CONTINUA FOR WHICH THE SET FUNCTION $T$ IS CONTINUOUS$^1$

BY

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Abstract. The set-valued set function $T$ has been studied extensively as an aid to classifying metric and Hausdorff continua. It is a consequence of earlier work of the author with H. S. Davis that $T$, considered as a map from the hyperspace of closed subsets of a compact Hausdorff space to itself, is upper semicontinuous. We show that in a continuum for which $T$ is actually continuous (in the exponential, or Vietoris finite, topology) semilocal connectedness implies local connectedness, and raise the question of whether any nonlocally connected continuum for which $T$ is continuous must be indecomposable.

1. Definitions and notation. The letters $S$ and $Z$ will denote compact Hausdorff spaces. The definition of the set-function $T$ and the notion of $T$-additivity, [1] and [2], are assumed. A continuum $S$ is $T$-symmetric iff for each pair of closed sets $A, B \subseteq S$, $A \cap T(B) = \emptyset$ whenever $B \cap T(A) = \emptyset$. $S$ is point $T$-symmetric iff this definition holds whenever $A$ and $B$ are singletons. (Compare this with Definition 1.1 of [4].) $S$ is almost connected im kleinen [3] at $x \in S$ provided every open set containing $x$ contains also a continuum with nonempty interior; $S$ is connected im kleinen at $x$ iff this $W$ can always be chosen to be a continuum neighborhood of $x$. Observe that $S$ is connected im kleinen at $p$ if and only if: $p \in A$ iff $p \in T(A)$ for every closed set $A \subseteq S$ [2]. A closed set $A \subseteq S$ is a closed domain [5, p. 74] iff $A = \text{Cl Int } (A)$. If in addition $A$ is connected, $A$ is called a continuum domain. $S$ will be called semilocally connected at $p$ iff $T(p) = \{p\}$. (See [6, p. 19] and [2].)

$\mathcal{F}(S)$ denotes the space of nonempty closed subsets of $S$ and $\mathcal{W}(S)$ the space of nonempty subcontinua of $S$ with the usual exponential topology [5]. $T$ is of course defined for all subsets of $S$. The phrase "$T$ is continuous for $S$" will mean that

$$T|_{\mathcal{F}(S)}: \mathcal{F}(S) \rightarrow \mathcal{F}(S)$$

is continuous. For $O \subseteq S$, define

$$\mathcal{F}(O) = \{A \in \mathcal{F}(S) : A \subseteq O\}, \quad \mathcal{I}(O) = \{A \in \mathcal{F}(S) : A \cap O \neq \emptyset\}.$$

Finally, the set function $aT$ is defined by: $p \in S - aT(X)$ iff there exists a finite collection of continua, $\{W_i\}_{i=1}^n$, such that $p \in \text{Int } \bigcap_{i=1}^n W_i$ while $X \cap \bigcap_{i=1}^n W_i = \emptyset$.

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2. **Introduction.** The first result is an easy consequence of Theorem A of [1]. The proof is left to the reader.

**Lemma 1.** \( T : \mathcal{F}(S) \to \mathcal{F}(S) \) is an upper semicontinuous mapping.

This suggests that the continuity-related properties of, and points of discontinuity of \( T \) may be interesting. It is the purpose of this paper to examine the extreme case, when \( T \) is continuous for the continuum \( S \). There are two trivial cases where this is true: (1) \( S \) is an indecomposable continuum. Here \( T(A) = S \) for all \( A \in \mathcal{F}(S) \), (2) \( S \) is connected im kleinen, or locally connected, in which case \( T(A) = A \) for each \( A \in \mathcal{F}(S) \). The question of whether these are the only possibilities appears to be difficult. It is shown here that if \( T \) is continuous and \( S \) is not connected im kleinen, then it also is neither almost connected im kleinen nor semilocally connected.

3. **Preliminary lemmas.**

**Lemma 2.** \( T \) is idempotent on \( S \) iff for every subcontinuum \( W \subseteq S \) and \( x \in \text{Int } W \), there is a continuum \( M \) with \( x \in \text{Int } M \subseteq M \subseteq \text{Int } W \).

**Indication of Proof.** It is clear that this condition implies \( T^2 = T \). To obtain the converse, apply the idempotency of \( T \) to \( S - W \).

**Corollary 1.** If \( T \) is idempotent on \( S \), \( W \subseteq S \) is a continuum, and \( K \) is a component of \( \text{Int } W \), then \( K \) is open.

