STRONG RENEWAL THEOREMS WITH INFINITE MEAN

BY

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Abstract. Let $F$ be a nonarithmetic probability distribution on $(0, \infty)$ and suppose $1 - F(t)$ is regularly varying at $\infty$ with exponent $a$, $0 < a \leq 1$. Let $U(t) = \sum F^n(t)$ be the renewal function. In this paper we first derive various asymptotic expressions for the quantity $U(t+h) - U(t)$ as $t \to \infty$, $h > 0$ fixed. Next we derive asymptotic relations for the convolution $U^*(z)(t)$, $t \to \infty$, for a large class of integrable functions $z$. All of these asymptotic relations are expressed in terms of the truncated mean function $m(t) = \int_0^t [1 - F(x)] \, dx$, $t$ large, and appear as the natural extension of the classical strong renewal theorem for distributions with finite mean. Finally in the last sections of the paper we apply the special case $a = 1$ to derive some limit theorems for the distributions of certain waiting times associated with a renewal process.

1. Principal theorems. Let $F$ be a probability measure concentrated on $[0, \infty)(^2)$ and let $U$ be the associated renewal measure defined for any measurable set $I$ by

$$U(I) = \sum_{n=0}^{\infty} F^n[I]$$

where $F^n$ denotes the $n$-fold convolution of $F$ with itself ($F^0$ is the probability measure concentrated at the origin). The series (1.1) converges to a finite number for every bounded $I$. (For this and other elementary properties of $U$ see [3, VI. 6]; for a probabilistic interpretation of $U$ see §9 in this paper.) We write $U(x)$ for $U([0, x])$ and we shall henceforth ignore the distinction between $U$ the measure and $U$ the function. (This convention applies to other measures as well.)

The main results of this paper deal primarily with the differences $U(t+h) - U(t)$ for $h > 0$ fixed, and $t \to \infty$. The principal assumption is that $F$ has the form

$$1 - F(t) = t^{-a}L(t), \quad t > 0,$$

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(^2) We assume, however, that not all the mass is at the origin.

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where $0 \leq \alpha \leq 1$ (fixed) and $L$ is a slowly varying function\(^{(3)}\). Unless otherwise indicated, we also assume $F$ is nonarithmetic; that is, we exclude the possibility that $F$ concentrates the entire mass on the multiples of some positive real number. For $\alpha \neq 1$, the arithmetic versions of Theorems 1 and 2 below were treated by A. Garsia and J. Lamperti, [5] (nothing was known in the case $\alpha = 1$). See §2(ii) for further discussion. Define the “truncated mean” function

\[(1.3) \quad m(t) = \int_0^t (1 - F(x)) \, dx = t(1 - F(t)) + \int_0^t xF(dx).\]

**Theorem 1.** Let $F$ satisfy (1.2) with $\frac{1}{2} < \alpha \leq 1$. Then for every $h > 0$ and as $t \to \infty$

\[(1.4) \quad U(t+h) - U(t) \sim C_\alpha h/m(t)\]

where $C_\alpha = [\Gamma(\alpha)\Gamma(2-\alpha)]^{-1}$.

**Theorem 2.** If $0 < \alpha \leq \frac{1}{2}$ then

\[(1.5) \quad \lim \inf_{t \to \infty} m(t)(U(t+h) - U(t)) = C_\alpha h.\]

**Remark.** When $\alpha \neq 1$, $m(t) \sim (1 - \alpha)^{-1} t^{1 - \alpha} L(t)$, $t \to \infty$ (see Lemma 1, §3) and

\[\frac{\Gamma(\alpha)\Gamma(2-\alpha)}{\Gamma(1-\alpha)\Gamma(\alpha + 1)} = \frac{\Gamma(1-\alpha)}{\Gamma(\alpha + 1)} \frac{\Gamma(\alpha + 1)}{\Gamma(1-\alpha)}.\]

It follows that (1.4) is equivalent to

\[(1.6) \quad \lim_{t \to \infty} t^{1 - \alpha} L(t)(U(t+h) - U(t)) = \frac{\sin \pi \alpha}{\pi} h.\]

The results of Theorems 2, 3, and 4 may be restated in an analogous fashion.

Let $z$ be a nonnegative function on $[0, \infty)$. For $h > 0$ write

\[\sigma^+ = h \sum_{k=1}^{\infty} \sup \{z(x) : (k-1)h \leq x < kh\}\]

and similarly define $\sigma^-$ with inf in place of sup. Following Feller [3, p. 348], we say that $z$ is directly Riemann integrable (dri) if the series defining the upper sum $\sigma^+$ converges and $\sigma^+ - \sigma^- \to 0$ as $h \to 0$. It follows immediately that a dri function is bounded, measurable, and (Lebesgue) integrable.

**Theorem 3.** Let $z$ be a nonnegative dri function on $[0, \infty)$ which satisfies

\[(1.7) \quad z(t) = O(1/t), \quad t > 0.\]

If $F$ has the form (1.2) with $\frac{1}{2} < \alpha \leq 1$ then

\[(1.8) \quad \int_0^t z(t-y)U(dy) \sim \frac{C_\alpha}{m(t)} \int_0^\infty z(x) \, dx.\]

\(^{(3)}\) A measurable ultimately positive function $L$ on $[0, \infty)$ is regularly varying with exponent $\rho$ if as $t \to \infty$, $L(xt)/L(t) \to x^\rho$ for all $x > 0$. When $\rho = 0$, i.e., $L(xt)/L(t) \to 1$, we also say $L$ is slowly varying. We assume as known the various properties of slowly varying functions as described in [3, pp. 272-274], or in [6]. Note that the function $L$ in (1.2) must be bounded on bounded subintervals of $[0, \infty)$.  

Theorem 4. Let \( z \geq 0 \) be a dri function (not necessarily satisfying (1.7)). If \( F \) satisfies (1.2) with \( \alpha \neq 0 \) then

\[
\lim \inf_{t \to \infty} m(t) \int_0^t z(t-y) U(dy) = C_{\alpha} \int_0^\infty z(x) \, dx.
\]

Remarks. 1. Define a complex valued \( z \) to be dri if \(|z|\) is dri as defined above. With this definition it follows readily from Theorem 3 that (1.8) holds for any dri \( z \) satisfying (1.7).

2. Any piecewise continuous function on \([0, \infty)\) vanishing off a compact interval is dri and certainly satisfies (1.7). In particular, taking \( z(x) = 1 \) for \( 0 \leq x < h \), and \( z(x) = 0 \) elsewhere we have by (1.8)

\[
U(t+h) - U(t) = \int_0^h z(t+h-x) U(dx) \sim \frac{C_{\alpha} h}{m(t)} \quad \text{as } t \to \infty.
\]

That \( m(t+h) \sim m(t) \), \( t \to \infty \), \( h \) fixed, follows easily from monotonicity and regular variation of \( m \), see Lemma 1.) Thus Theorem 3 is equivalent to Theorem 1 (we use Theorem 1 to prove Theorem 3). Similarly Theorem 4 (with \( 0 < \alpha \leq \frac{1}{2} \)) is equivalent to Theorem 2.

For a generalization of (1.8) to nonintegrable but regularly varying \( z \) see §2(iii).

§§3–8 of this paper are concerned with the proofs of Theorems 1–4. In §9 we give an application of the special case \( \alpha = 1 \) to obtain some curious limit theorems for the spent and residual waiting times of a renewal process.

2. Notes. (i) Let \( m \) and \( U \) be defined as in §1 and let \( \hat{m} \) and \( \hat{U} \) be their Laplace transforms:

\[
\hat{m}(\lambda) = \int_0^\infty e^{-\lambda x} (1 - F(x)) \, dx, \quad \hat{U}(\lambda) = \int_0^\infty e^{-\lambda x} U(dx).
\]

If in addition \( \hat{F} \) is the transform of \( F \) then by (1.1) and (1.3)

\[
\hat{m}(\lambda) = \frac{1 - \hat{F}(\lambda)}{\lambda}, \quad \hat{U}(\lambda) = \frac{1}{\lambda} \frac{1}{1 - \hat{F}(\lambda)}
\]

and hence \( \hat{U}(\lambda) \hat{m}(\lambda) = 1/\lambda \). Using this relation and Karamata’s Tauberian theorem, \([3, p. 420]\), we conclude the following:

Theorem 5. Let \( 0 \leq \alpha \leq 1 \). Each of statements (a) and (b) which follow implies the other and both imply the asymptotic relation (2.1).

(a) \( m \) is regularly varying with exponent \( 1 - \alpha \).

(b) \( U \) is regularly varying with exponent \( \alpha \).

(2.1) \[ U(t) \sim \frac{1}{\Gamma(\alpha + 1)\Gamma(2 - \alpha)} \frac{1}{(t/m(t))}. \]

By Lemma 1 statement (a) is true when \( F \) satisfies (1.2). (The converse is also true provided \( \alpha \neq 1 \); if (a) is true for some \( 0 \leq \alpha < 1 \), then (1.2) holds for some slowly
varying $L$, cf. [3, p. 422].) When $\alpha \neq 1$ in (1.2) we see as in the remark following Theorem 2 that (2.1) is equivalent to

$$U(t) \sim \frac{\sin \pi \alpha}{\pi \alpha} \frac{t^\alpha}{L(t)}, \quad t \to \infty,$$

(when $\alpha=0$, $(\sin \pi \alpha)/\pi \alpha \equiv 1$). For a proof of (2.2) when $0<\alpha<1$ cf. [3, p. 446]. See also Teugels [10]. When $1/2<\alpha \leq 1$ (2.1) may also be derived from Theorem 1 (1.4). We shall not do this however. Theorem 1 cannot be proved from (2.1).

(ii) Let $F$ be an arithmetic distribution on $(0, \infty)$ which we suppose, without loss of generality, has span 1. (A distribution has span $b>0$ if it is concentrated on the multiples of $b$ and $b$ is the largest such number.) The renewal measure $U$ defined by (1.1) is also arithmetic with span 1. Denote by $f_n$ and $u_n$ the mass assigned to the integer $n$ by $F$ and $U$. If $F$ satisfies (1.2), i.e.,

$$1 - F(n) = \sum_{n+1}^{\infty} f_k = n^{-\alpha} L(n)$$

for some $0<\alpha<1$ and slowly varying $L$, then (Lamperti-Garsia, 1962) for $1/2<\alpha<1$

$$\lim_{n \to \infty} n^{1-\alpha} L(n) u_n = \frac{\sin \pi \alpha}{\pi}$$

while for $0<\alpha \leq 1/2$ the limit must be replaced by lim inf. However (2.3) does hold when $0<\alpha \leq 1/2$ provided the limit is taken excluding a set of integers having density $0$.

These authors did not consider the case $\alpha=1$ (nor, for that matter, $\alpha=0$). The appropriate and true conclusion for $\alpha=1$ is

$$\lim_{n \to \infty} m(n) u_n = 1$$

where, as before,

$$m(n) = \int_0^n (1-F(x)) \, dx = \sum_{k=1}^n \sum_{f=k}^n f_j = \sum_{k=1}^n j f_j, \quad n \to \infty.$$

The proof of (2.3) and (2.4) starts with the following representation for $u_n$ (see [5] or [8, pp. 98–99]): let $\phi(\theta) = \sum f_k e^{ik\theta}$ and put $W(\theta) = \text{Re} [1-\phi(\theta)]^{-1}$ then provided $F$ has an infinite mean

$$u_n = \frac{1}{\pi} \text{Re} \int_0^\pi \frac{e^{-in\theta}}{1-\phi(\theta)} \, d\theta = \frac{2}{\pi} \int_0^\pi W(\theta) \cos n\theta \, d\theta$$

for $n \geq 1$. (When the mean $\mu$ is finite (2.5) holds with $u_n$ replaced by $u_n-1/\mu$.) The lack of a similar formula for $U(t+h)-U(t)$ when $F$ is nonarithmetic constitutes the chief difficulty in the proof of Theorem 1.

