

ELEMENTS WITH TRIVIAL CENTRALIZER IN WREATH PRODUCTS

BY

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Abstract. Groups with self-centralizing elements have been investigated in recent papers by Kappe, Konvisser and Seksenbaev. In particular, if $G = A \text{ wr } B$ is a wreath product some necessary and some sufficient conditions have been given for the existence of self-centralizing elements and for $G = \langle S_G \rangle$, where S_G is the set of self-centralizing elements. In this paper S_G and the set R_G of elements with trivial centralizer are determined both for restricted and unrestricted wreath products. Based on this the size of $\langle S_G \rangle$ and $\langle R_G \rangle$ is found in some cases, in particular if A and B are p -groups or if B is not periodic.

1. Introduction. An element x is said to have trivial centralizer in the group G if $\langle x, y \rangle$ is cyclic for each $y \in c_G x$. An element $x \in G$ is self-centralizing in G if $c_G x = \langle x \rangle$. Clearly self-centralizing elements have trivial centralizer but the converse is not true. The existence of a self-centralizing element x has a profound effect on the structure of the group. For example, if x is self-centralizing in G then the center of G is cyclic since $Z_1 G \subseteq c_G x = \langle x \rangle$, and there are other less obvious relations between some of the invariants of the group [1], [2], [3]. In many cases G is generated by the set S_G of all self-centralizing elements or the set R_G of all elements with trivial centralizer in G [2]. For the particular case of restricted wreath products Seksenbaev [5] has given some necessary and some sufficient conditions for $\langle S_G \rangle = G$ and $\langle R_G \rangle = G$, mainly for finite p -groups of odd order. In Theorem 1 of this paper we give a complete description of S_G and R_G for a wreath product $G = A \text{ wr } B$ with nontrivial factors, and based on this we obtain the following results on the size of $\langle S_G \rangle$, $\langle R_G \rangle$ and a related group P_G for the restricted wreath product $G = A \text{ wr } B$ of $A \neq 1$ and $B \neq 1$.

THEOREM 2. Define $P_H = \langle xy \mid x, y \in S_H \rangle$ for any group H . If A and B are p -groups, B cyclic and $S_A \neq \emptyset$ then

- (a) $G / \langle S_G \rangle \cong A / A' P_A$.
- (b) $|G : P_G| = 2 |A : A' P_A|$ for $p = 2$.

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(c) Suppose $B_i \neq 1$ are cyclic p -groups and let $A_1 = A, A_{i+1} = A_i \text{ wr } B_i, W = A_{k+1}$. Then

$$|W:P_W| = 2^k |A:A'P_A| \text{ for } p = 2 \text{ and } W/\langle S_W \rangle \cong A/A'\langle S_A \rangle \text{ for } p \neq 2.$$

THEOREM 3. Let B^* be the subgroup generated by the elements of infinite order in B and let \bar{A} be the base group of $G = A \text{ wr } B$. If B is not a torsion group then $\langle S_G \rangle = \bar{A}B^*$.

2. Definitions and notations. Throughout this paper we will always assume that $A \neq 1, B \neq 1$ and $G = A \text{ wr } B$. Our notation for the standard wreath product essentially follows [4]. Let F be the group of all functions on B with values in A and define $f^b \in F$ for $b \in B$ and $f \in F$ by $f^b(x) = f(xb^{-1})$ for all $x \in B$. The unrestricted wreath product $A \text{ Wr } B$ is the semidirect product FB . The support of $f \in F$ is the set of all $x \in B$ with $f(x) \neq 1$. For a subgroup H of A we define

$$\bar{H} = \{f \in F \mid f(x) \in H \text{ for all } x \in B \text{ and } f \text{ has finite support}\}.$$

The restricted wreath product $A \text{ wr } B$ is the semidirect product $\bar{A}B$. The natural homomorphism of $A \text{ wr } B$ onto $A \text{ wr } B/\bar{A} \cong B$ is denoted by μ . For each $a \in A$ we define a function $\gamma_a \in \bar{A}$ by $\gamma_a(1) = a$ and $\gamma_a(x) = 1$ for all $x \neq 1$ in B . The mapping $\gamma: a \rightarrow \gamma_a$ is then an isomorphic embedding of A in \bar{A} and $A \text{ wr } B$ is generated by A^γ and B .

For a given element $bf \in A \text{ wr } B$ with $b \in B$ and $f \in \bar{A}$ we define an element $h_x \in A$ for each $x \in B$ by

$$h_x = \prod_{i=-\infty}^{+\infty} f(xb^i) \text{ if } b \text{ has infinite order,}$$

$$h_x = f(xb)f(xb^2) \cdots f(xb^{|b|}) \text{ if } b \text{ has finite order.}$$

For any integer i we have $(bf)^i = b^i \cdot f^{b^{i-1}} \cdots f$. In particular, if b has finite order let $d = (bf)^{|b|} \in \bar{A}$. Then

$$d(x) = f(xb^{1-|b|}) \cdots f(x) = h_x \text{ for all } x \in B.$$

3. Some preliminary results. In the following lemma some useful information about elements commuting with a fixed element $bf \in G = A \text{ wr } B$ is collected.

LEMMA 1. Let $b, c \in B$ and $g, f \in \bar{A}$. Denote by T a left transversal of $\langle b \rangle$ in B .

- (i) $[cg, bf] = 1$ if and only if $g = f^{-c}g^b f$ and $[c, b] = 1$.
- (ii) If b has infinite order then $[g, bf] = 1$ implies $g = 1$.
- (iii) Suppose b has infinite order and $c \in c_b b$. There exists an element $g \in \bar{A}$ such that $[c^{-1}g, bf] = 1$ if and only if $h_{tc} = h_t$ for all $t \in T$.
- (iv) Assume b has finite order. For any $a \in c_A h_t$ and $t \in T$ define $g_t \in \bar{A}$ inductively by

$$g_t(t) = a,$$

$$g_t(tb^i) = g_t(tb^{i-1})^{f^{(tb^i)}} \text{ for } 0 < i < |b|,$$

$$g_t(x) = 1 \text{ for } x \notin t\langle b \rangle.$$

Then $[g_t, bf] = 1$.

