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GLOBAL DIMENSION OF ORDERS(1)

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Abstract. We prove that the finitistic global dimension (fGD) of an order in a quaternion algebra over the quotient field of a Dedekind domain is one. Examples are given of orders of global dimension $n-1$ in $n \times n$ matrices over the quotient field of a discrete valuation ring.

1. **Introduction.** Let R be an integral domain with quotient field K and E a finite dimensional associative K algebra with 1. By an *R order*, P, in E we mean a subring of E, containing 1, which is a finitely generated R module and which spans E over K. Much of the time we shall be concerned with the case of R being a DVR.

Maximal orders (i.e. orders contained in no others) in a central simple algebra over the quotient field of a DVR are known to be hereditary rings; see, for example, Auslander, Goldman [1, Theorem 2.3]. In addition, in such algebras there are a wealth of nonmaximal orders which are hereditary rings. Harada [3], [4] gives a quite complete account of these. In [7] Silver has mentioned the possibility that in a semisimple algebra over the quotient field of a DVR the only possible global dimensions for orders are one and infinity. In §2 below we shall see that this is true in case the algebra is a quaternion algebra but in §3 we shall show that this is in general false.

2. To begin we mention in the form of some preliminary lemmas several of the facts that we shall use. Projective dimension over a ring R, symbolized by d_R, will mean left projective dimension. Of course, since orders are left and right Noetherian their left and right global dimensions are the same; the omission of left or right when referring to global dimension will not cause confusion.

LEMMA 1. *Let R be a commutative ring, L a left Noetherian R algebra, and G a R algebra which is a flat R module. If A is a finitely generated left L module and B is any left L module then* $G \otimes_R \mathrm{Ext}_L^n(A, B) \simeq \mathrm{Ext}_{G \otimes_R L}^n(G \otimes_R A, G \otimes_R B)$.

Proof. Auslander, Goldman [1, Lemma 2.4].

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LEMMA 2. *Suppose P is an order in an algebra over the quotient field of a* DVR *and N is the radical of P. Then GD $P=1+\mathrm{d}_P N=\mathrm{d}_P P/N$.*

Proof. Silver [7, Corollary 4.6].

LEMMA 3. *Suppose R is any ring and $0\to A\to B\to C\to 0$ is an exact sequence of R modules. Then $\mathrm{d}_R B\leqq \max(\mathrm{d}_R A, \mathrm{d}_R C)$ with equality unless perhaps $\mathrm{d}_R C=1+\mathrm{d}_R A$.*

Proof. Well known.

LEMMA 4. *Suppose R is any ring and x is a nonzero-divisor with $Rx=xR$. If A is a R/Rx module of finite R/Rx projective dimension then $\mathrm{d}_R A=1+\mathrm{d}_{R/Rx}A$.*

Proof. Kaplansky [5, Theorem 5.3].

LEMMA 5. *Suppose R is left Noetherian, x is a central nonzero-divisor contained in the radical of R, and A is a finitely generated R module on which x is not a zero-divisor. Then $\mathrm{d}_R A=\mathrm{d}_{R/Rx}A/xA$.*

Proof. Kaplansky [5, Theorem 5.6].

Lemma 1 shows that in a discussion of the projective dimension of finitely generated modules over an order P in an algebra over the quotient field of a DVR, R, we may assume that R is complete. For if A is such a module then A has dimension n if and only if $\mathrm{Ext}_P^{n+1}(A, B)=0$ for all P modules B and there is some B with $\mathrm{Ext}_P^n(A, B)\neq 0$. In fact, all this is true if B is restricted to be a finitely generated module. But in this case $\mathrm{Ext}_P^i(A, B)$ is a finitely generated R module so that its completion is just $\hat{R}\otimes_R \mathrm{Ext}_P^i(A, B)$, where $\hat{R}$ is the completion of R. But the completion of a module is zero if and only if the module is zero. Since the global dimension of a ring is determined by the dimensions of its finitely generated modules, the global dimension of P is unchanged upon the completion of P.

Lemma 1 also shows that Ext localizes when the first argument is finitely generated, so that the results we obtain about orders over DVR's are easily globalized to results about orders over Dedekind domains.

Suppose S is any ring. We define lfGD S to be the supremum of the dimensions of finitely generated left S modules of finite projective dimension.

LEMMA 6. *Suppose E is a finite dimensional algebra over a field and E has only one idempotent*, 1. *Then* lfGD $E=0$.

Proof. We must show that there are no finitely generated modules of dimension one. Let M be a finitely generated E module and let $0\to K\to F\to M\to 0$ be a minimal resolution of M. If N is the radical of E then $K\subset NF$. As E is local, finitely generated projectives are free. Thus K cannot be projective for K is annihilated by a power of N.