Here is a brief proof of (2.4): from (2.5)

$$\frac{\pi}{2} u_n = \left( \int_0^{B/n} + \int_{B/n}^{n/2} \right) W(\theta) \cos n\theta \, d\theta = J_1 + J_2.$$
As in the latter part of the proof of Theorem 1, see (5.10) and (5.11), we get

\[ \lim_{n \to \infty} m(n)J_1 = \pi/2, \quad \limsup_{n \to \infty} m(n)|J_1| = O(1/B). \]

(The first limit follows directly from Lemma 4, \( a = 1 \).) Hence

\[ \lim_{n \to \infty} m(n)u_n = \lim_{B \to \infty} \lim_{n \to \infty} (2/\pi) m(n)(J_1 + J_2) = 1. \]

J. A. Williamson [11] has extended the results of Lamperti and Garsia [5] to include distributions not necessarily restricted to the positive integers nor to 1-dimension. He does not, however, consider nonarithmetic distributions. He also gives examples showing that (2.3) and its generalization to \( d \)-dimensions cannot hold when \( \alpha \leq d/2 \) without making further assumptions on \( F \). In this connection, see also [5, §3.4].

(iii) Suppose the positive function \( z \) on \( (0, \infty) \) is nondecreasing and regularly varying with exponent \( \beta > 0 \). Consider the integral

\[ U^*z(t) = \int_0^t z(t-x)U(dx) = \int_0^1 z(t(1-y))U(tdy). \]

By Theorem 5 \( U(ty)/U(t) \to y^\alpha \) and it follows that the measure \( U(tdy)/U(t) \) converges weakly as \( t \to \infty \) to the measure with density \( \alpha y^{\alpha-1} \). Furthermore

\[ f_t(y) = z(t(1-y))/z(t) \to (1-y)^\beta, \quad t \to \infty \]

and the convergence is uniform in \( y, 0 \leq y \leq 1 \), since each \( f_t(y) \) is monotone in \( y \) and the limit function \( (1-y)^\beta \) is continuous. We see therefore that

\[ \frac{U^*z(t)}{z(t)U(t)} = \int_0^1 \frac{z(t(1-y))}{z(t)} \frac{U(tdy)}{U(t)} \to \alpha \int_0^1 (1-y)^\beta y^{\alpha-1} \, dy \]

as \( t \to \infty \). Now \( tz(t) \sim (1 + \beta) \int_0^1 z(x) \, dx \) by Karamata's theorem on regular variation, [3, p. 273]. Hence using (2.1) we see that (2.7) may be put in the equivalent form

\[ \int_0^t z(t-x)U(dx) \sim \frac{D(\alpha, \beta)}{m(t)} \int_0^t z(x) \, dx, \quad t \to \infty, \]

where

\[ D(\alpha, \beta) = \frac{\Gamma(1+\beta)}{\Gamma(1+\alpha)\Gamma(2-\alpha)} \int_0^1 (1-y)^\beta y^{\alpha-1} \, dy = \frac{\Gamma(2+\beta)}{\Gamma(\alpha+\beta+1)\Gamma(2-\alpha)}. \]

Notice that the proof of (2.7) and (2.8) did not depend on the renewal nature, (1.1), of \( U \); (2.8) remains true when \( U > 0 \) is any nondecreasing function regularly varying with exponent \( \alpha, 0 < \alpha \leq 1 \), and \( m \) is any function satisfying (2.1).

J. Teugels [10] gave a proof of (2.8) when \( z > 0 \) is nonincreasing and regularly varying with exponent \( \beta \) where \( -1 < \beta \leq 0 \). The proof is much complicated by the fact that convergence in (2.6) is no longer uniform: when \( \beta < 0 \) the function
(1 − y)β is not bounded at y = 1. (Teugels imposes a supplementary and rather technical condition on U, in addition to regular variation, which seems to me to be unnecessary; compare the proof in Feller [3, p. 447], of a result where similar problems arise.) Again the proof makes no use of the renewal properties of U.

The regular variation of z with exponent β > −1 and to a lesser extent the monotonicity of z is clearly essential to the proof of (2.8). In particular, the condition β > −1 cannot be dropped. When β > −1, the integral \( \int_0^t z(x) \, dx \) occurring in (2.8) diverges to \( \infty \) as \( t \to \infty \), while for \( \beta < -1 \), \( \int_0^x z(x) \, dx \) is finite for all large enough \( A \). In this case, \( \beta < -1 \), Theorem 3, §1, usually applies and leads to results directly opposed to (2.8). For example, let \( z(t) = t^{-5} \), \( t > 1 \) and \( z(t) = 1 \), \( t \leq 1 \) (\( z \) is regularly varying with exponent \( \beta = -5 \)). Then \( \int_0^\infty z(x) \, dx = 5/4 \) and, provided \( \alpha > \frac{1}{2} \), Theorem 3 gives \( m(t) U^* z(t) \to C_a 5/4 < \infty \) as \( t \to \infty \). On the other hand, if (2.8) were true we would get \( m(t) U^* z(t) \to D(\alpha, -5) 5/4 = \infty \).

One last remark. As noted before, one could prove Theorem 5 from Theorem 1 (and Lemma 1) at least for \( \frac{1}{2} < \alpha \leq 1 \). Since (2.8) depends only on Theorem 5 for the regular variation of U and since Theorem 3 is equivalent to Theorem 1, we see that (2.8) could be derived from Theorem 3, at least in principle, when the only data given, besides the function z, is that U is the renewal function of a distribution F of the form (1.2). In no way, however, can Theorem 3 be proved from (2.8).

(iv) The classical “strong” and “weak” renewal theorems assert respectively

(2.9) \[ U(t+h) - U(t) \to h/\mu \quad (h > 0) \]
(2.10) \[ (1/\mu) U(t) \to 1/\mu \]

as \( t \to \infty \), for any (nonarithmetic) distribution \( F \) on \((0, \infty)\) with mean \( \mu \leq \infty \) \((1/\mu \) is interpreted as 0 when \( \mu = \infty \)). Since \( m(t) \to \mu \) as \( t \to \infty \) we may rewrite (2.9) and (2.10) as

\[ U(t+h) - U(t) \sim h/m(t), \quad U(t) \sim t/m(t) \]

provided \( \mu < \infty \). Thus apart from the constant \( C_a \) in (1.4) and \( \Gamma(\alpha + 1) \Gamma(2-\alpha) \)^{-1} = \( C_a/\alpha \) in (2.1), Theorems 1 and 5 are the natural generalizations of these classical theorems.

(v) It should be pointed out that when \( \alpha = 1 \) in (1.2), i.e., if \( F \) has the form \( 1 - F(t) = L(t)/t \) for some slowly varying \( L \), then \( F \) may or may not have a finite mean. For an example when \( \mu < \infty \) consider \( L(t) = \log(t+2)^{-3} \sim (\log t)^{-3} \). For \( \mu = \infty \), consider \( L(t) \sim \text{const} > 0 \).

As noted in (iv), the classical theorems already imply Theorem 1 (and 5) when \( \mu < \infty \). Hence we shall assume from now on that \( \mu = \infty \) when \( \alpha = 1 \) in (1.2).

3. Properties of distributions satisfying (1.2). Let \( F \) be of the form (1.2) (when \( \alpha = 1 \) we assume in addition that \( F \) have infinite expectation, see §2). Let \( \phi \) be the characteristic function of \( F \):

\[ \phi(\theta) = \int_0^\infty e^{ix\theta} F(dx). \]
Lemma 1. The function $m$ defined by (1.3) is regularly varying with exponent $1 - \alpha$, and as $t \to \infty$

$$t(1 - F(t))/m(t) = t^{1 - \alpha} L(t)/m(t) \to 1 - \alpha.$$  

We shall need the following immediate consequence of Lemma 1: let $\eta > 0$, then provided $\alpha > 1/2$ and $B > 0$,

$$\lim_{t \to \infty} t^{-1} m^2(t) \int_{\eta}^{1/B} m^{-2}(x) \, dx = \frac{1}{(2\alpha - 1) B^{2\alpha - 1}}.$$  

Note. The restriction to $\alpha > 1/2$ in (3.2) partly explains the failure (at least of the proof) of Theorems 1 and 3 when $\alpha \leq 1/2$. See equation (5.11).

Proof. This lemma is a direct consequence of Karamata's theorem on regularly varying functions, see Feller [3, p. 273]. The relation (3.2) likewise follows from this theorem. To see this, define $Z(x) = m^{-2}(x)$ for $x \geq \eta$, $Z(x) = 0$, $0 \leq x < \eta$. Since $m$ is regularly varying with exponent $1 - \alpha$, $Z$ varies regularly with exponent $-2(1 - \alpha) = 2\alpha - 2$. Hence, according to the theorem,

$$\lim_{t \to \infty} \frac{t Z(t)}{\int_{\eta}^{1/B} Z(x) \, dx} = \lim_{t \to \infty} \frac{(t/B) Z(t/B)}{\int_{\eta}^{1/B} Z(x) \, dx} = 1 + 2\alpha - 2 = 2\alpha - 1.$$  

But $Z(t/B) \sim (1/B)^{2\alpha - 2} Z(t)$, $t \to \infty$ (by definition of regular variation). Therefore

$$\int_{\eta}^{1/B} m^{-2}(x) \, dx \sim (2\alpha - 1)^{-1} (t/B) Z(t/B) \sim t m^{-2}(t)/(2\alpha - 1) B^{2\alpha - 1}$$

as $t \to \infty$ which proves (3.2).

Lemma 2. As $\theta \to 0+$

$$1 - \phi(\theta) \sim e^{-\frac{\pi \theta}{2}} \Gamma(2 - \alpha) \theta m(1/\theta) \quad (\alpha \neq 0).$$

When $\alpha = 1$ we have in addition to (3.3)

$$\Re (1 - \phi(\theta)) \sim \frac{\pi}{2} \theta L(1/\theta), \quad \theta \to 0+. $$

Proof. Suppose $0 < \alpha < 1$. Then by (3.1) $m(1/\theta) \sim (1 - \alpha)^{-1} \theta^{\alpha - 1} L(1/\theta)$, $\theta \to 0+$. Since $\Gamma(2 - \alpha)/(1 - \alpha) = \Gamma(1 - \alpha)$ we see that (3.3) is equivalent to

$$1 - \phi(\theta) \sim e^{-\frac{\pi \theta}{2}} \Gamma(1 - \alpha) \theta^{\alpha} L(1/\theta), \quad \theta \to 0+.$$  

Stated in this form (3.3) is well known so we omit the proof. See Garsia and Lamperti [5], or Feller [3, Problems 12 and 13, p. 562]. (There is a slight misprint in the latter reference.)

When $\alpha = 1$, (3.3) and (3.4) do not seem to be as well known. Here then is a brief proof. For any $A$, $\theta > 0$, write

$$1 - \phi(\theta) = \left( \int_0^{A/\theta} + \int_{A/\theta}^{\alpha} \right) (1 - e^{\eta \theta}) F(dy) = J_1 + J_2$$
\[ |J_0| = \left| \int_{A/\theta}^{\infty} (1-e^{ix})F(dy) \right| \leq 2(1-F(A/\theta)), \]
\[ J_1 = \int_{0}^{A/\theta} (1-e^{iy})F(dy) = -(1-e^{iA})(1-F(A/\theta)) - i \int_{0}^{A} e^{ix}(1-F(x/\theta)) \, dx. \]

But \(1-F(t) = L(t)/t\) with \(L\) slowly varying. Hence

\[ 1 - \phi(\theta) = o\left(\frac{\theta L(A/\theta)}{A}\right) - i \int_{0}^{A} e^{ix}(1-F(x/\theta)) \, dx. \]  
(The bound in the 0 term is \(\leq 4\) in magnitude.)