(v) If $h_t = 1$ for all $t \in T$ then bf and b are conjugate.

(vi) Assume $g, g^* \in \bar{A}$ commute with bf , b has finite order and $g(t) = g^*(t)^\delta$ for some integer δ . Then $g(x) = g^*(x)^\delta$ for all $x \in t\langle b \rangle$.

REMARK 1. For later applications in §5 it should be noted that (i), (iv) and (v) provided b has finite order hold also in the unrestricted wreath product $X = A \text{ Wr } B$.

Proof. (i) $1 = [cg, bf] \equiv [c, b] \pmod{\bar{A}}$ and $B \cap \bar{A} = 1$ imply $[c, b] = 1$. Hence

$$1 = [cg, bf] = g^{-1}c^{-1}f^{-1}b^{-1}cgbf = g^{-1}c^{-1}f^{-1}cb^{-1}gbf = g^{-1}f^{-c}g^bf.$$

Conversely if $[c, b] = 1$ and $g = f^{-c}g^bf$ then $[cg, bf] = g^{-1}f^{-c}g^bf = g^{-1}g = 1$.

(ii) If $[g, bf] = 1$ then by (i), $g(x) = f^{-1}(x)g(xb^{-1})f(x)$ and by iteration $g(x)$ is conjugate to $g(xb^i)$ for all $x \in B$ and all integers i . Since g has finite support there is some x_t in each coset $t\langle b \rangle$ such that $g(x_t) = 1$ hence $g = 1$.

(iii) Suppose there is some $g \in A$ such that $[c^{-1}g, bf] = 1$. From (i) we have $g(x) = f^{-1}(xc)g(xb^{-1})f(x)$ for all $x \in B$ hence by iteration for all $t \in T$ and all integers $j > 0$

$$g(tb^j) = f^{-1}(tb^j c) \dots f^{-1}(tb^{-j+1}c)g(tb^{-j})f(b^{-j+1}) \dots f(tb^j).$$

Since g and f have finite support and b has infinite order there exists an integer $N \geq 0$ such that $1 = f(tb^j) = f(tb^j c) = g(tb^j)$ for all $t \in T$ and all j with $|j| \geq N$. Thus if $c \in c_B b$ we have

$$\begin{aligned} 1 &= g(tb^N) = f^{-1}(tcb^N) \dots f^{-1}(tcb^{-N+1})g(tb^{-N})f(tb^{-N+1}) \dots f(tb^N) \\ &= h_{tc}^{-1}h_t \quad \text{for all } t \in T. \end{aligned}$$

Conversely assume $h_{tc} = h_t$ for all $t \in T$ and let $N \geq 0$ be an integer such that $f(tb^j) = f(tb^j c) = 1$ for all $t \in T$ and all j with $|j| \geq N$. Define a function $g: B \rightarrow A$ inductively by

$$\begin{aligned} g(tb^{-j}) &= 1 \quad \text{for all } t \in T \text{ and all } j \geq N, \\ g(tb^i) &= f^{-1}(tb^i c)g(tb^{i-1})f(tb^i) \quad \text{for } i > -N. \end{aligned}$$

By construction we have $g(x) = f^{-1}(xc)g(xb^{-1})f(x)$ for all $x \in B$ and $[c, b] = 1$ by assumption, hence $[c^{-1}g, bf] = 1$ by (i). To show $g \in \bar{A}$ observe that for $j \geq N$ we have $g(tb^j) = g(tb^N) = h_{tc}^{-1}g(tb^{-N})h_t = h_{tc}^{-1}h_t = 1$. Further

$$S = \{t \mid t \in T \text{ and } f(z) \neq 1 \text{ for some } z \in t\langle b \rangle\}$$

and

$$S^* = \{t \mid t \in T \text{ and } f(z) \neq 1 \text{ for some } z \in tc\langle b \rangle\}$$

are finite sets since $f(z) \neq 1$ for only finitely many $z \in B$ and Tc is also a left transversal of $\langle b \rangle$ in B if $[c, b] = 1$. Thus $g(z) = 1$ unless z belongs to the finite set of elements of the form

$$sb^j \quad (s \in S; |j| \leq N), \quad s^*b^j \quad (s^* \in S^*; |j| \leq N).$$

This proves $g \in \bar{A}$ and hence (iii).

(iv) By (i) we have to prove $g_t(x) = f^{-1}(x)g_t(xb^{-1})f(x)$ for all $x \in B$. This is immediate from the definition for $x \notin t\langle b \rangle$ and $x = tb^i$ with $0 < i < |b|$. For $x = t = tb^{|b|}$ we have inductively

$$f^{-1}(tb^{|b|})g_t(tb^{|b|-1})f(tb^{|b|}) = \dots = g_t(t)^{f(tb)\dots f(tb^{|b|})} = g_t(t)^{h_t} = g_t(t),$$

since $g_t(t) = a \in c_A h_t$. Thus $g_t(x) = f^{-1}(x)g_t(xb^{-1})f(x)$ for all $x \in B$.

(v) If $b = 1$ then $1 = h_t = f(t)$ for all $t \in T$, hence $f = 1$ and $1 = bf = f$. Hence we may assume $b \neq 1$ and if b has finite order define $k \in \bar{A}$ inductively for all $t \in T$ by

$$k(t) = 1, \\ k(tb^i) = f^{-1}(tb^i)k(tb^{i-1}) \quad \text{for } 0 < i < |b|.$$

Then $(k^{-b}fk)(tb^i) = k^{-1}(tb^{i-1})f(tb^i)k(tb^i) = 1$ for $0 < i < |b|$. Further for $t = tb^{|b|}$ we have $(k^{-b}fk)(t) = k^{-1}(tb^{|b|-1})f(t)k(t) = k^{-1}(tb^{|b|-1})f(t)$ and $k(tb^{|b|-1}) = f^{-1}(tb^{|b|-1}) \dots f^{-1}(tb)k(t)$. Hence $(k^{-b}fk)(t) = h_t = 1$ and $k^{-b}fk = 1$.

