \) such that
\( h_s^m = 1 \) for all \( s \in T \). With \( a = h_t \) we see from (*) also that \( (|h_s|, |h_t|) = 1 \) since \( (\alpha - \beta, \beta) = 1 \). Since \( h_s^m = 1 \) for all \( s \in T \) this implies that there exists some \( t \in T \) with \( h_t = 1 \). Suppose \( h_s \neq 1 \) for some \( s \in T \). Then for \( a = h_s \in A = c_A h_t \) (*) implies \( h_s^a = 1 \) and \( 1 = h_s^a \)
a contradiction since \( (\alpha, \beta) = 1 \). This proves (ii).

To prove (iii) observe that \( b^f \) and \( b \) are conjugate by (ii) and Lemma 1(v). In particular \( b \neq 1 \) and \( b \in R_X \cap B \subseteq R_B \).

Finally from (ii) and (*) we get \( a^\alpha = 1 \) for \( a \in c_A h_t = A \) with \( (\alpha, |b|\beta) = 1 \) which proves (iv).

The sufficiency of conditions (i) to (iv) follows as in the proof of Theorem 1(2).

(b) Since \( S_X \subseteq R_X \) we get from condition (ii) of (a) that \( h_t = 1 \) for all \( t \in T \). Then Lemma 1(v) implies that \( b^f \) and \( b \) are conjugate, hence \( b \in S_X \). Then for some \( 1 \neq a \in A \) define \( k \in \mathbb{A} \subseteq F \) by \( k(x) = a \) for \( x \in \langle b \rangle \) and \( k(x) = 1 \) for \( x \notin \langle b \rangle \), and observe \( [k, b] = 1 \), but \( \langle b \rangle \cap F = B \cap F = 1 \).

**Corollary 4.** If \( A \) and \( B \) are \( p \)-groups then \( R_X = \{1\} \cup S_X \).

**References**


**State University of New York, Binghamton, New York 13901**

**University of Cincinnati, Cincinnati, Ohio 45221**