We prove (3.3) first. From (3.1) and slow variation of \(L\) we get

\[ L(A/\theta) \sim L(1/\theta) = o(m(1/\theta)), \quad \theta \to 0+. \]

Hence from (3.6)

\[ \lim_{\theta \to 0+} \frac{1-\phi(\theta)}{m(1/\theta)} = -i \lim_{\theta \to 0+} \int_{0}^{A} e^{ix} \left(1-F(x/\theta)\right) \frac{dx}{\theta m(1/\theta)} \]
provided the latter limit exists. Now by Lemma 1 \(m\) is slowly varying (=regularly varying with exponent 0); also \(m(0)=0\). Hence, the measure \(Q_\theta\) on \([0,A]\) with distribution function \(Q_\theta(y)=m(y/\theta)/m(1/\theta)\) converges weakly as \(\theta \to 0+\) to the measure which assigns unit mass to the origin. Whence, for any continuous \(g\) on \([0,A]\)

\[ \int_{0}^{A} g(x)Q_\theta(dx) = \int_{0}^{A} g(x) \left(1-F(x/\theta)\right) \frac{dx}{\theta m(1/\theta)} \]
as \(\theta \to 0+.\) Taking \(g(x)=e^{ix}\) we see that the right-hand side of (3.7) equals \(-i\). This proves (3.3).

Note. The preceding proof requires only minor changes to apply in the case \(0<\alpha<1\). In particular, a term \(O(1/A^\alpha)\) must be added to the right side of (3.7); also \(Q_\theta\) converges to the measure with density \((1-\alpha)x^{-\alpha}\). In (3.7) one lets \(\theta \to 0+\) followed by \(A \to \infty\). The remainder of the proof is then an evaluation of an improper integral.

To prove (3.4), take real parts in (3.6). Then

\[ \text{Re} \left( \frac{1-\phi(\theta)}{\theta L(1/\theta)} \right) = o\left(\frac{1}{A}\right) + \int_{0}^{A} \sin x \cdot \frac{L(x/\theta)}{L(1/\theta)} \, dx. \]
(The bound in the 0 term is \(\leq 8\) for all \(0<\theta \leq \theta_4\) sufficiently small.) Letting \(\theta \to 0+\) and then \(A \to \infty\) we see that

\[ \lim_{\theta \to 0} \frac{\text{Re} \left( \frac{1-\phi(\theta)}{\theta L(1/\theta)} \right)}{\theta L(1/\theta)} = \lim_{A \to \infty} \lim_{\theta \to 0} \int_{0}^{A} \sin x \cdot \frac{L(x/\theta)}{L(1/\theta)} \, dx. \]
provided the iterated limit exists. Since $L$ is slowly varying, we get from the Kara-
mata theorem mentioned earlier

$$\int_0^t L(u) \, du \sim tL(t), \quad t \to \infty.$$  

Hence, for every $y \geq 0$,

$$\lim_{\theta \to 0^+} \int_0^\theta \frac{L(x/\theta)}{L(1/\theta)} \, dx = \lim_{\theta \to 0^+} \frac{\theta}{L(1/\theta)} \int_0^\theta L(u) \, du = y.$$  

That is, the measure with density $L(x/\theta)/L(1/\theta)$, $x \geq 0$, converges weakly as $\theta \to 0$ to Lebesgue measure. Hence for any continuous function $f$ and any compact interval $[0, A]$, say,

$$\lim_{\theta \to 0^+} \int_0^A f(x) \left( \frac{L(x/\theta)}{L(1/\theta)} \right) \, dx = \int_0^A f(x) \, dx.$$  

Letting $f(x) = (\sin x)/x$ and returning to (3.8) we have

$$\lim_{\theta \to 0^+} \frac{\text{Re} \left( 1 - \phi(x) \right)}{\theta L(1/\theta)} = \lim_{A \to \infty} \int_0^A \frac{\sin x}{x} \, dx = \frac{\pi}{2},$$

which proves (3.4).

For the purposes of the next two lemmas put

$$W(x) = \text{Re} \left( \frac{1}{1 - \phi(x)} \right) = \frac{\text{Re} \left( 1 - \phi(x) \right)}{|1 - \phi(x)|^2}.$$  

Note that $IP$ is positive since $\text{Re} \left( 1 - \phi(x) \right) = \int_0^\theta (1 - \cos xt)F(dt) > 0$, and symmetric: $W(-x) = W(x)$. Also, $W$ is unbounded (hence undefined) at all $x$ for which $\phi(x) = 1$ (in particular at $x = 0$); at all other $x$ $W$ is continuous.

**Lemma 3.** As $\theta \to 0^+$

$$\int_0^\theta W(x) \, dx \sim \frac{\cos (\pi \alpha/2)}{(1 - \alpha)\Gamma(2 - \alpha)} \cdot \frac{1}{m(1/\theta)}.$$

When $\alpha = 1$ the constant on the right is replaced by

$$\frac{\pi}{2} \left( \lim_{\alpha \to 1} \frac{\cos (\pi \alpha/2)}{(1 - \alpha)\Gamma(2 - \alpha)} \right).$$

**Remark.** The integrability of $W$ over bounded intervals containing the origin is, of course, part of the conclusion. This fact, however, is true for any distribution on $(0, \infty)$ (and for some distributions on the entire line); see [3, p. 578].

**Proof.** A simple calculation using (3.9) and the asymptotic relations (3.3), (3.4) and (3.5) gives

$$W(x) \sim \frac{k_0 L(1/x)}{x^2 - \alpha m^2(1/x)}, \quad x \to 0^+,$$
where \( k_\alpha \) is the constant occurring on the right in (3.10) \((k_1 = \pi/2)\). Next note that the function \( 1/m(1/x), x > 0 \) is absolutely continuous on any interval bounded away from 0 and \( \infty \). So, by the chain rule and (1.2)

\[
\frac{d}{dx} \left( \frac{1}{m(1/x)} \right) = \frac{1-F(1/x)}{x^2 m^2(1/x)} = \frac{L(1/x)}{x^{2-\alpha} m^2(1/x)}
\]

for almost all \( x \). (The exceptional set is at most countable.)

Consider \( 0 < \epsilon < 1 \) fixed but arbitrary. By (3.11) there is a \( \lambda = \lambda(\epsilon) > 0 \) such that

\[
W(x) > (1 \pm \epsilon) k_\alpha \frac{L(1/x)}{m(1/x)}
\]

whenever \( 0 < x \leq \lambda \). Integrating these inequalities from \( x = \delta \) to \( x = \theta \) and using (3.12) yields

\[
\int_\delta^\theta W(x) \, dx > (1 \pm \epsilon) k_\alpha \int_\delta^\theta \frac{1}{m(1/\theta)} - \frac{1}{m(1/\delta)}
\]

for \( 0 < \delta \leq \theta \leq \lambda \). Now let \( \delta \to 0 \), then \( m(1/\delta) \to \infty \) \((\mu = \infty \) recall), hence

\[
(1-\epsilon) k_\alpha \int_0^\theta W(x) \, dx < (1+\epsilon) \frac{k_\alpha}{m(1/\theta)}
\]

whenever \( 0 < \theta \leq \lambda \). This concludes the proof.

By Lemmas 1 and 3, as \( t \to \infty \)

\[
m(t) \int_0^\theta W(y/t) \, dy = \int_0^\theta W(x) \, dx \to k_\alpha \theta^{1-\alpha}
\]

for all \( \theta > 0 \) and it follows that the measure with density \( q_t(y) = (m(t)/t)W(y/t) \) converges weakly as \( t \to \infty \) to a measure which when \( \alpha = 1 \) is concentrated at the origin with total mass \( k_1 = \pi/2 \) and when \( 0 < \alpha < 1 \) is absolutely continuous with density \((1-\alpha)k_\alpha \theta^{-\alpha}\). Denote the limit measure by \( E_\alpha \). Then for any function \( f \) continuous on a compact interval, \([0,B]\), say,

\[
m(t) \int_0^{Bt} f(t\theta) W(\theta) \, d\theta = \int_0^B f(y) q_t(y) \, dy \to \int_0^B f(y) E_\alpha(\,dy), \quad t \to \infty.
\]

Taking \( f(y) = \cos y \) we have

**Lemma 4.** Let \( W \) be given by (3.9). Then for any \( B > 0 \)

\[
\lim_{t \to \infty} m(t) \int_0^{Bt} W(\theta) \cos \theta \, d\theta = \frac{\cos(\pi \alpha/2)}{\Gamma(2-\alpha)} \int_0^B \frac{\cos y}{y^{\alpha}} \, dy, \quad \alpha \neq 1,
\]

\[
= \pi/2, \quad \alpha = 1.
\]

**Lemma 5.** (i) For all \( \theta_1 \neq \theta_2 \)

\[
|\phi(\theta_2) - \phi(\theta_1)| \leq 2 |\theta_2 - \theta_1| m(1/|\theta_2 - \theta_1|).
\]
(ii) If \( F \) is nonarithmetic, then for each \( A > 0 \), there is a number \( k > 0 \), which may depend on \( A \), such that

\[
(3.16) \quad \theta m(1/\theta) \leq k|1 - \phi(\theta)| \quad \text{for } 0 < \theta \leq A.
\]

If \( F \) is arithmetic with span \( h \), (3.16) is true provided \( A < 2\pi/h = \text{period of } \phi \).

**Proof.** (i) Fix \( B > 0 \). Then

\[
\left| \phi(\theta_2) - \phi(\theta_1) \right| = \left| \left( \int_0^B + \int_B^\infty \right) (e^{ix\theta_2} - e^{ix\theta_1}) F(dx) \right|
\]

\[
\leq \int_0^B |e^{ix\theta_2} - e^{ix\theta_1}| F(dx) + 2(1 - F(B))
\]

\[
\leq |\theta_2 - \theta_1| \int_0^B xF(dx) + 2(1 - F(B)).
\]

But \( 0 \leq \int_0^B xF(dx) = m(B) - B(1 - F(B)) \) by (1.3). Hence setting \( B = |\theta_2 - \theta_1|^{-1} \) we get

\[
|\phi(\theta_2) - \phi(\theta_1)| \leq B^{-1}|m(B) - B(1 - F(B))| + 2(1 - F(B)) = B^{-1}m(B) + 1 - F(B) \leq 2B^{-1}m(B)
\]

which proves (3.15). (Note that (1.2) was not used; (3.15) holds for any \( F \) on \([0, \infty)\).)

(ii) If \( F \) is nonarithmetic then \( 1 - \phi(\theta) > 0 \) for all \( \theta \neq 0 \). By Lemma 2 as \( \theta \to 0 + \)

\[
\theta m(1/\theta)/|1 - \phi(\theta)| \to 1/\Gamma(2 - \alpha)
\]

and it follows that the function

\[
\beta(\theta) = \theta m(1/\theta)|1 - \phi(\theta)|^{-1}, \quad \theta \neq 0
\]

\[
= (\Gamma(2 - \alpha))^{-1}, \quad \theta = 0
\]

is continuous on \([0, A]\). Taking \( k = \max \{\beta(\theta) : 0 \leq \theta \leq A\} \) gives (3.16).