If b has infinite order there is some N such that $f(tb^i) = 1$ for all $t \in T$ and all integers $|i| \geq N$. Defining $k \in \bar{A}$ by $k(tb^{-i}) = 1$ for $i \geq N$ and

$$k(tb^i) = f^{-1}(tb^i)k(tb^{i-1})$$

for $i > -N$ we will have $k(tb^i) = f^{-1}(tb^i) \dots f^{-1}(tb^{-N})$, hence for all $i > N$

$$k(tb^i) = k(tb^{N+1}) = (f(tb^{-N}) \dots f(tb^N))^{-1} = h_i^{-1} = 1.$$

This proves $k \in \bar{A}$ and $k(tb^i) = f^{-1}(tb^i)k(tb^{i-1})$ for all i , and hence $(k^{-b}fk)(tb^i) = k^{-1}(tb^{i-1})f(tb^i)k(tb^i) = 1$ for all integers i and all $t \in T$.

We have now $k^{-b}fk = 1$ in both cases, and

$$b = (k^{-b}fk)b = b^{-1}k^{-1}bfbk = (bf)^{kb}$$

shows that b and bf are conjugate in G .

(vi) From Lemma 1(i) we get

$$g(x) = f^{-1}(x)g(xb^{-1})f(x) \quad \text{and} \quad g^*(x) = f^{-1}(x)g^*(xb^{-1})f(x)$$

and inductively that for each $x \in t\langle b \rangle$ there is some $a \in A$ such that $g(x) = g(t)^a$, $g^*(x) = g^*(t)^a$. Hence $g(x) = g(t)^a = g^*(t)^{\delta a} = (g^*(t)^a)^\delta = g^*(x)^\delta$.

LEMMA 2. (a) A finite abelian group $H = \langle a_1, \dots, a_k, w \rangle$ is cyclic if $\langle a_i, w \rangle$ is cyclic for $i = 1, \dots, k$ and $(|a_i|, |a_j|) = 1$ for all $i \neq j$.

(b) The finite abelian group $H = \langle u, v \rangle$ is cyclic provided one of the following conditions is satisfied:

(i) H has a subgroup W such that H/W is cyclic and $(|u|, |W|) = 1$.

(ii) There is some integer $\alpha \neq 0$ such that $\langle u^\alpha, v \rangle$ is cyclic and $(|H/\langle u \rangle|, \alpha) = 1$.

(c) The abelian group $H = \langle u, v \rangle$ with $u \neq 1, v \neq 1$ is cyclic if and only if there exist integers α, γ with $(\alpha, \gamma) = 1$ and $u^\alpha = v^\gamma$.

Proof. (a) follows immediately from the main theorem for finite abelian groups. We note also the following consequence of the main theorem: if m and n are integers with $(m, n) = 1$ then G is cyclic if and only if $G^m = \langle g^m \mid g \in G \rangle$ and $G^n = \langle g^n \mid g \in G \rangle$ are cyclic. Apply with $m = |W|$, $n = |u|$ for (i), with $m = |H/\langle u \rangle|$, $n = \alpha$ for (ii) and with $m = \alpha$, $n = \gamma$ for (c). For the converse in (c) let $H = \langle w \rangle$, $u = w^\sigma$, $v = w^\tau$ and ω the least common multiple of σ and τ . Then $\omega = \sigma\lambda = \tau\mu$ with $(\lambda, \mu) = 1$ and $u^\lambda = w^\omega = v^\mu$.

LEMMA 3. *Let $g, f \in \bar{A}$, $1 \neq b \in B$ and $g \neq 1$. If $\langle g, bf \rangle$ is cyclic then b has finite order and there are integers α, β with $(\alpha, |b|\beta) = 1$ and $g^\alpha = d^\beta$, where $d = (bf)^{|b|}$.*

Proof. By Lemma 2(c) there are integers α, γ with $(\alpha, \gamma) = 1$ and $g^\alpha = (bf)^\gamma$. Since $g \in \bar{A}$ and $\langle b \rangle \cap \bar{A} = 1$ this implies that b has finite order and $|b|$ divides γ , hence $\gamma = |b|\beta$.

4. Determination of S_G and R_G . We now state and prove

THEOREM 1. *Let $f \in \bar{A}$ and $b \in B$ such that $bf \neq 1$ and let T be a left transversal of $\langle b \rangle$ in B .*

(a) *The element bf has trivial centralizer in $G = A \text{ wr } B$ if and only if one of the following conditions is satisfied:*

- (1.1) *b has infinite order.*
- (1) (1.2) *For each element $c \in c_B b$ satisfying $h_{tc} = h_t$ for all $t \in T$ the subgroup $\langle c, b \rangle$ is cyclic.*
 - (2.1) *b has finite order.*
 - (2.2) *$h_t = 1$ for all $t \in T$.*
 - (2) (2.3) *$b \neq 1$ has trivial centralizer in B .*
 - (2.4) *A is periodic and elements in A have order prime to $|b|$.*
 - (3.1) *b has finite order.*
 - (3.2) *$h_t \neq 1$ has trivial centralizer in A for all $t \in T$.*
 - (3) (3.3) *$c_A h_t / \langle h_t \rangle$ is a p' -group for all primes p dividing $|b|$.*
 - (3.4) *If $B \neq \langle b \rangle$, then $c_A h_t$ is periodic for all $t \in T$ and $(|h_s|, |y|) = 1$ for all $s \neq t$ in T and all $y \in c_A h_t$.*

(b) *The element bf is self-centralizing in G if and only if one of the following conditions is satisfied:*

- (4.1) *b has infinite order.*
- (4) (4.2) *$c \in \langle b \rangle$ if and only if $c \in c_B b$ and $h_{tc} = h_t$ for all $t \in T$.*
 - (5.1) *b has finite order.*
 - (5.2) *h_t is self-centralizing in A for all $t \in T$.*
 - (5) (5.3) *If $B \neq \langle b \rangle$, then h_t has finite order for all $t \in T$ and $(|h_s|, |h_t|) = 1$ for all $s \neq t$ in T .*

REMARK 2. It should be noted that in cases (3) and (5) the group B is actually finite. Indeed, since f has finite support only finitely many $h_t = f(tb) \cdots f(tb^{|\mathbf{b}|})$ are nontrivial, hence (3.2) or (5.2) imply that T and hence B is finite.