THEOREM 7. *Let E be a quaternion algebra over the quotient field K of a* DVR *R whose maximal ideal is (t), and let P be an order in E. Then* lfGD $P=1$.

Proof. We may assume that R is complete. Suppose that P has only one idempotent, 1. Then P/tP has only one idempotent for any others would lift to P as R is complete. By Lemma 6, lfGD $P=0$ and Lemmas 4 and 5 show that lfGD $P=1$.

Suppose now that P has one nonidentity idempotent. It must now be the case that E is 2×2 matrices over K for if E is a division algebra it contains no nonidentity idempotents. P has two orthogonal primitive idempotents, which, by conjugation, we may assume are the matrix units $\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right)$. Thus P has the form

$$\begin{pmatrix} R & (t^n) \\ (t^m) & R \end{pmatrix}.$$

Conjugating by the element

$$\begin{pmatrix} t^{-m} & 0 \\ 0 & 1 \end{pmatrix}$$

allows us to assume that P has the form

$$\begin{pmatrix} R & (t^n) \\ R & R \end{pmatrix}, \qquad n \geqq 0.$$

In case $n=0$, P is maximal whence the result is clear. In case $n\geqq 1$ it is easily seen that all the P/tP's are isomorphic. Now if $n=1$ then P is hereditary: its radical is generated as a left ideal by the nonzero-divisor $\left(\begin{smallmatrix}0 & t\\ 1 & 0\end{smallmatrix}\right)$. Thus employing Lemmas 4 and 5 we see that in any case lfGD $P/tP=0$ and, for any n, lfGD $P=1$.

It is obvious that the case of more orthogonal primitive idempotents does not occur.

COROLLARY 8. *An order in a quaternion algebra over the quotient field of a* DVR *has global dimension one or infinity.*

In case $\frac{1}{2}$ is in R we sketch another proof of Theorem 7. E has an involution $*$ which induces an R valued quadratic form on every order in E. This form may be diagonalized so that every order is expressed as

$$P = R1+Ri+Rj+Rk \qquad \text{(sum direct)}$$

where $ij=-ji=ak$, a in R; i^2, j^2 in R; (j^2) contained in (i^2). Scrutiny yields that if (i^2) is contained in (t) then P has only one idempotent. If i^2 and j^2 are units then P is maximal. If $(i^2)=R$ and $(j^2)=(t)$ then P is a nonmaximal hereditary order. If $(i^2)=R$ and (j^2) is contained in (t^2) then GD $P=$infinity, lfGD $P=1$.

3. We now present some examples. In what follows it will from time to time be claimed that some module of an order is or is not projective. These modules will all be contained in E and will contain a basis of E. In fact, there is a simple procedure for checking the projectivity of such modules. Note that if A is such then $\mathrm{Hom}_P(A, P)=\{x\in E \mid Ax\subset P\}$ if A is a left P module, and the dual basis lemma for projective modules says that A is projective if and only if 1 is in $\mathrm{Hom}_P(A, P)A$.

LEMMA 9. *Suppose S is a left Noetherian ring and X is a 2-sided ideal of S contained in the radical of S. Then*

$$\mathrm{lGD}\, S \leqq \mathrm{r.w.d}_S S/X + \mathrm{lGD}\, S/X.$$

Proof. Small [8, Theorem 1].

We shall be considering a special kind of order in matrix algebras called a *tiled* order. This is an order P such that $e_{ii}Pe_{jj} \subset P$, $i, j = 1, \ldots, n$, where the e_{ii} are matrices with 1 in the iith position and zeros elsewhere. A tiled order is simply one which can be represented as (A_{ij}) where A_{ij} is an ideal in R. For Theorems 10–12 R will be a DVR with quotient field K and maximal ideal (t).

THEOREM 10. *The tiled order $P = (A_{ij})$ in $M_n(K)$, where $A_{ij} = R$ for $i \geqq j$ and $A_{i,i+k} = (t^k)$ for $i = 1, \ldots, n$ and $1 \leqq k \leqq n-i$ has global dimension two.*

Proof. Written out P is just the order

$$\begin{pmatrix} R & (t) & (t^2) & \cdots & (t^{n-1}) \\ R & R & (t) & \cdots & (t^{n-2}) \\ \vdots & & & & \\ R & R & R & \cdots & R \end{pmatrix}.$$

The radical of P is just P with the main diagonal entries replaced by (t). Let I be the ideal which is P with all R's replaced by (t)'s. I is a left and right projective 2-sided ideal of P contained in the radical of P. P/I is just $n \times n$ triangular matrices over $R/(t)$ which is hereditary. Thus $\mathrm{GD}\, P \leqq 2$ by Lemma 9. But $\mathrm{GD}\, P \neq 1$ since the radical is not projective.