4. An inversion formula for the renewal measure. Define the symmetric renewal measure

\[
V\{I\} = \frac{1}{2}(U\{I\} + U\{-I\})
\]

where \( U \) is given by (1.1) and \(-I = \{x : -x \in I\}\). In this section we establish the following

**Formula.** Suppose \( F \) is nonarithmetic and has an infinite mean. Then for any continuous function \( g \) with compact support whose Fourier transform

\[
\gamma(x) = \int_{-\infty}^{\infty} e^{ix\theta} g(\theta) \, d\theta
\]

satisfies

\[
\gamma(x) = O(1/|x|^2), \quad |x| \to \infty,
\]

we have

\[
\int_{-\infty}^{\infty} e^{-ix\lambda} \gamma(x) V\{t+dx\} = \int_{-\infty}^{\infty} e^{-i\theta g(\theta + \lambda)} \Re \left( \frac{1}{1 - \phi(\theta)} \right) d\theta
\]
for all real \( \lambda \) and \( t \). Here, as elsewhere, \( \phi \) is the characteristic function of \( F \). Note that the integral on the right in (4.3) only extends over a bounded interval. For examples of \( g \) and \( \gamma \) see §5.

**Lemma 6.** Let \( \gamma \) be any continuous function satisfying (4.2). Then for every \( t \) the integral

\[
\int_{-\infty}^{\infty} |\gamma(x-t)| V(dx)
\]

is finite.

**Proof.** Since \( \int_{-1}^{1} |\gamma(x-t)| V(dx) < \infty \) and since \( |\gamma(x-t)| \) is bounded by a constant (which may depend on \( t \) but not \( x \)) times \( 1/x^2 \), it suffices to show

\[
(4.4) \quad \int_{|x| \geq 1} \frac{1}{x^2} V(dx) = \int_{1}^{\infty} \frac{1}{x^2} U(dx) < \infty.
\]

From (2.10) it follows that \( U(x) \leq k_1 x \) for some constant \( k_1 < \infty \) and all \( x \geq 1 \). Therefore integrating by parts in (4.4) we get

\[
\int_{1}^{\infty} \frac{1}{x^2} U(dx) = \lim_{A \to \infty} \left( \frac{U(A)}{A^2} - U(1) + 2 \int_{1}^{A} \frac{U(x)}{x^3} dx \right)
\]

\[
= -U(1) + 2 \int_{1}^{\infty} \frac{U(x)}{x^3} dx \leq 2k_1 \int_{1}^{\infty} \frac{1}{x^2} dx < \infty
\]

which proves (4.4) and the lemma.

For \( 0 \leq s < 1 \) let \( V_s \) be the finite symmetric measure

\[
V_s(dx) = \frac{1}{2} \sum_{n=0}^{\infty} s^n(\Phi^n(dx) + \Phi^n(-dx))
\]

and note that

\[
(4.5) \quad V_s[I] \uparrow V[I] \quad \text{as} \quad s \uparrow 1
\]

for every measurable \( I \) bounded or not.

Since

\[
\Phi(-\theta) = \overline{\Phi(\theta)}
\]

we have

\[
\int_{-\infty}^{\infty} e^{ix\theta} V_s(dx) = \frac{1}{2} \sum_{n=0}^{\infty} s^n(\Phi^n(\theta) + \Phi^n(-\theta)) = \text{Re} \left( \frac{1}{1-s\Phi(\theta)} \right)
\]

and an application of Fubini's theorem gives

\[
\int_{-\infty}^{\infty} \gamma(x) V_s(dx) = \int_{-\infty}^{\infty} g(\theta) \text{Re} \left( \frac{1}{1-s\Phi(\theta)} \right) d\theta \quad (0 \leq s < 1)
\]

for any (Lebesgue) integrable function \( g \) with \( \gamma \) given by (4.1). Replacing \( g \) by

\[
g_s(\theta) = e^{-s\theta} g(\theta + \lambda)
\]
and \( y \) by

\[
\gamma_1(x) = \int_{-\infty}^{\infty} e^{i\lambda x} g_1(\theta) \, d\theta = e^{-i\lambda(x-t)\gamma(x-t)}
\]

we get

(4.6) \[
\int_{-\infty}^{\infty} e^{-i\lambda(x-t)\gamma(x-t)} V_s(dx) = \int_{-\infty}^{\infty} e^{-i\lambda(x-t)} g(\theta+\lambda) \, \Re \left( \frac{1}{1-s\phi(\theta)} \right) \, d\theta.
\]

**Lemma 7.** For any continuous function \( h \) with compact support

(4.7) \[
\lim_{s \to 1} \int_{-\infty}^{\infty} h(\theta) \, \Re \left( \frac{1}{1-s\phi(\theta)} \right) \, d\theta = \int_{-\infty}^{\infty} h(\theta) \, \Re \left( \frac{1}{1-\phi(\theta)} \right) \, d\theta
\]

provided \( F \) is nonarithmetic and has infinite expectation.

**Proof.** We base the proof on the following proposition due to Feller and Orey [4]:

**Proposition.** The measure whose density is

\[
\frac{1}{1 + \bar{\theta}^2} \Re \left( \frac{1}{1-s\phi(\theta)} \right)
\]

converges weakly and in variation to a finite measure as \( s \to 1^- \). In every interval excluding the origin the limit measure is automatically absolutely continuous with density given by

\[
\frac{1}{1 + \bar{\theta}^2} \Re \left( \frac{1}{1-\phi(\theta)} \right).
\]

If \( \beta \) is the mass assigned to the origin by the limit then \( \beta = \pi|\mu| > 0 \) when \( \mu \) (the mean of \( F \)) is finite and \( \beta = 0 \) in case \( \mu = \infty \).

We omit the proof. (Besides the Feller-Orey paper, see also Breimann [1, p. 221], and Feller [3, p. 578].) The proposition implies, among other things, that

\[
\lim_{s \to 1^-} \int_{-\infty}^{\infty} \frac{f(\theta)}{1 + \bar{\theta}^2} \Re \left( \frac{1}{1-s\phi(\theta)} \right) \, d\theta = \beta f(0) + \int_{-\infty}^{\infty} \frac{f(\theta)}{1 + \bar{\theta}^2} \Re \left( \frac{1}{1-\phi(\theta)} \right) \, d\theta
\]

for every continuous function \( f \) with compact support. In our case \( \beta = 0 \), and (4.7) follows by setting \( f(\theta) = (1 + \theta^2) h(\theta) \).

**Proof of formula (4.3).** The very strong convergence (4.5) of the measures \( V_s \) to \( V \) implies

(4.8) \[
\lim_{s \to 1^-} \int_{-\infty}^{\infty} f(x) V_s(dx) = \int_{-\infty}^{\infty} f(x) V(dx)
\]

for every \( f \) integrable with respect to \( V \). (In fact, if \( f \) is nonnegative the integral on the left is nondecreasing as a function of \( s \) and one can show (4.8) holds even if \( f \) is not integrable.)
Suppose now \( g \) and \( \gamma \) satisfy (4.1) and (4.2) with \( g \) continuous and vanishing off a compact set. Then by Lemma 6
\[
e^{-tA(x-\theta)\gamma(x-t)}
\]
is integrable with respect to \( V\{dx\} \) for every \( t \) and \( \lambda \). Hence by (4.6) and (4.8)
\[
\int_{-\infty}^{\infty} e^{-tA\gamma(x)}V\{t+dx\} = \int_{-\infty}^{\infty} e^{-tA\gamma(x)}V\{dx\}
= \lim_{\lambda \to \infty} \int_{-\infty}^{\infty} e^{-t\lambda g(\theta + \lambda)} \operatorname{Re} \left( \frac{1}{1-\phi(\theta)} \right) d\theta.
\]
Formula (4.3) now follows from Lemma 7.

5. Proof of Theorem 1.

1°. Introduce measures \( \mu_t, t>0 \), by
\[
\mu_t(I) = 2m(t)\mathcal{V}\{I+t\} = m(t)(\mathcal{U}\{I+t\} + \mathcal{U}\{-I-t\})
\]
where \( I \) is measurable and \( I+t=\{x : x-t \in I\} \). Since \( U \) is concentrated on \([0, \infty)\) it follows by taking \( I=\{0, h\} \) in (5.1) that
\[
U(t+h) - U(t) = (1/m(t))\mu_t(I).
\]
Therefore to prove Theorem 1 it suffices to show
\[
\mu_t(I) \to C_\alpha |I|, \quad t \to \infty,
\]
for every bounded interval \( I \) where \( |I| \) denotes the length of \( I \) and
\[
C_\alpha = [\Gamma(\alpha)\Gamma(2-\alpha)]^{-1}.
\]
For each \( \alpha > 0 \) put \( \gamma_\alpha(0)=1 \) and
\[
\gamma_\alpha(x) = 2(1-\cos(\alpha x))/\alpha^2 x^2.
\]

**Lemma 8.** Let \( \{\mu_t\}, t>0, \) be a family of measures such that \( \mu_t(I) < \infty \) for every compact set \( I \) and all \( t \). Suppose for some constant \( C \)
\[
\lim_{t \to \infty} \int_{-\infty}^{\infty} e^{-t\lambda \gamma_\alpha(x)} \mu_t(dx) = C \int_{-\infty}^{\infty} e^{-t\lambda \gamma_\alpha(x)} dx
\]
for every \( \alpha > 0 \) and all real \( \lambda \). Then \( C^{-1}\mu_t \) converges weakly to Lebesgue measure:
\( \mu_t(I) \to C|I| \) for every bounded interval \( I \).

(We defer the proof until §6.)

Now \( \gamma_\alpha \) is the Fourier transform (4.1) of the function
\[
g_\alpha(\theta) = (1/\alpha)(1-|\theta|/\alpha), \quad \text{ when } |\theta| \leq \alpha
\]
\[
= 0, \quad \text{ when } |\theta| > \alpha.
\]
Whence by the Fourier inversion theorem
\[
\int_{-\infty}^{\infty} e^{-t\lambda \gamma_\alpha(x)} dx = 2\pi g_\alpha(\lambda).
\]
Clearly we may also apply our inversion formula (4.3) to obtain

\[(5.7) \quad \int_{-\infty}^{\infty} e^{-t(x+y)2\alpha}(x)\mu_t(dx) = 2m(t) \int_{-\infty}^{\infty} e^{-t\theta^2g_\alpha(\theta+\lambda)}W(\theta) \, d\theta\]

where \(W(\theta) = \text{Re} \left[\frac{1}{1-\phi(\theta)}\right]^{-1}\). Note that the integral on the right extends from \(\theta = -a-\lambda\) to \(\theta = a-\lambda\). From (5.6) and (5.7) we see that (5.4) in our case is equivalent to

\[(5.8) \quad \lim_{t \to \infty} m(t) \int_{-\infty}^{\infty} e^{-t\theta g_\alpha(\theta+\lambda)}W(\theta) \, d\theta = \pi C g_\alpha(\lambda)\]

and, by Lemma 8, the proof of (5.2) (and Theorem 1) will be completed when we establish (5.8), with \(C = C_\alpha\) for every \(a > 0\) and all real \(\lambda\).

2°. Let \(B > 1\) be fixed but otherwise arbitrary, and write the integral in (5.8) as the sum \(J_1 + J_2\) where

\[\begin{align*}
J_1(t, B) &= \int_{-B^2t}^{B^2t} e^{-t\theta g_\alpha(\theta+\lambda)}W(\theta) \, d\theta \\
J_2(t, B) &= \int_{B^2t}^{\infty} e^{-t\theta g_\alpha(\theta+\lambda)}W(\theta) \, d\theta
\end{align*}\]

The last integral follows by making the substitution \(\theta \to -\theta\) in the integral \(\int_{-\infty}^{B^2t}\), using the evenness of the functions \(g_\alpha\) and \(W\) and noting that \(g_\alpha\) vanishes outside the interval \((-a, a)\). We will show

\[(5.10) \quad m(t)J_1(t, B) = g_\alpha(\lambda) \frac{2 \cos \frac{\pi \alpha}{2}}{\Gamma(2-\alpha)} \int_0^B \cos x \frac{x^\alpha}{x^\alpha} \, dx, \quad \alpha \neq 1\]

\[\quad = \pi g_\alpha(\lambda), \quad \alpha = 1\]

and

\[(5.11) \quad \limsup_{t \to \infty} m(t)|J_2(t, B)| = O\left(\frac{1}{B^2-1}\right), \quad \frac{1}{2} < \alpha \leq 1\]

which lead directly to (5.8).