Proof. (1) Suppose bf has trivial centralizer, b has infinite order and $c \in c_B b$ satisfies $h_{tc} = h_t$ for all $t \in T$. From Lemma 1(iii) follows the existence of some $g \in \bar{A}$ such that $[c^{-1}g, bf] = 1$ and $bf \in R_G$ implies $\langle c^{-1}g, bf \rangle$ is cyclic. Let $H = \langle c, b \rangle$. Then $H^\mu = \langle c^{-1}g, bf \rangle^\mu$ is cyclic and $H^\mu = H\bar{A}/\bar{A} \cong H/H \cap \bar{A}$. But $H \cap \bar{A} \subseteq B \cap \bar{A} = 1$ so $H \cong H^\mu$ is cyclic.

Conversely suppose b and f satisfy conditions (1.1) and (1.2) and $w \in G$ commutes with bf , where $w = c^{-1}k$ for some $c \in B$ and $k \in \bar{A}$. Lemma 1(i) gives $c \in c_B b$ and $h_{tc} = h_t$ for all $t \in T$ from Lemma 1(iii), hence $\langle c, b \rangle$ is cyclic by (1.2). Let $K = \langle c^{-1}k, bf \rangle$. Then $K^\mu = \langle c, b \rangle^\mu$ is cyclic and $K^\mu = K\bar{A}/\bar{A} \cong K/K \cap \bar{A}$. But $[g, bf] = 1$ for all $g \in K \cap \bar{A}$ since K is abelian and so from Lemma 1(ii) we have $K \cap \bar{A} = 1$. Hence $K \cong K^\mu$ is cyclic.

(2) Suppose b has finite order and $h_t = 1$ for all $t \in T$. By assumption $bf \neq 1$. Then $b \neq 1$ since $b = 1$ implies $f(t) = h_t = 1$ for all $t \in T$, hence $f = 1 = bf$. To prove (2.3) observe that b is conjugate to $bf \in R_G$ by Lemma 1(v) and hence $b \in B \cap R_G \subseteq R_B$.

Finally for (2.4) let $a \in A$ and for each $t \in T$ define $k_t \in \bar{A}$ by

$$k_t(x) = a \quad \text{for } x \in t\langle b \rangle, \quad k_t(x) = 1 \quad \text{for } x \notin t\langle b \rangle.$$

Then $[k_t, b] = 1$ by construction and $b \in R_G$ implies $\langle k_t, b \rangle$ cyclic. Hence k_t and a have finite order since $b \neq 1$ has finite order. If $r = (|a|, |b|)$ then $r = (|k_t|, |b|)$ and there are subgroups of order r in $\langle k_t, b \rangle$, $\langle k_t \rangle$ and $\langle b \rangle$. But there is only one subgroup H of order r in $\langle k_t, b \rangle$ since $\langle k_t, b \rangle$ is cyclic hence $H \subseteq \langle k_t \rangle \cap \langle b \rangle \subseteq \bar{A} \cap B = 1$. This proves $1 = r = (|a|, |b|)$ and thus (2.4).

Conversely assume conditions (2) are satisfied. Since b and bf are conjugate by (2.2) and Lemma 1(v) it is sufficient to show $b \in R_G$. Let $g \in \bar{A}$ and $c \in B$ be such that $[cg, b] = 1$. Then $[g, b] = 1 = [c, b]$ by Lemma 1(i) and $\langle c, b \rangle$ is a finite cyclic group by (2.3) and (2.1). Hence if $K = \langle cg, b \rangle$ then $K/K \cap \bar{A} \cong K\bar{A}/\bar{A} = K^\mu = \langle c, b \rangle^\mu$ is finite cyclic, $K \cap \bar{A}$ is finite abelian by (2.4) and $(|K \cap \bar{A}|, |b|) = 1$ by (2.4). Thus K is finite and Lemma 2(i) shows K cyclic.

(3) Assume b has finite order and $h_t \neq 1$ for some $t \in T$. Suppose $h_s = 1$ for some $s \in T$. Then $s \neq t$ and $h_t \in c_A h_s = A$. Define $g_s \in \bar{A}$ as in Lemma 1(iv) with $g_s(s) = h_t$. Then $\langle g_s, bf \rangle$ is abelian, hence cyclic and $g_s^\alpha = d^\beta$ with $(\alpha, \beta) = 1$ by Lemma 3. This gives $h_t^\alpha = 1$ for the argument s and $1 = h_t^\beta$ for the argument t , a contradiction since $(\alpha, \beta) = 1$ and $h_t \neq 1$.

To prove (3.2) let $a \in c_A h_t$ and define $g_t \in \bar{A}$ as in Lemma 1(iv) such that $g_t(t) = a$. Then $[bf, g_t] = 1$ and $\langle bf, g_t \rangle$ is cyclic since $bf \in R_G$. Hence also the subgroup $H = \langle d, g_t \rangle$ is cyclic, $H = \langle h \rangle$ with $h \in \bar{A}$, $d = h^t$ and $g_t = h^t$. So $h_t = d(t) = h(t)^t$, $a = g_t(t) = h(t)^t$ which proves that $\langle h_t, a \rangle$ is cyclic for all $a \in c_A h_t$.

To prove (3.3) we may assume $b \neq 1$ since the condition is vacuous for $b = 1$.

Defining $g_t \in \bar{A}$ as in Lemma 1(iv) for $1 \neq a \in c_A h_t$ with $g_t(t) = a$ we have $[g_t, bf] = 1$ and hence $\langle g_t, bf \rangle$ is cyclic. From Lemma 3 we obtain $g_t^\alpha = d^\beta$ with $(\alpha, |b|\beta) = 1$. For the argument t this gives $a^\alpha \in \langle h_t \rangle$, hence $c_A h_t / \langle h_t \rangle$ is periodic and $(\alpha, |b|) = 1$ proves (3.3).