**Proof.** Let \( p \in K \). Let \( M \) be a continuum neighborhood of \( p \) with \( M \subseteq \text{Int } W \). Then \( M \subseteq K \) so that \( p \in \text{Int } K \).

**Corollary 2.** If \( T \) is idempotent on \( S \), \( x \in S \), and \( W \) is a continuum neighborhood of \( x \), then \( x \) has a continuum neighborhood \( M \subseteq W \) which is a continuum domain.

**Proof.** Let \( M \) be the closure of that component of \( \text{Int } W \) containing \( x \).

**Lemma 3.** If \( S \) is a continuum for which \( T \) is continuous, then \( T \) is idempotent on \( S \) also.

**Proof.** Let \( W \subseteq S \) be a subcontinuum and \( x \in \text{Int } W \). Now, \( T^{-1}(\mathcal{F}(S - \text{Int } W)) \) is a closed set by continuity of \( T \) and

\[
\mathcal{F}(S - W) \subseteq T^{-1}(\mathcal{F}(S - \text{Int } W))
\]

by definition of \( T \). Since, for \( W \neq S \), \( S - \text{Int } W \) is a limit point of \( \mathcal{F}(S - W) \), it follows that \( T(S - \text{Int } W) \subseteq S - \text{Int } W \). Then, \( x \) has a continuum neighborhood \( M \) missing \( S - \text{Int } W \). Thus \( M \subseteq \text{Int } W \). If \( W = S \), it suffices to choose \( M = S \), so that in either case the proof is complete by Lemma 2.

**Lemma 4.** If \( S \) is a continuum for which \( T \) is idempotent and in which \( T(p, q) \) is a continuum for all \( p, q \in S \), then \( S \) is indecomposable.
Proof. Suppose not. Then by Corollary 2, there is a nonempty proper continuum domain \( W \subseteq S \). Let \( p_0 \) and \( q_0 \) be any two points in \( S - \text{Int} \, W \). Since \( T(p_0, q_0) \cap \text{Int} \, W = \emptyset \) and \( T(p_0, q_0) \) is connected, \( p_0 \) and \( q_0 \) lie in the same component of \( S - \text{Int} \, W \). Thus, \( S - \text{Int} \, W \) is a continuum. By Lemma 2, there is a continuum \( M \), with nonempty interior, such that \( M \subseteq \text{Int} (S - \text{Int} \, W) = S - W \). Then, let \( S - (\text{Int} \, W \cup \text{Int} \, M) = L \) and let \( p_1, q_1 \) be any two points in \( L \). \( T(p_1, q_1) \) is a continuum contained in \( L \), so that \( L \) is a continuum also. Now suppose \( p \in \text{Int} \, M \) and \( q \in \text{Int} \, W \). Then \( T(p, q) \) is not a continuum since it misses \( \text{Int} \, L \), a contradiction.

The proofs of the next two lemmas are left to the reader. They involve standard compactness arguments.

**Lemma 5.** If \( A \subseteq S \) is closed, \( aT(A) = \bigcup_{p \in A} T(p) \).

**Lemma 6.** \( S \) is \( T \)-additive iff \( T(A) = aT(A) \) for every closed \( A \subseteq S \).

**Lemma 7.** If \( T \) is continuous for \( S \), so is \( aT \).

Proof. It is clear that \( aT \) is upper semicontinuous. (Mimic the proof of Theorem A of [1].) Thus suppose \( O \) is open in \( S \), and \( A \in aT^{-1}(\mathcal{G}(O)) \), or \( aT(A) \cap O \neq \emptyset \). Then by Lemma 5 there is a \( p \in A \) with \( T(p) \cap O \neq \emptyset \). By continuity of \( T \), there is an open set \( U \subseteq S \) containing \( p \) such that, for all \( x \in U \), \( T(x) \cap O \neq \emptyset \). Then if \( B \in \mathcal{G}(U) \), \( aT(B) \cap O \neq \emptyset \), so that \( A \in \mathcal{G}(U) \subseteq aT^{-1}(\mathcal{G}(O)) \). Thus \( aT^{-1}(\mathcal{G}(O)) \) is open.

**Lemma 8.** If \( S \) is a point \( T \)-symmetric continuum for which \( T \) is continuous, then \( T(p, q) = T(p) \cup T(q) \) for every \( p, q \in S \).