3°. **Proof of (5.10).** It is clear from (5.5) that

\[(5.12) \quad |g_\alpha(\theta_2) - g_\alpha(\theta_1)| \leq \frac{1}{a^2} |\theta_2 - \theta_1|\]

for all \(\theta_1, \theta_2\). Hence

\[m(t) \left|J_1(t, B) - g_\alpha(\lambda) \int_{-B^2t}^{B^2t} e^{-t\theta}W(\theta) \, d\theta\right| \leq m(t) \int_{-B^2t}^{B^2t} |g_\alpha(\theta+\lambda) - g_\alpha(\lambda)|W(\theta) \, d\theta \leq \frac{2B}{a^2} \frac{m(t)}{t} \int_0^B W(\theta) \, d\theta = O\left(\frac{1}{t}\right)\]
where the $O(1/t)$ follows from (3.10) and Lemma 1. Thus
\[
\lim_{t \to \infty} m(t) J_1(t, B) = g_a(\lambda) \lim_{t \to \infty} m(t) \int_{-B/2t}^{B/2t} e^{-u\theta} W(\theta) \, d\theta
\]
\[
= 2g_a(\lambda) \lim_{t \to \infty} m(t) \int_0^{B/2t} W(\theta) \cos \theta \, d\theta
\]
and (5.10) now follows from Lemma 4.

4°. Proof of (5.11). Let
\[
h_1(\theta) = e^{-u\theta} g_a(\theta + \lambda) + e^{u\theta} g_a(\theta - \lambda),
\]
\[
h_2(\theta) = e^{-u\theta} g_a(\theta + \pi/t + \lambda) + e^{u\theta} g_a(\theta + \pi/t - \lambda).
\]
Then $h_1(\theta + \pi/t) = -h_2(\theta)$ and making the change of variables $\theta \to \theta + \pi/t$ in (5.9) gives
\[
J_2(t, B) = \int_{B/2t}^{A} h_1(\theta) W(\theta) \, d\theta = \int_{(B - \pi)/2t}^{A} -h_3(\theta) W(\theta + \pi/t) \, d\theta
\]
(note that the integrand in the last written integral vanishes for $A - \pi/t \leq \theta$).

Adding these integrals we get
\[
2J_2 = -\int_{B - \pi/t}^{B/2t} h_2(\theta) W(\theta + \pi/t) \, d\theta + \int_{B/2t}^{A} \left[ h_1(\theta) W(\theta) - h_2(\theta) W(\theta + \pi/t) \right] \, d\theta.
\]

Now $|h_1(\theta)| \leq 2/a$ and from (5.12) we have
\[
|h_1(\theta) - h_2(\theta)| \leq \left| g_a(\theta + \lambda) - g_a\left(\theta + \lambda + \frac{\pi}{t}\right) \right| + \left| g_a(\theta - \lambda) - g_a\left(\theta - \lambda + \frac{\pi}{t}\right) \right| \leq \frac{2\pi}{a^2 t}.
\]

Thus
\[
|h_1(\theta) W(\theta) - h_2(\theta) W(\theta + \frac{\pi}{t})| \leq |h_1(\theta) - h_2(\theta)| W(\theta) + \left| W(\theta) - W\left(\theta + \frac{\pi}{t}\right) \right| |h_2(\theta)|
\]
\[
\leq \frac{2\pi}{a^2 t} W(\theta) + \frac{2}{a} \left| W\left(\theta + \frac{\pi}{t}\right) - W(\theta) \right|.
\]

Applying these inequalities in (5.13) gives
\[
|J_2| \leq \frac{1}{a} \int_{(B - \pi)/2t}^{B/2t} W\left(\theta + \frac{\pi}{t}\right) d\theta + \frac{\pi}{a^2 t} \int_{B/2t}^{A} W(\theta) \, d\theta
\]
\[
+ \frac{1}{a} \int_{B/2t}^{A} \left| W\left(\theta + \frac{\pi}{t}\right) - W(\theta) \right| d\theta.
\]

From Lemma 3 it is clear that
\[
\lim_{t \to \infty} m(t) \int_{(B - \pi)/2t}^{B/2t} W\left(\theta + \frac{\pi}{t}\right) d\theta = k_a[(B + \pi)^{1-a} - B^{1-a}] = O\left(\frac{1}{B^a}\right).
\]

Also, since $W$ is integrable on $[0, A]$, $A < \infty$,
\[
\frac{\pi}{a^2 t} \int_{B/2t}^{A} W(\theta) \, d\theta = O\left(\frac{m(t)}{t}\right) \to 0 \quad \text{as} \quad t \to \infty.
\]
(That $m(t)/t \to 0$, $t \to \infty$, follows from Lemma 1, §3, in our case, but is true for any $F$ on $[0, \infty)$ with $m$ given by (1.3)). Hence from (5.14)

$$\limsup_{t \to \infty} m(t) |J_2(t, B)| = a^{-1} \limsup_{t \to \infty} m(t) \int_{B/t}^{\mathcal{A}} \left| W\left(\theta + \frac{\pi}{t}\right) - W(\theta) \right| d\theta + O\left(\frac{1}{B^a}\right).$$

But $O(B^{-a}) = O(B^{1-2a})$ ($B > 1$, $0 \leq a \leq 1$), so the proof of (5.11) will be complete when we show

$$\limsup_{t \to \infty} m(t) \int_{B/t}^{\mathcal{A}} \left| W\left(\theta + \frac{\pi}{t}\right) - W(\theta) \right| d\theta = O\left(\frac{1}{B^{2a-1}}\right).$$

By Lemma 5 (i) we get

$$\left| W\left(\theta + \frac{\pi}{t}\right) - W(\theta) \right| = \left| \Re \frac{\phi(\theta + \pi/t) - \phi(\theta)}{[1 - \phi(\theta + \pi/t)][1 - \phi(\theta)]} \right| \leq \frac{2(\pi/t)m(t/\pi)}{|1 - \phi(\theta + \pi/t)||1 - \phi(\theta)|}.$$

Applying this estimate and the Cauchy-Schwarz inequality to the integral in (5.15) gives

$$\int_{B/t}^{\mathcal{A}} \left| W\left(\theta + \frac{\pi}{t}\right) - W(\theta) \right| d\theta \leq \frac{2\pi}{t} m\left(\frac{1}{\pi}\right) \left( \int_{B/t}^{\mathcal{A}} \frac{d\theta}{|1 - \phi(\theta + \pi/t)|^2} \right)^{1/2} \left( \int_{B/t}^{\mathcal{A}} \frac{d\theta}{|1 - \phi(\theta)|^2} \right)^{1/2} < 8 m(t/\pi) \int_{B/t}^{\mathcal{A}} \frac{d\theta}{|1 - \phi(\theta)|^2} \quad (\pi/t \leq \mathcal{A}).$$

Again by Lemma 5(ii) there is a constant $k < \infty$ such that

$$1/|1 - \phi(\theta)| \leq k/\theta m(1/\theta)$$

for $0 < \theta \leq 2A$. Consequently

$$\int_{B/t}^{\mathcal{A}} \frac{d\theta}{|1 - \phi(\theta)|^2} \leq k^2 \int_{B/t}^{\mathcal{A}} \frac{d\theta}{\theta^2 m^2(1/\theta)} = k^2 \int_{\eta}^{\mathcal{A}} \frac{dx}{m^2(x)}$$

where $\eta = 1/2A$. Combining (5.16) and (5.17) we get

$$\limsup_{t \to \infty} m(t) \int_{B/t}^{\mathcal{A}} \left| W\left(\theta + \frac{\pi}{t}\right) - W(\theta) \right| d\theta \leq 8k^2 \lim_{t \to \infty} \frac{m^2(t)}{t} \int_{\eta}^{\mathcal{A}} \frac{dx}{m^2(x)} \leq \frac{1}{(2a-1)B^{2a-1}} \quad (\alpha > \frac{1}{2})$$

where the last equality comes from (3.2). This completes the proof of (5.15) and hence of (5.11).

5°. The proof of (5.8) with $C = C_a = [\Gamma(a)\Gamma(2-a)]^{-1}$ is now almost immediate. Let

$$\Delta(t) = \left| m(t) \int_{-\infty}^{\infty} e^{-u^a g_a(\theta + \lambda)} W(\theta) d\theta - \pi C_a g_a(\lambda) \right| = \left| m(t) (J_1 + J_2) - \pi C_a g_a(\lambda) \right|$$

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and suppose \( \alpha \neq 1 \). Then by (5.10) and (5.11)

\[
\limsup_{t \to \infty} \Delta(t) \leq \lim_{t \to \infty} \left| m(t) J_1 - \frac{\pi g_\alpha(\lambda)}{\Gamma(\alpha) \Gamma(2-\alpha)} \right| + \limsup_{t \to \infty} m(t) |J_2|
\]

(5.18)

\[
= \frac{g_\alpha(\lambda)}{\Gamma(2-\alpha)} \left| 2 \cos \left( \frac{\pi \alpha}{2} \right) \int_0^B \cos \frac{\pi x}{x^a} \, dx - \frac{\pi}{\Gamma(\alpha)} \right| + O \left( \frac{1}{B^{2\alpha-1}} \right).
\]

Now as \( B \to \infty \), \( \int_0^B x^{-\alpha} \cos x \, dx \to \sin \left( \frac{\pi \alpha}{2} \right) \Gamma(1-\alpha) \), hence

\[
\lim_{B \to \infty} \left| 2 \cos \left( \frac{\pi \alpha}{2} \right) \int_0^B \cos \frac{\pi x}{x^a} \, dx - \frac{\pi}{\Gamma(\alpha)} \right| = \left| \sin \left( \frac{\pi \alpha}{2} \right) \Gamma(1-\alpha) - \frac{\pi}{\Gamma(\alpha)} \right| = 0.
\]

Therefore taking the limit in (5.18) as \( B \to \infty \) we get

\[
\limsup_{t \to \infty} \Delta(t) = \lim_{B \to \infty} \limsup_{t \to \infty} \Delta(t) = 0
\]

which proves (5.8) when \( \alpha \neq 1 \). When \( \alpha = 1 \) the proof of (5.8), with \( C = C_1 = 1 \), from (5.10) and (5.11) is even simpler so we omit it. Theorem 1 now follows from Lemma 8.

6. Proof of Lemma 8. There is no loss in generality in supposing \( C = 1 \). Taking \( \lambda = 0 \) in (5.4) and (5.6) we see that as \( t \to \infty \)

\[
\Delta(t) = \int_{-\infty}^{\infty} \gamma_\alpha(x) \mu_t(dx) \to \int_{-\infty}^{\infty} \gamma_\alpha(x) \, dx = \frac{2\pi}{a} > 0.
\]

Hence (5.4) implies that the characteristic function of the probability measure

\[
P_t(dx) = \frac{1}{\Delta_t(a)} \gamma_\alpha(x) \mu_t(dx)
\]

converges pointwise to the characteristic function of the probability measure

\[
P(dx) = (a/2\pi) \gamma_\alpha(x) \, dx.
\]

Consequently, by the continuity theorem for characteristic functions \( P_t \) converges weakly to \( P \) as \( t \to \infty \). Whence

\[
(6.1) \quad \lim_{t \to \infty} \int_{-\infty}^{\infty} B(x) \gamma_\alpha(x) \mu_t(dx) = \int_{-\infty}^{\infty} B(x) \gamma_\alpha(x) \, dx
\]

for every bounded continuous function \( B \) on \( \mathbb{R}^1 \) and for every \( a > 0 \).