Finally for (3.4) assume $B \neq \langle b \rangle$ and let $1 \neq y \in c_A h_t$. Define $1 \neq g_t \in \bar{A}$ as in Lemma 1(iv) with $g_t(t) = y$. Then $\langle g_t, bf \rangle$ is abelian, hence cyclic, and $g_t^\alpha = d^\beta$ with $(\alpha, \beta) = 1$ by Lemma 3. For the argument $s \neq t$ this gives $1 = h_s^\beta$, hence h_s has finite order for all $s \in T$. For the argument t we get $y^\alpha = h_t^\beta$. Since $(\alpha - \beta, \beta) = 1$ this implies $(|h_s|, |h_t|) = 1$ for $y = h_t \in c_A h_t$. But $|y|$ divides $\alpha|h_t|$ for all $y \in c_A h_t$, hence $(|h_s|, |y|)$ divides $(|h_s|, \alpha|h_t|) = (\beta, \alpha) = 1$.

Conversely assume that conditions (3) are satisfied, let $cg \in c_G(bf)$ and suppose $c \notin \langle b \rangle$. Then also $[cg, d] = 1$ where $d = (bf)^{|b|}$, and Lemma 1(i) gives $d = g^{-1}d^c g$ hence $d(x) = d(xc^{-1})^{g(x)}$ for all $x \in B$. Since $[d, bf] = 1$ we have also

$$d(x) = d(xb^{-1})^{f(x)} \quad \text{for all } x \in B,$$

so $d(xc^{-1})$ is conjugate to $d(z)$ for each $z \in x\langle b \rangle$. Let $tc^{-1} = sb^\beta$ with $s \in T$ and an integer β . Then $h_s = d(s)$ and $h_t = d(t)$ are conjugate, so (3.4) implies $s = t$, hence $c \in \langle b \rangle$, say $c = b^i$ for some integer i . Since $cg(bf)^{-1} \in \bar{A}$ and $\langle cg(bf)^{-1}, bf \rangle = \langle cg, bf \rangle$ we see that in order to prove that bf has trivial centralizer in G it suffices to show that $\langle g, bf \rangle$ is cyclic for each $g \in \bar{A}$ with $[g, bf] = 1$. Lemma 2(ii) with $u = bf, v = g, \alpha = |b|$ and $d = u^\alpha$ gives just that provided that we can show

- (a) $\langle d, g \rangle$ is cyclic;
- (b) $\langle bf, g \rangle / \langle bf \rangle$ is a p' -group for all primes p dividing $|b|$.

Since $\langle bf \rangle \cap \bar{A} = \langle (bf)^{|b|} \rangle = \langle d \rangle$ then

$$\langle bf \rangle \cap \langle g \rangle = \langle bf \rangle \cap \langle \bar{A} \rangle \cap \langle g \rangle = \langle d \rangle \cap \langle g \rangle.$$

Then

$$\langle bf, g \rangle / \langle bf \rangle \cong \langle g \rangle / \langle bf \rangle \cap \langle g \rangle = \langle g \rangle / \langle d \rangle \cap \langle g \rangle$$

shows that (b) can be replaced by

- (b*) $\langle g \rangle / \langle d \rangle \cap \langle g \rangle$ is a p' -group for each prime p dividing $|b|$.

To show (a) assume first $B = \langle b \rangle$. Then $d(t) = h_t$ and (3.2) gives $\langle d(t), g(t) \rangle$ is cyclic, say $k_t = d(t)^\alpha g(t)^\beta, d(t) = k_t^i$ and $g(t) = k_t^j$ with integers α, β, i, j and $k_t \in A$. Let $k = d^\alpha g^\beta \in \bar{A}$ and observe that d, g and k commute with bf . From Lemma 1(vi) we get $d = k^i$ and $g = k^j$, so $\langle d, g \rangle = \langle k \rangle$ is cyclic.

For $B \neq \langle b \rangle$ the elements $h_s, g(s)$ have finite order for all $s \in T$ by (3.4). Since g commutes with bf we have from Lemma 1(i) that $g(s)$ and $g(x)$ are conjugate if $x \in s\langle b \rangle$. For each $t \in T$ define $d_t \in \bar{A}$ by $d_t(x) = d(x)$ for $x \in t\langle b \rangle, d_t(x) = 1$ for $x \notin t\langle b \rangle$. Then d_t commutes with d_s, g and bf , only finitely many d_t are nontrivial and d is the product of all d_t . Let Π be the set of all primes dividing $|h_t|$ and decompose $\langle g \rangle = \langle m \rangle \times \langle n \rangle$ with $m, n \in \bar{A}$ such that $\langle m \rangle$ is a Π -group and $\langle n \rangle$ is a Π' -group. Since $(|g(s)|, |h_t|) = 1$ by (3.4) for $s \neq t$ and $|g(x)| = |g(s)|$ for $x \in s\langle b \rangle$

it follows that $m(x)=1$ for $x \notin t\langle b \rangle$. By (3.2) the abelian group $\langle m(t), d_t(t) \rangle \subseteq \langle g(t), h_t \rangle$ is cyclic, say $\langle m(t), d_t(t) \rangle = \langle k_t \rangle$ with $k_t = m(t)^\alpha d_t(t)^\beta$. Let $k = m^\alpha d_t^\beta$, and observe that $m \in \langle g \rangle$, d_t and k commute with bf . Since $m(x) = d_t(x) = k(x) = 1$ for $x \notin t\langle b \rangle$ we get then from Lemma 1(vi) that $\langle m, d_t \rangle = \langle k \rangle$ is a cyclic Π -group. Hence $\langle d_t, g \rangle = \langle d_t, m \rangle \times \langle n \rangle$ is cyclic. Finally $(|d_s|, |d_t|) = 1$ for $s \neq t$ in T by (3.4), so Lemma 2(a) can be applied to prove (a).

To show (b*) assume first $B = \langle b \rangle$ and p divides $|b|$. Then by (3.3) there is some integer i such that $(p, i) = 1$ and $g(t)^i \in \langle h_t \rangle$, say $g(t)^i = d(t)^j$. But Lemma 1(vi) gives $g(x)^i = d(x)^j$ for all $x \in B$ so $g^i \in \langle d \rangle \cap \langle g \rangle$.