Proof. For each \( p \in S \), define

\[
A(p) = \{ q : T(p, q) \in \mathcal{W}(S) \}, \quad B(p) = \{ q : T(p, q) = T(p) \cup T(q) \}.
\]

It follows from the continuity of \( T \) and \( aT \) and the fact that \( \mathcal{W}(S) \) is closed in \( \mathcal{F}(S) \) that both \( A(p) \) and \( B(p) \) are closed in \( S \). If \( x \in T(p) \), \( T(x) = T(p) = T(x, p) \) by point \( T \)-symmetry and idempotency. Since \( T(p) \) is a continuum (by Corollary 1 of [1]), \( x \in A(p) \cap B(p) \). Also, if \( x \in A(p) \cap B(p) \), \( T(p, x) = T(p) \cup T(x) \), and this set is a continuum. Hence \( T(p) \cap T(x) \neq \emptyset \). Let \( q \in T(p) \cap T(x) \). Then \( x \in T(q) \subseteq T^2(p) = T(p) \). Thus, \( A(p) \cap B(p) = T(p) \).

Now, suppose there is a \( p \in S \) such that \( B(p) \neq T(p) \). Let \( y \in A(p) \) and \( x \in B(p) \) be arbitrary points. Then \( T(x, y) = T(x) \cup T(p) \) and \( T(x) \cap T(p) = \emptyset \). Hence, \( T(x) \cap A(p) = \emptyset \), since otherwise \( (T(x) \cap A(p)) \cup (T(x) \cap B(p)) \) is a separation of \( T(x) \). Let \( U \) be an open set with \( U \cap A(p) = \emptyset \) while \( T(x) \subseteq U \). Now suppose \( q \in \text{Bd}(U) \). Since \( q \notin T^2(x, p) \), there is a continuum \( W \) with \( q \in \text{Int} \, W \) and \( W \cap T(x, p) = \emptyset \). Then \( W \subseteq B(p) \), since otherwise \( (A(p) \cap W) \cup (B(p) \cap W) \) is a separation of \( W \). Therefore, \( y \notin W \), and \( q \notin T(x, y) \). Then

\[
T(x, y) = (T(x, y) \cap U) \cup (T(x, y) \cap (S-U)) \, \text{sep}
\]
and by Corollary 2 of [1], $T(x, y) = T(x) \cup T(y)$, so that $x \in B(y)$. Thus, $B(p) - T(p) \subseteq B(y)$, and since $B(y)$ is closed and $p \in \text{Cl} (B(p) - T(p))$, (If not, $p \in \text{Int} A(p)$, and since $A(p)$ is a continuum, there is a continuum $M$ with $p \in \text{Int} M \subseteq \text{Int} A(p)$. Hence $M$ misses some $q \in T(p)$, and $p \notin T(q)$, contradicting the point $T$-symmetry of $S$.) it follows that $p \in B(y)$, or that $y \in B(p)$. But $y \in A(p)$, so that $y \in T(p)$, and $A(p) = T(p)$. By contraposition, if $A(p) \neq T(p)$, $B(p) = T(p)$, so that for each $p \in S$ either $A(p) = S$ or $B(p) = S$. Suppose that there is a $p \in S$ such that $A(p) = S$. Let $q \in S$ be arbitrary. Either $q \in T(p)$, in which case $A(p) = A(q) = S$; or $q \notin T(p)$, in which case $q \in A(p)$ so that $p \in A(q)$. Since $p \notin T(q)$, $A(q) \neq T(q)$, so that $A(q) = S$. Thus, either $A(p) = S$ for every $p \in S$ or $B(p) = S$ for every $p \in S$. If $B(p) = S$ for all $p$, the lemma is proved, so suppose $A(p) = S$ for every $p$. By definition of $A(p)$ and Lemma 4, $S$ is indecomposable, so that $B(p) = S$ for all $p$ in this case also.

**Lemma 9.** If $S$ is a point $T$-symmetric continuum for which $T$ is continuous, then $S$ is $T$-additive.

**Proof.** By Lemma 6, it suffices to prove that $T(A) = aT(A)$ for every $A \in \mathcal{F}(S)$. Since both $T$ and $aT$ are continuous, and the set $\{A \in \mathcal{F}(S) : A$ is finite$\}$ is dense in $\mathcal{F}(S)$, it suffices to prove that $aT(A) = T(A)$ for finite sets $A$. Thus, suppose $M$ is a finite set of smallest cardinal number such that $T(M) \neq \bigcup_{p \in M} T(p)$.