For any continuous function \( f \) with compact support, write

\[
\lambda(f) = \int_{-\infty}^{\infty} f(x) \mu_t(dx), \quad \lambda(f) = \int_{-\infty}^{\infty} f(x) \, dx.
\]

Let \( I \) be a bounded interval and let \( \varepsilon > 0 \) be arbitrary but fixed. We can find continuous functions \( f^+ \) and \( f^- \) both with compact support such that

(i) \( 0 \leq f^- \leq 1 \), \( f^-(x) = 0 \) for \( x \notin I \),

(ii) \( |I| \leq \lambda(f^-) + \varepsilon \).
(iii) \( f^+ \geq 0, f^+(x) = 1 \) for \( x \) in \( I \),
(iv) \( \lambda(f^+) \leq |I| + \epsilon \).

Now choose \( a > 0 \) so small that

\[
f^+(x) = f^-(x) = 0 \quad \text{when} \quad |x| \geq \pi/4a.
\]

Then since

\[
\gamma_a(x) = 2 \left( 1 - \cos \frac{ax}{a^2} \right) > 0 \quad \text{for} \quad |x| < \pi/2a
\]

it follows that \( B^+ = f^+/\gamma_a \) and \( B^- = f^-/\gamma_a \) are continuous functions on \( \mathbb{R}^1 \) with compact support (hence bounded). Therefore by (6.1)

\[
\lambda(f^+) = \int_{-\infty}^{\infty} B^+(x) \gamma_a(x) \mu(x) \, dx = \int_{-\infty}^{\infty} B^+(x) \gamma_a(x) \, dx = \lambda(f^+).
\]

From (i) and (iii) it is clear that

\[
\lambda(f^-) = \frac{1}{4} \int_{-\infty}^{\infty} f^+(x) \gamma_a(x) \mu(x) \, dx
\]

for all \( t > 0 \). Letting \( t \to \infty \) and using (6.2) we get

\[
\lambda(f^-) \leq \liminf_{t \to \infty} \mu_t(I) \leq \limsup_{t \to \infty} \mu_t(I) \leq \lambda(f^+),
\]

and hence by (ii) and (iv)

\[
|I| - \epsilon \leq \liminf_{t \to \infty} \mu_t(I) \leq \limsup_{t \to \infty} \mu_t(I) \leq |I| + \epsilon.
\]

Since this holds for every \( \epsilon > 0 \) it follows that

\[
\mu_t(I) \to |I|, \quad t \to \infty,
\]

which completes the proof.

7. Proof of Theorem 2.

Let our first task is to show

\[
\liminf_{t \to \infty} m(t)(U(t+h) - U(t)) \geq C \, h \quad (h > 0),
\]

or, equivalently,

\[
\liminf_{t \to \infty} t^{1-a} L(t)(U(t+h) - U(t)) \geq \frac{\sin \pi \alpha}{\pi} \, h.
\]

(See remark following the statement of Theorem 2.)

Condition (1.2) with \( 0 < \alpha < 1 \) is necessary and sufficient for \( F \) to be in the domain of attraction of the unique (apart from a scale factor) stable distribution with exponent \( \alpha \) concentrated on \([0, \infty)\). Thus if a sequence \( \{B_n\} \) is chosen so that

\[
0 < B_n \uparrow \infty \quad \text{and} \quad n(1 - F(B_n)) \equiv nB_n^{-a}L(B_n) \to 1
\]

as \( n \to \infty \), then

\[
F^*(B_n, x) \to \int_0^x q_a(y) \, dy \quad (n \to \infty, \ x \geq 0)
\]
where \( q_a > 0 \) and satisfies

\[
\int_0^\infty e^{-\lambda y} q_a(y) \, dy = \exp \left[ -\lambda^a \Gamma(1 - \alpha) \right], \quad \lambda \geq 0.
\]

In addition to (7.3) a local limit theorem for nonarithmetic distributions due to C. Stone [9] implies the somewhat stronger result

\[
F^{\kappa'}(t + h) - F^{\kappa'}(t) = (h|B_k)q_a(t|B_k) + \delta_k|B_k
\]

where \( \delta_k \to 0 \) as \( k \to \infty \) uniformly in \( t > 0 \) ((7.3) only allows \( F^{\kappa'}(t + h) - F^{\kappa'}(t) \sim hB_k^{-1}q_a(tB_k^{-1}) \) for \( t \) and \( h \) fixed). Using (7.4) we prove (7.2) almost exactly as Garsia and Lamperti [5] prove the analogous inequality in the arithmetic case. Thus from (1.1) and (7.4)

\[
U(t + h) - U(t) > \sum_{k=n}^r \left( F^{\kappa'}(t + h) - F^{\kappa'}(t) \right)
\]

\[
= h \sum_{k=n}^r \frac{1}{B_k} q_a \left( \frac{t}{B_k} \right) + \frac{\delta_k}{B_k}.
\]

Let \( 0 < A < C < \infty \), and choose \( n = \lceil At^\alpha/L(t) \rceil, r = \lceil Ct^\alpha/L(t) \rceil \). Then, as in [5], we have both

\[
t^{1-a}L(t) \sum_{n}^{r} \frac{\delta_k}{B_k} = o(1), \quad t \to \infty
\]

and, writing \( x_k = kL(t)/t^{\alpha}, n \leq k \leq r, \)

\[
t^{1-a}L(t) \sum_{n}^{r} \frac{1}{B_k} q_a \left( \frac{t}{B_k} \right) \sim \sum_{A \leq x_k \leq C} x_k^{-1/\alpha} q_a(x_k^{-1/\alpha})(x_{k+1} - x_k)
\]

\[
\to \int_A^C x^{-1/\alpha} q_a(x^{-1/\alpha}) \, dx
\]

as \( t \to \infty \). Hence for any \( \epsilon > 0 \)

\[
t^{1-a}L(t)(U(t + h) - U(t)) \geq \int_A^C x^{-1/\alpha} q_a(x^{-1/\alpha}) \, dx - \epsilon
\]

for all \( t \) sufficiently large. In other words

\[
\liminf_{t \to \infty} t^{1-a}L(t)(U(t + h) - U(t)) \geq \int_A^C x^{-1/\alpha} q_a(x^{-1/\alpha}) \, dx,
\]

and (7.2) now follows by letting \( A \to 0, C \to \infty \) and noting

\[
\int_0^\infty x^{-1/\alpha} q_a(x^{-1/\alpha}) \, dx = \alpha \int_0^\infty y^{-\alpha} q_a(y) \, dy = \frac{\sin \pi \alpha}{\pi}.
\]

2°. To complete the proof of Theorem 2 we need the following lemma (also needed in the proof of Theorem 3).

**Lemma 9.** Let \( z \) be any nonnegative integrable (but not necessarily dri) function on \([0, \infty)\). Then

\[
\liminf_{t \to \infty} m(t) \int_0^t z(t - y) U(dy) \leq C_\alpha \int_0^\infty z(x) \, dx \quad (0 < \alpha \leq 1).
\]
To finish the proof of Theorem 2 we set $z(x) = 1$ for $0 \leq x \leq h$, $z(x) = 0$ elsewhere. Noting that $m(t+h) \sim m(t)$ as $t \to \infty$ we get from (7.5)

$$
\liminf_{t \to \infty} m(t)(U(t+h) - U(t)) = \liminf_{t \to \infty} m(t+h)U^*z(t+h)
$$

(7.6)

$$
\leq C_a \int_0^\infty z(x) \, dx = C_a h.
$$

Together (7.1) and (7.6) give (1.5).

**Proof of Lemma 9.** Let $v(t) = U^*z(t) = \int_0^t z(t-x)U(dx)$. Then

$$
\delta(\lambda) = \int_0^\infty e^{-\lambda x} v(x) \, dx = \left(\int_0^\infty e^{-\lambda x} z(x) \, dx\right)\tilde{U}(\lambda) = \tilde{z}(\lambda)\tilde{U}(\lambda)
$$

where $\tilde{U}$ is defined as in §2(i). Since $U$ is regularly varying with exponent $a$ we have

$$
\tilde{U}(\lambda) \sim \Gamma(\alpha+1)U(1/\lambda) \quad \text{as } \lambda \to 0+
$$

by Theorem 1 in [3, p. 420]. Now $\tilde{z}(0) = \int_0^\infty z(x) \, dx < \infty$ and it follows that

$$
\delta(\lambda) \sim \tilde{z}(0)\Gamma(\alpha+1)U(1/\lambda), \quad \lambda \to 0+
$$

which, by the converse of the same Theorem 1 in [3], is the same as

(7.7)

$$
\int_0^t v(x) \, dx \sim \tilde{z}(0)U(t), \quad t \to \infty.
$$

Now by Theorem 5 in §2

(7.8)

$$
U(t) \sim (\Gamma(\alpha+1)\Gamma(2-\alpha))^{-1} t/m(t) = (C_a/\alpha)t/m(t)
$$

as $t \to \infty$; also, since $1/m$ is regularly varying with exponent $\alpha - 1 > -1$ we have for fixed $\eta > 0$

(7.9)

$$
\frac{1}{\alpha} \frac{t}{m(t)} \sim \int_\eta^t \frac{dx}{m(x)}, \quad t \to \infty
$$

(cf. [3, p. 273]). From (7.7), (7.8), and (7.9) it follows that

(7.10)

$$
\int_0^t v(x) \, dx \sim C_a \tilde{z}(0) \int_\eta^t \frac{dx}{m(x)}, \quad t \to \infty.
$$

Suppose, contrary to (7.5),

$$
\liminf_{t \to \infty} m(t)v(t) > C_a \tilde{z}(0).
$$

Then for some $\epsilon > 0$ and all $x \geq \eta$ sufficiently large

$$
v(x) \geq (1+\epsilon)C_a \tilde{z}(0)(1/m(x)).
$$

Hence

$$
\int_0^t v(x) \, dx \geq \int_\eta^t v(x) \, dx \geq (1+\epsilon)C_a \tilde{z}(0) \int_\eta^t \frac{dx}{m(x)}
$$

for all $t \geq \eta$. But this contradicts (7.10).

1°. Let $h > 0$. Throughout this section put $z_k(x) = 1$ when $(k-1)h \leq x < kh$, $z_k(x) = 0$ elsewhere, and let

$$v_k(t) = U^*z_k(t) = U(t-(k-1)h) - U(t-kh).$$

Since $m(t-kh) \sim m(t)$ for fixed $kh$, $t \to \infty$, we have by Theorems 1 and 2

$$\liminf_{t \to \infty} m(t)v_k(t) = C_\alpha h \quad (0 < \alpha \leq \frac{1}{2}),$$

(8.1)

$$\lim_{t \to \infty} m(t)v_k(t) = C_\alpha h \quad \left(\frac{1}{2} < \alpha \leq 1\right); \quad k = 1, 2, \ldots.$$

2°. Let $z \geq 0$ be any dri function on $[0, \infty]$. Then

$$\liminf_{t \to \infty} m(t)\int_0^t z(t-y)U(dy) \geq C_\alpha \int_0^\infty z(x)dx \quad (0 < \alpha \leq 1).$$

Theorem 4 follows immediately from (8.2) and Lemma 9.

To prove (8.2) let $\varepsilon > 0$ be arbitrary. We suppose $h > 0$ is so small that

$$\int_0^\infty z(x)dx - \frac{\varepsilon}{C_\alpha} < \sum_{k=1}^\infty a_k h$$

where $a_k = \inf\{z(x) : (k-1)h \leq x < kh\}$. Then by (8.1) and Fatou’s lemma

$$C_\alpha \int_0^\infty z(x)dx - \varepsilon < \sum_{k=1}^\infty a_k \liminf_{t \to \infty} m(t)v_k(t)$$

$$\leq \liminf_{t \to \infty} m(t)\sum_{k=1}^\infty a_k U^*z_k(t)$$

$$\leq \liminf_{t \to \infty} m(t)U^*z(t)$$

which implies (8.2) as $\varepsilon > 0$ is arbitrary.