If $B \neq \langle b \rangle$, then $c_A h_t$ is periodic for all $t \in T$ by (3.4). Suppose there is some prime p dividing $|b|$ and some $k \in \langle g \rangle$ such that $k^p \in \langle d \rangle$. In particular $k(t) \in \langle h_t \rangle$ for all $t \in T$ by (3.3), say $k(t) = h_t^{\alpha_t}$ with integers α_t . Now $(|h_s|, |h_t|) = 1$ for $s \neq t$ in T by (3.4), and the Chinese remainder theorem gives the existence of an integer α such that

$$\alpha \equiv \alpha_t \pmod{|h_t|} \quad \text{for all } t \in T.$$

Finally $d(t) = h_t$, so Lemma 1(vi) shows $k(x) = d(x)^\alpha$ for all $x \in B$. Hence there is no element of order p in $\langle g \rangle / \langle d \rangle \cap \langle g \rangle$, which proves (b*).

(4) Assume now that bf is a self-centralizing element in G and b has infinite order. If $c \in \langle b \rangle$ then $c \in c_B b$ and clearly $h_{xc} = h_x$ for all $x \in B$. Assume $c \in c_B b$ satisfies $h_{tc} = h_t$ for all $t \in T$. There exists $g \in \bar{A}$ by Lemma 1(iii) such that $[c^{-1}g, bf] = 1$ hence $c^{-1}g \in \langle bf \rangle$ since bf is self-centralizing. If $c^{-1}g = (bf)^i$ then in particular $c^{-1} = b^i$ so b is self-centralizing.

Conversely, assume (4.1) and (4.2) are satisfied and there are $c \in B$ and $g \in \bar{A}$ such that $[c^{-1}g, bf] = 1$. Then $[c, b] = 1$ from Lemma 1(i) and Lemma 1(iii) together with (4.2) implies $c = b^j$ with some integer j . But $k = (bf)^{-j}(c^{-1}g) \in \bar{A}$ commutes with bf , hence $k = 1$ by Lemma 1(ii) and so $c^{-1}g \in \langle bf \rangle$.

(5) Suppose b has finite order and let $a \in c_A h_t$. Define g_t as in Lemma 1(iv) with $g_t(t) = a$. Then $[g_t, bf] = 1$ and by assumption $g_t = (bf)^i$ with some integer i . But $g_t \in \bar{A}$ implies that $|b|$ divides i so $i = |b|j$ and $(bf)^i = (bf)^{|b|j} = d^j$. Hence $a = g_t(t) = d(t)^j = h_t^j$ which proves (5.2).

Since a self-centralizing element has trivial centralizer we may apply the results of (2) and (3) to prove (5.3). We have $c_A h_t = \langle h_t \rangle$ by (5.2) hence, $h_t \neq 1$ for all $t \in T$ and condition (3.4) gives (5.3).

Conversely, assume (5) is satisfied, cg commutes with bf and $c \notin \langle b \rangle$. Then $d = (bf)^{|b|}$ commutes with cg and bf so Lemma 1(i) implies

$$d = f^{-1}d^b f \quad \text{and} \quad d = g^{-1}d^c g.$$

In particular, if $tc^{-1} = sb^\beta$ with $s \in T$ and an integer β , then $h_t = d(t)$ is conjugate to $d(s) = h_s$ and hence $s = t$ by (5.3). But then $c = b^{-\beta} \in \langle b \rangle$, a contradiction. Thus $c \in \langle b \rangle$, say $c = b^i$ and $k = (bf)^{-i}(cg) \in \bar{A}$ commutes with bf and d . Since $d(t) = h_t$ is self-centralizing by (5.2) there is an integer α_t for each $t \in T$ such that $k(t) = d(t)^{\alpha_t}$.

From (5.3) we have $(|d(s)|, |d(t)|) = 1$ for $s \neq t$ in T , so by the Chinese remainder theorem there is an integer α with

$$\alpha \equiv \alpha_t \pmod{|d(t)|} \quad \text{for all } t \in T.$$

Then $k(t) = d(t)^\alpha$ for all $t \in T$ and Lemma 1(vi) implies $k(x) = d(x)^\alpha$ for all $x \in B$. Hence $k \in \langle d \rangle \subseteq \langle bf \rangle$, and bf is self-centralizing.

COROLLARY 1. *If B is a p -group and A is torsion-free or a p -group then $1 \neq bf \in R_G$ with $b \in B$ and $f \in \bar{A}$ implies $B = \langle b \rangle$.*

Proof. Since B is periodic we have only to consider conditions (2) and (3). Further under our hypothesis it is impossible to satisfy condition (2.4). Hence from (3.2) we get $h_t \neq 1$ for all $t \in T$. Thus $(|h_s|, |h_t|) \neq 1$ if A is a p -group, and hence $B = \langle b \rangle$ by (3.4). If A is torsion-free we observe that $1 \neq h_t \in c_A h_t$, hence $1 \neq c_A h_t \in A$ is not periodic and $B = \langle b \rangle$ follows again from (3.4). For later applications we note the following

COROLLARY 2. *If A and B are p -groups then $bf \in S_G$ if and only if $B = \langle b \rangle$ and $h_1 \in S_A$.*

COROLLARY 3. *If A and B are p -groups then*

- (i) $\{1\} \cup S_G = R_G$.
- (ii) $S_G \neq \emptyset$ if and only if B is cyclic and $S_A \neq \emptyset$.

Proof. (i) Clearly $\{1\} \cup S_G \subseteq R_G$. So let $1 \neq bf \in R_G$. From Corollary 1 we have $B = \langle b \rangle$, hence $b \neq 1$ and from (3.3) we see $c_A h_t = \langle h_t \rangle$. Thus condition (5) is satisfied for bf and hence $bf \in S_G$.