THEOREM 11. *The tiled order $P = (A_{ij})$ in $M_n(K)$ where $A_{ij} = R$ for $i \geqq j$; $A_{i,i+1} = (t)$ for $i = 1, \ldots, n-1$; $A_{i,i+k} = (t^2)$ for $1 < k \leqq n-i$ has global dimension $n-1$.*

Proof. N, the radical of P, is just P with the main diagonal entries replaced by (t)'s. We denote by P_i the ith column of P, by N_i the ith column of N. Of course each P_i is projective. We may compute $\mathrm{d}_P N$ by computing the maximum of the $\mathrm{d}_P N_i$. First note that N_1 is isomorphic to P_2 and so is projective. N_2 is not projective and if $2 \leqq i \leqq n-1$ then $N_i = tP_{i-1} + P_{i+1}$, and computing the kernel of the obvious resolution we find that it is just N_{i-1}. Thus $\mathrm{d}_P N_{n-1} = n-2$. Similar computations show that $\mathrm{d}_P N_n = n-3$. Thus $\mathrm{d}_P N = n-2$ and $\mathrm{GD}\, P = n-1$.

An exercise in a similar vein yields the global dimensions of tiled orders in 3×3 matrices over the quotient field of a DVR. Suppose P is such. By suitable conjugations we may assume that

$$P = \begin{pmatrix} R & (t^a) & (t^b) \\ R & R & (t^c) \\ R & (t^d) & R \end{pmatrix}$$

and computation yields

THEOREM 12. *The only tiled orders of finite global dimension, up to conjugation, in $M_3(K)$ are*

(1) $a=b=c=d=0$;

(2) $a=b=c=1$, $d=0$;

(3) $a=b=1$, $c=d=0$;

(4) $a=c=1$, $b=2$, $d=0$.

The first is maximal, the second two are nonmaximal hereditary, and the last has global dimension two.

It is perhaps interesting to note that two stunning facts about hereditary orders are not true for orders of finite global dimension in general, as an example that Fields has shown us proves [2]. The facts are that all hereditary orders in matrices are conjugate to tiled orders and that an order containing a hereditary order is hereditary, see [3], [4]. However, we present some conjectures about orders of finite global dimension over DVR's:

1. The maximum finite global dimension of an order in $n\times n$ matrices is $n-1$.
2. DCC on orders of finite global dimension in matrices.
3. If $P\subset Q$ are orders of finite global dimension with no orders between them then their global dimensions differ by at most one.
4. If $P=(A_{ij})$ is a tiled order with $A_{ij}=R$ for $i\geqq j$ then P has finite global dimension if and only if A_{ij} contains (t^{j-i}) for $i<j$.

We conclude with an example of orders over a regular local ring.

LEMMA 13. *If $P=(A_{ij})$ is a tiled order over a* DVR *with $A_{ij}=R$ for $i\geqq j$ and (t) contained in A_{ij} for $i<j$ then P is hereditary.*

Proof. One easily checks that the radical of P is projective.

THEOREM 14. *Let R be a regular local ring of dimension $n>0$ with maximal ideal M generated by a system of parameters $x, x_2, \ldots, x_n$. Let $P=(A_{ij})$ be a tiled order over R with $A_{ij}=R$ for $i\geqq j$ and M contained in A_{ij} for $i<j$. Then* GD $P=n$.

Proof. To avoid confusing verbiage we shall write the proof for the case $P=\left(\begin{smallmatrix} R & M \\ R & R\end{smallmatrix}\right)$. It will readily be seen that the proof generalizes for the more general order of the theorem.

We proceed by induction on n. If $n=1$, Lemma 14 is what we need. Suppose $n>1$. The radical of P is $\left(\begin{smallmatrix} M & M \\ R & M\end{smallmatrix}\right)$ which contains

$$X=\begin{pmatrix}(x) & (x)\\ (x) & (x)\end{pmatrix}.$$

X is a 2-sided ideal which is left and right projective.

$$P/X=\begin{pmatrix}R/(x) & M/(x)\\ R/(x) & R/(x)\end{pmatrix}$$

so by induction GD $P/X=n-1$. By Lemma 9, GD $P\leqq n$. On the other hand let $Q=\left(\begin{smallmatrix} R & R \\ R & R \end{smallmatrix}\right)$. Q is isomorphic to X so $\mathrm{d}_P Q=0$. Induction and Lemmas 4 and 5 show $\mathrm{d}_P Q/MQ=n\leqq \mathrm{GD}\, P$.

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