3°. From now on in addition to being dri we assume $z$ satisfies (1.7). That is for some constant $b < \infty$

$$0 \leq z(x) \leq b/x, \quad x > 0.$$  

(8.3)

We also assume $\frac{1}{2} < \alpha \leq 1$ in (1.2). Obviously our goal now is to show

$$\limsup_{t \to \infty} m(t)\int_0^t z(t-y)U(dy) \leq C_\alpha \int_0^\infty z(x)dx.$$  

(8.4)

4°. Fix $0 < \theta < 1$. Then

$$\limsup_{t \to \infty} m(t)\int_0^t z(t-y)U(dy) \leq \frac{bC_\alpha \theta^\alpha}{\alpha(1-\theta)}$$

and

$$\limsup_{t \to \infty} m(t)\int_0^t z(t-y)U(dy) \leq C_\alpha \int_0^\infty z(x)dx.$$  

(8.5)

(8.6)
Proof of (8.5). From (8.3)
\[ \int_0^\theta z(t-y)U(dy) \leq b \int_0^\theta \frac{1}{t-y} U(dy) \leq \frac{b}{(1-\theta)t} U(t\theta). \]
But \( U(t\theta) \sim \theta^p U(t) \sim \alpha^{-1} C_\theta \theta^p t/m(t) \) as \( t \to \infty \) by Theorem 5 and Lemma 1. Hence
\[ \limsup_{t \to \infty} m(t) \int_0^\theta z(t-y)U(dy) \leq \frac{b}{1-\theta} \lim_{t \to \infty} \frac{m(t)}{t} U(t\theta) = \frac{b C_\theta \theta^p}{\alpha(1-\theta)}. \]

Proof of (8.6). Let \( \epsilon > 0 \) be arbitrary and put \( b_k = \sup \{ z(x) : (k-1)h \leq x < kh \} \).
We assume \( h \) is so small that
\[ \sum_1^\infty b_k h < \int_0^\infty z(x) \, dx + \frac{\epsilon}{C_a}. \]
Let \( n \) be the largest integer satisfying \((n-1)h \leq t(1-\theta)\). Then \( z_k(t-y) = 0 \) for \( k \geq n+1 \) and all \( t \theta \leq y \leq t \), hence
\[ \int_0^t z(t-y)U(dy) \leq \sum_1^n b_k \int_0^t z_k(t-y)U(dy) \leq \sum_1^n b_k v_k(t). \]
Suppose for the moment that
\[ \lim_{t \to \infty} m(t) \sum_1^\infty b_k v_k(t) = C_a \sum_1^\infty b_k h. \]
Then by (8.8) and (8.7)
\[ \limsup_{t \to \infty} m(t) \int_0^t z(t-y)U(dy) \leq C_a \sum_1^\infty b_k h < C_a \int_0^\infty z(x) \, dx + \epsilon \]
which yields (8.6) on letting \( \epsilon \to 0 \).
Let \( \beta_t(k) = b_k m(t)v_k(t) \) for \( k = 1, 2, \ldots, n \) and \( \beta_t(k) = 0 \) for \( k \geq n+1 \) then
\[ m(t) \sum_1^n b_k v_k(t) = \sum_1^n \beta_t(k), \]
and since, by (8.1), \( \beta_t(k) \to C_\theta h b_k, k = 1, 2, \ldots, t \to \infty \),
we see that to establish (8.9) it will suffice to find numbers \( T \) and \( B \) so that
\[ \beta_t(k) = Bb_k \quad \text{for all } k \geq 1 \text{ and all } t \geq T. \]
First choose \( s_0 \) so that \( s \geq s_0 \) implies
\[ U(s+h) - U(s) < 2C_\theta h/m(s). \]
Next from \( m(t\theta - h) \sim m(t\theta) \sim \theta^{1-a} m(t) \) as \( t \to \infty \), we find a \( t_0 \) so that for all \( t \geq t_0 \)
\[ m(t) < 2\theta^{a-1} m(t\theta - h). \]
Suppose now that \( t \geq t_0, t\theta - h \geq s_0 \) and \( 1 \leq k \leq n \). Noting that \( t \theta - h \leq t - kh \), by definition of \( n \), we get
\[ m(t) < 2\theta^{a-1} m(t\theta - h) \leq 2\theta^{a-1} m(t-kh) \]
and
\[ v_k(t) = U(t-kh+h) - U(t-kh) < 2C_\theta h/m(t-kh), \]
that is, \( m(t)\nu_k(t) < 4C_nh^{\alpha-1} \). Since \( \beta(k) = 0 \) for \( k > n \) we see that (8.10) holds with \( T = \max (s_0 + h)/(\theta, t_0) \) and \( B = 4C_nh^{\alpha-1} \). This completes the proof of (8.6).

5°. From (8.5) and (8.6) we have

\[
\lim \sup_{t \to \infty} m(t)U^*z(t) = \lim \sup_{t \to \infty} m(t)\left( \int_0^{t^\theta} + \int_{t^\theta}^{t^\phi} \right) z(t-y)U(dy)
\]

\[
= O\left( \frac{\theta^a}{1-\theta} \right) + C_n \int_0^\infty z(x) \, dx
\]

whenever \( 0 < \theta < 1 \). Letting \( \theta \to 0 \) gives (8.4).

Theorem 3 is evident from (8.2) and (8.4).

9. An application. In this section we study the asymptotic behavior of the spent and residual waiting times associated with a renewal process whose waiting time distribution has the form (1.2) with \( \alpha = 1 \).

A renewal process with waiting time distribution \( F \) is any sequence \( \{S_n\}, n \geq 0 \) of the form \( S_0 = 0, S_n = X_1 + \cdots + X_n, n \geq 1 \), where the \( X_n \) are positive mutually independent random variables with common distribution \( F \). The \( S_n \) are usually interpreted as consecutive points on a time axis and are called renewal epochs. The \( X_n \) are then called waiting times. In this context \( U(I) = \sum F(I) = \sum P(S_n \in I) \) is clearly the expected number of renewal epochs falling in \( I \).

Our interest here is in two auxiliary random variables \( Y_t \) and \( Z_t \) called, respectively, the spent and residual (or excess) waiting time at epoch \( t \) defined as follows: let \( N_t = \max \{n : S_n \leq t\} \) ( = the number of renewal epochs in \( (0, t] \)). Then

\[
Y_t = t - S_{N_t}, \quad Z_t = S_{N_t+1} - t.
\]

When the distribution \( F \) has a finite mean, \( Y_t \) and \( Z_t \) have nondegenerate limit distributions:

\[
\lim_{t \to \infty} P\{Y_t > y, Z_t > z\} = \frac{1}{\mu} \int_{y+z}^\infty [1 - F(u)] \, du
\]

(see [3, p. 371, problem 3], or [2, Theorem 1]).

In general when \( \mu = \infty \) the most one can say is \( Y_t \to \infty \) and \( Z_t \to \infty \) in probability. However, if \( F \) has the form (1.2) with \( 0 < \alpha < 1 \), then Lamperti [7] and Dynkin [2] have shown that \( Y_t/t \) and \( Z_t/t \) have nontrivial limit distributions:

\[
\lim_{t \to \infty} P\left\{ \frac{Y_t}{t} > y, \frac{Z_t}{t} > z \right\} = \frac{\sin \pi \alpha}{\pi} \int_y^{t^1} (z+u)^{-\alpha} (1-u)^{\alpha-1} \, du,
\]

for \( 0 \leq z < \infty \) and \( 0 \leq y \leq 1 \). See also Feller [3, p. 447]. These writers show that (1.2) with \( 0 < \alpha < 1 \) is in fact necessary and sufficient for \( Y_t/t \) and \( Z_t/t \) to have nontrivial limit distributions. (Dynkin proves that if \( Y_t/\beta(t) \) (or \( Z_t/\beta(t) \)) has a nontrivial limit distribution where \( \beta(t) \) is regularly varying and approaches infinity as \( t \to \infty \), then (1.2) holds for some \( 0 < \alpha < 1 \) and \( \beta(t)/t \to \text{const.} \)).

When \( \alpha = 1 \) in (1.2) \( F \) may or may not have a finite mean (see §2(v)), but in either case it is quite straightforward to show that \( Y_t/t \to 0 \) and \( Z_t/t \to 0 \) in probability.
(see (9.4) for the precise rate). But as noted above if $\mu = \infty$ we also have $Y_t$ and $Z_t \to \infty$ (in probability) so one might expect that some nonlinear normalization such as $\lambda(Y_t)/\beta(t)$ where $\lambda(t), \beta(t) \to \infty$ will yet produce a nontrivial limit distribution.

**Theorem 6.** Let $F$ have the form
\[ 1 - F(t) = L(t)/t, \quad t > 0, \]
where $L$ is slowly varying at $\infty$ and suppose the mean of $F$ is infinite. Then for $0 \leq a \leq 1, b \geq 0$
\[ \lim_{t \to \infty} P\left( \frac{m(Y_t)}{m(t)} \leq a, \frac{m(Z_t)}{m(t)} \leq b \right) = \min\{a, b\} \]
where $m$ is the function defined by (1.3).

The limit distribution in (9.2) is just the uniform distribution concentrated on the diagonal of the unit square, consequently we have the following.

**Corollary.** $(m(Y_t) - m(Z_t))/m(t) \to 0$ in probability as $t \to \infty$, and for $0 < \theta < 1$
\[ \lim_{t \to \infty} P\left( \frac{m(Y_t)}{m(t)} \leq \theta \right) = \lim_{t \to \infty} P\left( \frac{m(Z_t)}{m(t)} \leq \theta \right) = \theta. \]

**Remarks.** 1. Since $Z_t$ and $Y_t \to \infty$ in probability it is clear that the function $m$ in these results may be replaced by any function $m_1$ such that $m_1(t) \uparrow \infty$ and $m_1(t)/m(t) \to k \neq 0$ as $t \to \infty$.

2. It should be pointed out that for any $F$ on $(0, \infty)$ with a finite mean (9.3) (but not (9.2)) is still valid. To see this consider for example $Y_t$. Let $\rho$ be the continuous inverse of $m$: $\rho(m(t)) = t, m(\rho(x)) = x, 0 \leq x < \mu$. From (9.1),
\[ \lim_{t \to \infty} P\{Y_t \leq y\} = \mu^{-1} \int_0^y [1 - F(x)] \, dx = m(y)/\mu; \]
hence
\[ \lim_{t \to \infty} P\{m(Y_t)/m(t) \leq \theta\} = \lim_{t \to \infty} P\{Y_t \leq \rho(\theta \mu)\} = m(\rho(\theta \mu))/\mu = \theta \quad (0 < \theta < 1). \]

Our last result gives precise information about the distribution of $Y_t/t$ and $Z_t/t$ for large $t$.

**Theorem 7.** Let $F$ be as in Theorem 6 and let $0 \leq a \leq 1, b \geq 0, a + b \neq 0$. Then as $t \to \infty$
\[ \lim_{t \to \infty} \frac{G_t(a, b)}{L(t)/m(t)} \sim \frac{1 + b}{a + b} \log \left( \frac{1 + b}{a + b} \right). \]
(Note that $L(t)/m(t) \to 0$ as $t \to \infty$ by Lemma 1.)