(ii) Suppose $S_G \neq \emptyset$ and $bf \in S_G$. Then $B = \langle b \rangle$ and $S_A \neq \emptyset$ since $h_1 \in S_A$ by Corollary 2. Conversely assume $B = \langle b \rangle$, $a \in S_A$ and let $f = \gamma_a$. Then

$$h_1 = f(b) \cdots f(b^{b^1}) = a \in S_A,$$

and $bf \in S_G$ by Corollary 2, hence $S_G \neq \emptyset$.

5. The size of S_G and P_G . We introduce a new characteristic subgroup P_H which will be useful to compute S_G if G is an iterated wreath product.

DEFINITION. $P_H = \langle xy \mid x, y \in S_H \rangle$.

By definition $P_H \subseteq \langle S_H \rangle$, and if H is a p -group, $p \neq 2$ then $\langle x \rangle = \langle x^2 \rangle$ shows $P_H = \langle S_H \rangle$. The generalized quaternion groups are examples for $P_H = \langle S_H \rangle$, while for the dihedral groups of 2-power order $P_H \neq \langle S_H \rangle$.

LEMMA 4. *Suppose A and $B = \langle b \rangle$ are p -groups and $S_A \neq \emptyset$. Then*

- (i) $G' \subseteq P_G$.
- (ii) $(A'P_A)^\gamma = A' \cap P_G = A' \cap \langle S_G \rangle$.

Proof. (i) Let $u, v, w \in A$, $uw = 1$ and $1 \neq c \in B = \langle b \rangle$. If $a \in S_A$ then $\gamma_u \gamma_w^c b \in S_G$ by Corollary 2, hence

$$\gamma_u \gamma_w^c = \gamma_u \gamma_w^c (\gamma_a^c)^{-1} = (\gamma_u \gamma_w^c b) (\gamma_a^c b)^{-1} \in P_G.$$

But the elements $\gamma_u \gamma_w^c \subseteq P_G$ generate G' [4, Corollary 4.5, p. 350] hence $G' \subseteq P_G$.

(ii) If $a_1, a_2 \in S_A$ then $a_1^2 b \in S_G$ and $(a_2^{-1})^2 b \in S_G$ by Corollary 2 thus $(a_1 a_2)^2 = a_1^2 a_2^2 = a_1^2 b ((a_2^{-1})^2 b)^{-1} \in P_G$ hence $P_A^2 \subseteq P_G$. Since $(A')^2 \subseteq G'$ and $G' \subseteq P_G$ by Lemma 4(i) we have $(A'P_A)^2 \subseteq A' \cap P_G$.

Conversely let $h \in A' \cap P_G$. Then

$$h = (b_1 f_1) \cdots (b_r f_r) \quad \text{where } r \text{ is even,}$$

$B = \langle b_1 \rangle = \cdots = \langle b_r \rangle$ and $u_i = f_i(b_1) \cdots f_i(b_i^{b_1}) \in S_A$ by Corollary 2. Since

$$c' f_i c f_{i+1} = c' c f_i^c f_{i+1} \quad \text{for every } c', c \in B$$

and $h \in A' \subseteq \bar{A}$ we may rewrite

$$h = f_1^* \cdots f_r^*,$$

with f_i^* conjugate to f_i under B . In particular $f_i^*(b) \cdots f_i^*(b^{b_1}) \equiv f_i(b) \cdots f_i(b^{b_1}) \equiv u_i \pmod{A'}$. Now $h(b) \cdots h(b^{b_1}) \equiv \prod_{i=1}^r (f_i^*(b) \cdots f_i^*(b^{b_1})) \equiv u_1 \cdots u_r \pmod{A'}$, and so $h(b) \cdots h(b^{b_1}) \in A'P_A$ since r is even. But $h \in A'$ implies $h(b^i) = 1$ for $0 < i < |b|$, so $h(1) = h(b^{b_1}) \in A'P_A$, or $h \in (A'P_A)^2$. This proves $(A'P_A)^2 = A' \cap P_G$.

For $p \neq 2$ we have $P_G = \langle S_G \rangle$. To prove $A' \cap P_G = A' \cap \langle S_G \rangle$ we may hence assume $p = 2$. Suppose $h \in A' \cap \langle S_G \rangle$. By Corollary 2

$$h = (b_1 f_1) \cdots (b_r f_r),$$

with $b_i f_i \in S_G$, $B = \langle b \rangle = \langle b_1 \rangle = \cdots = \langle b_r \rangle$ and $f_i \in \bar{A}$. Since $h \in A' \subseteq \bar{A}$ and $h \equiv b_1 \cdots b_r \pmod{\bar{A}}$, we have $b_1 \cdots b_r \in \bar{A} \cap B = 1$. But $B = \langle b \rangle = \langle b_i \rangle$ and $p = 2$ imply that each b_i is an odd power of b , hence r is even because b has even order. Thus $h \in P_G$, which proves

$$A' \cap \langle S_G \rangle = A' \cap P_G,$$

since trivially $P_G \subseteq \langle S_G \rangle$.

We can now prove Theorem 2 announced in the introduction.

Proof. (a) We first observe that $G = A' \langle S_G \rangle$ since $\langle A', b \rangle = G$ and $\gamma_a b \in S_G$ by Corollary 2 for $B = \langle b \rangle$ and $a \in S_A$. Hence $G / \langle S_G \rangle \cong A' / A' \cap \langle S_G \rangle$, and $A' \cap \langle S_G \rangle = (A'P_A)^2$ by Lemma 4(ii) implies

$$G / \langle S_G \rangle \cong A / A'P_A.$$

(b) By Corollary 2 an element $x \in S_G$ has the form $x = b^* f$ with $f \in \bar{A}$ and $B = \langle b^* \rangle$ where b^* is an odd power of b . In particular $P_G \subseteq \bar{A} \langle b^2 \rangle$. But $\gamma_a b \in S_G$ for $a \in S_A$ so $b^2 \in \bar{A}P_G$. Hence $\bar{A}P_G = \bar{A} \langle b^2 \rangle$ and

$$|G : \bar{A}P_G| = |G : \bar{A} \langle b^2 \rangle| = 2.$$

Further $A'P_G$ is normal in G since $G' \subseteq A'P_G$ by Lemma 4(i). But \bar{A} is generated by conjugates of A' , hence $\bar{A}P_G = A'P_G$ and $\bar{A}P_G / P_G = A'P_G / P_G \cong A' / A' \cap P_G \cong A / A'P_A$ by Lemma 4(ii). This proves