**Proof.** From (9.7) it follows that
\[ G_t(a, b) = P\{Y_t > ta, Z_t > tb\} = \int_0^{t-a} [1 - F(t + tb - x)]U(dx) \]
\[ = \int_0^{1-a} [1 - F(t(1 + b - y))]U(tdy). \]
We now argue as in the proof of (2.8): By Lemma 1 and Theorem 5 (with $\alpha=1$)
\[ [1-F(t)]U(t) \sim L(t)/m(t), \quad t \to \infty, \]
so
\[ G_t(a, b) \frac{m(t)}{L(t)} \sim \int_0^{1-a} \frac{1-F(t(1+b-y))}{1-F(t)} U\{dy\}, \quad t \to \infty. \]
Now
\[ f_t(y) = \frac{1-F(t(1+b-y))}{1-F(t)} \to \frac{1}{1+b-y} \quad \text{as} \quad t \to \infty \]
and the convergence is uniform in $0 \leq y \leq 1-a$ (provided $a+b \neq 0$) since each $f_t(y)$ is monotone in $y$ and since the limit $1/(1+b-y)$ is continuous on $0 \leq y \leq 1-a$.

Also, since $U(\{ty\})/U(t) \to y$, the measure $U\{tdy\}/U(t)$ converges weakly to Lebesgue measure as $t \to \infty$.

From these remarks we see that
\[ P\{Y_t > ta, Z_t > tb\} \frac{m(t)}{L(t)} \to \int_0^{1-a} \frac{1}{1+b-y} dy, \quad t \to \infty; \]
and (9.4) follows.

**Proof of Theorem 6.** Since we use Theorem 1 we shall assume $F$ is nonarithmetic.

Theorem 6 is still true when $F$ is arithmetic, and, though certain of the details in
the present proof must be slightly modified, the essential points are the same. (Of
course one uses (2.4) rather than Theorem 1 in the arithmetic case.)

Let $\rho$ be the strictly increasing continuous inverse of the function $m$: $\rho(m(t)) = m(\rho(t)) = t$. Since $F$ has infinite expectation, $m(t) \to \infty$ as $t \to \infty$ so $\rho$ is defined on
$[0, \infty)$. Fix $0 < a < 1$, $b > 0$ and let
\[ a_t = \rho(\alpha m(t)), \quad b_t = \rho(bm(t)). \]
We will prove
\[ \lim_{t \to \infty} P\{Y_t \leq a_t, Z_t > b_t\} = \max \{a, b\} - b \]
which is evidently the same as (9.2).

Our starting point in proving (9.6) is the following equation
\[ P\{Y_t \leq a, Z_t > b\} = \int_{t-a}^t [1-F(t+b-y)]U\{dy\}. \]
Here is a probabilistic derivation: By definition $Y_t = t - S_{N_t}$, $Z_t = S_{N_t+1} - t$ where $N_t = n$ if and only if $S_n \leq t < S_{n+1}$. Hence the joint event $\{Y_t \leq a, Z_t > b\}$ occurs if and only if for some (unique) $n$, $S_n = y$ with $t-a \leq y \leq t$ and then $Z_t = S_{n+1} - t = X_{n+1} + y - t > b$. By independence of $S_n$ and $X_{n+1}$, the conditional probability of the second event is simply $P\{X_{n+1} > t+b-y\} = 1-F(t+b-y)$. Multiplying this by $F^n\{dy\}$, the distribution of $S_n$, and summing over all $t-a \leq y \leq t$ we get
\[ P\{Y_t \leq a, Z_t > b, N_t = n\} = \int_{t-a}^t [1-F(t+b-y)]F^n\{dy\}. \]
Summing over all \( n \geq 0 \) gives (9.7) since \( \sum F^n = U \).

**Lemma 10.** (i) Let \( a_t \) be defined by (9.5) with \( 0 < a < 1 \). Then

(9.8) \[
\frac{a_t}{t} \to 0 \quad \text{but} \quad a_t \to \infty \quad \text{as} \quad t \to \infty.
\]

(ii) Let \( \epsilon, \delta > 0 \). Then there is a \( T > 0 \) such that for all \( t \geq T \) and all \( \frac{1}{2}t \leq y \leq 2t \) we have

(9.9) \[
\frac{1 - \epsilon}{m(t)} \delta < U(y + \delta) - U(y) < \frac{1 + \epsilon}{m(t)} \delta.
\]

(We prove Lemma 10 later.)

Let \( \epsilon, \delta > 0 \) with \( 0 < \epsilon < 1 \) be fixed but arbitrary. By Lemma 10, \( a_t \to \infty \) and \( (t - a_t)/t \to 1 \) as \( t \to \infty \). Hence by choosing \( T_1 \) sufficiently large we may assume that both (9.9) and the inequalities

(9.10) \[
\frac{1}{2}t + 10\delta < t - a_t < t < 2t - 10\delta, \quad a_t > 100\delta,
\]

hold simultaneously for all \( t \geq T_1 \). Let \( t \geq T_1 \) and consider the partition \( 0 = y_0 < y_1 < y_2 < \cdots \) of \( [0, \infty) \) where \( y_k = k\delta \). Write

\[
\Delta U_k = U(y_{k+1}) - U(y_k) = U(y_k + \delta) - U(y_k)
\]

and let \( y_r \) and \( y_n \) be chosen as in the following diagram

(9.11)

0 \hspace{1cm} t-a_t-\delta \hspace{1cm} t-a_t \hspace{1cm} y_r \hspace{1cm} y_{r+1} \hspace{1cm} y_{n-1} \hspace{1cm} y_n \hspace{1cm} t+\delta

(\( y_r \leq t - a_t \), \( y_n \leq t \)). Since \( y_r > t - a_t - \delta \) and \( y_n < t + \delta \) it follows from (9.9) and (9.10) that

(9.12) \[
\frac{1 - \epsilon}{m(t)} \delta < \Delta U_k < \frac{1 + \epsilon}{m(t)} \delta, \quad k = r, r+1, \ldots, n-1, n.
\]

Now let \( f(y) = 1 - F(t + b_t - y) \), \( 0 \leq y \leq t + b_t \). Then \( f \) is nonnegative, nondecreasing and bounded by 1. Consequently by (9.7), (9.11) and (9.12)

\[
P(Y_t \leq a_t, Z_t > b_t) = \int_{t-a_t}^t f(y)U(dy) \leq \sum_{k=r}^{n-1} f(y_k)\Delta U_k < \frac{1 + \epsilon}{m(t)} \sum_{k=r}^{n-1} f(y_{k+1})\delta
\]

\[
= \frac{1 + \epsilon}{m(t)} \sum_{k=r+1}^{n+1} f(y_k)\delta \leq \frac{1 + \epsilon}{m(t)} \int_{y_{r+1}}^{y_{n+1}} f(y) dy
\]

\[
\leq \frac{1 + \epsilon}{m(t)} \int_{t-a_t}^{t+2\delta} f(y) dy \leq \frac{1 + \epsilon}{m(t)} \int_{t-a_t}^t f(y) dy + \frac{4\delta}{m(t)}.
\]

A similar calculation gives

\[
P(Y_t \leq a_t, Z_t > b_t) > \frac{1 - \epsilon}{m(t)} \int_{t-a_t}^t f(y) dy - \frac{4\delta}{m(t)}
\]
But
\[ \int_{t-a_t}^{t} f(y) \, dy = \int_{t-a_t}^{t} [1 - F(t+b_t-y)] \, dy = m(a_t+b_t) - m(b_t) \]
\[ = m(a_t+b_t) - bm(t). \]
Therefore for all \( t \geq T_1 \)
\[ P\{ Y_t \leq a_t, Z_t > b_t \} \leq (1+\varepsilon) \left( \frac{m(a_t+b_t)}{m(t)} - b \right) \pm \frac{4\delta}{m(t)} \]
Assume for the moment
\[ \lim_{t \to \infty} \frac{m(a_t+b_t)}{m(t)} = \max \{ a, b \}. \]
Then since \( m(t) \to \infty \) as \( t \to \infty \) we conclude from (9.13) and (9.14):
\[ (1-\varepsilon)(\max \{ a, b \} - b) \leq \lim \inf P\{ Y_t \leq a_t, Z_t > b_t \} \]
\[ \leq \lim \sup P\{ Y_t \leq a_t, Z_t > b_t \} \]
\[ \leq (1+\varepsilon)(\max \{ a, b \} - b) \]
and (9.6) follows.
It remains to prove (9.14). Let \( c = \max \{ a, b \} \) and \( c_t = \rho(cm(t)) \). Then \( cm(t) \leq m(c_t) \leq m(2c_t) \), or
\[ c \leq \frac{m(a_t+b_t)}{m(t)} \leq \frac{m(2c_t)}{m(t)} = \frac{(m(2c_t)/m(c_t))c}{m(t)} \]
Now \( m \) is slowly varying by Lemma 1 and \( c_t \to \infty \) by Lemma 10, hence
\[ \frac{m(2c_t)}{m(c_t)} \to 1 \]
as \( t \to \infty \). Letting \( t \to \infty \) in (9.15) gives (9.14). This completes the proof of Theorem 6.

**Proof of Lemma 10.** (i) Since both \( \rho(t) \to \infty \) and \( m(t) \to \infty \) it is clear that \( a_t = \rho(am(t)) \to \infty \) as \( t \to \infty \) for any \( a > 0 \). Let \( 0 < a < b \) we show
\[ \frac{\rho(am(t))}{\rho(bm(t))} = \frac{a_t}{b_t} \to 0, \quad t \to \infty. \]
To get (9.8) take \( b=1, \ 0 < a < 1 \) in (9.16).
Suppose (9.16) fails. Then for some \( 0 < \theta < 1 \) and some sequence \( t_n \to \infty \) we have \( \theta \leq \frac{a_{t_n}}{b_{t_n}} \leq 1 \) for all \( n \). Hence \( m(\theta b_{t_n}) \leq m(a_{t_n}) < m(b_{t_n}) \), or since \( m(a_t) = am(t) \), \( m(b_{t_n}) = bm(t) \),
\[ (9.17) \]
\[ m(\theta b_{t_n})/m(b_{t_n}) \leq a/b < 1. \]
But \( m(\theta b_{t_n})/m(b_{t_n}) \to 1 \) as \( t_n \to \infty \), since \( m \) is slowly varying and \( b_{t_n} \to \infty \), so (9.17) leads to the contradiction \( 1 \leq a/b < 1 \). Hence (9.16) must be true.
(ii) Let \( \varepsilon, \varepsilon_1, \varepsilon_2, \delta \) be positive numbers with \( \varepsilon_1, \varepsilon_2 < 1 \). Since \( m \) is slowly varying there is a \( t_1 > 0 \) such that
\[ 1 - \varepsilon_1 < \frac{m(t/2)}{m(2t)} < 1 + \varepsilon_1 \quad \text{for all } t \geq t_1. \]
By Theorem 1, \( \alpha = 1 \), we can find \( t_2 > 0 \) so that

\[(9.19) \quad (1 - \epsilon_2) \frac{\delta}{m(y)} < U(y + \delta) - U(y) < (1 + \epsilon_2) \frac{\delta}{m(y)}, \quad \text{for } y \geq t_2.\]

Suppose now that \( \frac{1}{2} t \geq \max \{ t_1, t_2 \} \) and \( \frac{1}{2} t \leq y \leq 2t \). Then since \( m \) is increasing

\[m(t/2)/m(2t) \leq m(t)/m(y) \leq m(2t)/m(t/2).\]

Consequently \( 1 - \epsilon_1 < m(t)/m(y) < 1/(1 - \epsilon_1) \) by (9.18), and from (9.19) it follows that

\[(1 - \epsilon_1)(1 - \epsilon_2) \frac{\delta}{m(t)} < U(y + \delta) - U(y) < \frac{1 + \epsilon_2}{1 - \epsilon_1} \frac{\delta}{m(t)} \cdot\]

By (pre) choosing \( \epsilon_1, \epsilon_2 \) so that \( (1 - \epsilon_1)(1 - \epsilon_2) \geq 1 - \epsilon \) and \( (1 + \epsilon_2)/(1 - \epsilon_1) \leq 1 + \epsilon \) we get (9.9) with \( T = \max \{ 2t_1, 2t_2 \} \).

Bibliography


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