$$|G : P_G| = |G : \bar{A}P_G| | \bar{A}P_G / P_G | = 2 |A : A'P_A|.$$

(c) For $k=1$ this follows directly from (a) and (b). We proceed by induction on k . Let $V = A_{k+2} = W \text{ wr } B_{k+1}$. Then for $p=2$, $|V:P_V| = 2|W:W'P_W|$ by (b) and $W' \subseteq P_W$ by Lemma 4(i). By induction $|W:P_W| = 2^k|A:A'P_A|$ hence $|V:P_V| = 2^{k+1}|A:A'P_A|$. For $p \neq 2$ we have $V/\langle S_V \rangle \cong W/W'P_W$ by (a), $P_W = \langle S_W \rangle$ and $W' \subseteq P_W$ by Lemma 4(i), hence $V/\langle S_V \rangle \cong W/\langle S_W \rangle$ and by induction

$$V/\langle S_V \rangle \cong W/\langle S_W \rangle \cong A/A'P_A.$$

Proof of Theorem 3. Let b be an element of infinite order in B , $1 \neq a \in A$ and choose T so that $1 \in T$. We show first that if $f = \gamma_a$, then the element bf satisfies condition (4.2) of Theorem 1. Observe that

$$\begin{aligned} h_x &= 1 \quad \text{for } x \notin \langle b \rangle, \\ h_x &= a \quad \text{for } x \in \langle b \rangle. \end{aligned}$$

Hence $h_{tc} = h_t$ implies for $t=1$ that $h_{tc} \neq 1$ and so $c \in \langle b \rangle$. This proves by Theorem 1(4) that $b\gamma_a \in S_G$ and a similar argument shows $b^2\gamma_a \in S_G$ hence both b and γ_a are contained in $\langle S_G \rangle$. But this implies $B^* \subseteq \langle S_G \rangle$ and $A' \subseteq \langle S_G \rangle$. Since \bar{A} is generated by the conjugates of A' and $\langle S_G \rangle$ is normal in G , we have $\bar{A}B^* \subseteq \langle S_G \rangle$.

On the other hand, if $bf \in S_G$ then $b \in B^*$ for otherwise b has finite order and Remark 2 implies B is finite which contradicts the assumption that B is not a torsion group. Thus $S_G \subseteq \bar{A}B^*$, hence $\langle S_G \rangle = \bar{A}B^*$.

6. Unrestricted wreath products. The characterization of self-centralizing elements and elements with trivial centralizer is much easier for the unrestricted wreath product as the following theorem shows. In particular if A and B are p -groups it suffices to consider the restricted wreath product.

THEOREM 4. *Suppose B has infinite order and $X = A \text{ Wr } B$ is the unrestricted wreath product of A and B .*

(a) *Let $b \in B$ and $f \in F$ such that $bf \neq 1$, and T a left transversal of $\langle b \rangle$ in B . Then bf has trivial centralizer in X if and only if the following conditions are satisfied:*

- (i) b has finite order.
 - (ii) $h_t = 1$ for all $t \in T$.
 - (iii) $b \neq 1$ has trivial centralizer in B .
 - (iv) A is periodic and elements in A have order prime to $|b|$.
- (b) X has no self-centralizing elements.

Proof. (a) Suppose $bf \in R_X$. If b has infinite order, let $1 \neq a \in A$ and define $1 \neq g \in F$ by

$$g(1) = a, \quad g(x) = g(xb^{-1})^{f(x)} \quad \text{for } x \in \langle b \rangle$$

and $g(x) = 1$ for $x \notin \langle b \rangle$. Then by construction $g^{bf} = g$, so that $\langle g, bf \rangle$ must be cyclic. This contradicts Lemma 3 and this proves (i).

Let $1 \neq a \in c_A h_t$ and define g_t as in Lemma 1(iv) with $g_t(t) = a$. Then $\langle g_t, bf \rangle$ is

cyclic and by Lemma 3 $g^\alpha = d^\beta$ with integers α, β such that $(\alpha, |b|\beta) = 1$. In particular

$$(*) \quad a^\alpha = h_t^\beta, \quad 1 = h_s^\beta \quad \text{for all } s \neq t \text{ in } T.$$

Hence h_s has finite order, $c_A h_t$ is periodic and there is some integer $m > 0$ such that $h_s^m = 1$ for all $s \in T$. With $a = h_t$ we see from (*) also that $(|h_s|, |h_t|) = 1$ since $(\alpha - \beta, \beta) = 1$. Since $h_s^m = 1$ for all $s \in T$ this implies that there exists some $t \in T$ with $h_t = 1$. Suppose $h_s \neq 1$ for some $s \in T$. Then for $a = h_s \in A = c_A h_t$ (*) implies $h_s^\alpha = 1$ and $1 = h_s^\beta$ a contradiction since $(\alpha, \beta) = 1$. This proves (ii).

To prove (iii) observe that bf and b are conjugate by (ii) and Lemma 1(v). In particular $b \neq 1$ and $b \in R_X \cap B \subseteq R_B$.

Finally from (ii) and (*) we get $a^\alpha = 1$ for $a \in c_A h_t = A$ with $(\alpha, |b|\beta) = 1$ which proves (iv).

The sufficiency of conditions (i) to (iv) follows as in the proof of Theorem 1(2).

(b) Since $S_X \subseteq R_X$ we get from condition (ii) of (a) that $h_t = 1$ for all $t \in T$. Then Lemma 1(v) implies that bf and b are conjugate, hence $b \in S_X$. Then for some $1 \neq a \in A$ define $k \in \bar{A} \subseteq F$ by $k(x) = a$ for $x \in \langle b \rangle$ and $k(x) = 1$ for $x \notin \langle b \rangle$, and observe $[k, b] = 1$, but $\langle b \rangle \cap F \subseteq B \cap F = 1$.

COROLLARY 4. *If A and B are p -groups then $R_X = \{1\} \cup S_X$.*

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