As a consequence of Lemma 8, $M$ contains at least three points. $T(M)$ is a continuum, since if $A \cup B$ is a separation of $T(M)$ by Lemma 2 of [1] and the minimality of $M$,

$$T(M) = T(M \cap A) \cup T(M \cap B) = \bigcup_{p \in M \cap A} T(p) \cup \bigcup_{p \in M \cap B} T(p) = \bigcup_{p \in M} T(p)$$

contrary to the choice of $M$. Further, if $p, q \in M$ are distinct points, then $T(p) \cap T(q) = \emptyset$, since if not, then the point $T$-symmetry and idempotency yield $T(p) = T(q)$, and then

$$T(M) \subseteq T(M - \{p\}) \subseteq T(M - \{p\}) \subseteq aT(M - \{p\}) \subseteq aT(M)$$

and since always $aT(M) \subseteq T(M)$, this contradicts the choice of $M$.

Now, let $p \in M$ be arbitrary and set $N = M - \{p\}$. Then $N$ has at least two points, and since for distinct points $a, b \in N$, $T(a) \cap T(b) = \emptyset$, and $aT(N) = T(N)$, it follows that $T(N)$ is not a continuum. Set

$$L = \{x \in S : T(N \cup \{x\}) \neq aT(N \cup \{x\})\} = \{x \in S : T(N \cup \{x\}) \in \mathcal{W}(S)\}.$$  

$L \neq \emptyset$ since $N \subseteq L$. $K \neq \emptyset$ since $p \in K$. $L$ is closed since $T, aT$, and $\cup$ are continuous, and $K$ is closed since $\mathcal{W}(S)$ is closed in $\mathcal{F}(S)$, and $T$ and $\cup$ are continuous. If $y \in K \cap L$, then $T(y) \cap T(q) \neq \emptyset$ for every $q \in N$. By point $T$-symmetry and idempotency, $T(y) = T(q)$ for every $q \in N$, a contradiction to the fact that for
a, b ∈ N, if a ≠ b, then T(a) ≠ T(b). Thus K ∩ L = ∅. But if x /∈ L, T(N ∪ \{x\}) is a continuum by the argument applied to M, above, and x ∈ K. Hence, K ∪ L is a separation of the continuum S, and this contradiction completes the proof.

**Lemma 10.** If S is a continuum for which T is continuous, W ⊆ S is a continuum with nonvoid interior, and O is open in S with W ⊆ O, then there is a point p such that T(p) ⊆ O.

**Proof.** Either S−W is connected or it is not. If S−W is connected let p ∈ Int W. Then Cl (S−W) is a continuum neighborhood of every point outside W missing p, and T(p) ⊆ W ⊆ O. Thus, suppose M ∪ N is a separation of S−W. Then, if x ∈ M, T(x) ⊆ M, since N ∪ W is a continuum neighborhood of every point outside of M which misses x. Similarly, if x ∈ N, T(x) ⊆ N. Now let

\[ A = \{x : T(x) ∩ M ∩ (S−O) ≠ ∅\}, \quad B = \{x : T(x) ∩ M ≠ ∅\}. \]

A ⊆ B, B ≠ ∅ since M ⊆ B; and B is open while A is closed by continuity of T. Since N ∩ B = ∅, B ∩ S, so that A ≠ B by connectedness of S. Let x ∈ B−A. Then T(x) ∩ M ≠ ∅, but T(x) ∩ M ∩ (S−O) = ∅. Let p ∈ T(x) ∩ M. Then T(p) ⊆ M ∩ T(x); in particular,

\[ T(p) ∩ (S−O) ⊆ M ∩ T(x) ∩ (S−O) = ∅, \]

since M−M = O. Thus, T(p) ⊆ O.

The next lemma is due to Eugene Vanden Boss.

**Lemma 11.** A semilocally connected T-additive continuum S is connected im kleinen.

**Proof.** For A ⊆ S, A closed,

\[ T(A) = \bigcup_{p ∈ A} T(p) = \bigcup_{p ∈ A} \{p\} = A. \]

Hence S is connected im kleinen at each point, [2].

**Lemma 12.** If S is a T-additive continuum for which T is continuous, and W ⊆ S is a continuum domain, then T(W) = W.

**Proof.** Let L = {p : T(p) ⊆ W}. Let x ∈ W. Let M be an arbitrary continuum neighborhood of x. Then Int M ∩ Int W ≠ ∅. Let y ∈ Int M ∩ Int W. Then by idempotency and Lemma 2,

\[ y ∉ T(S−Int M) ∪ T(S−Int W); \]

by additivity,

\[ y ∉ T((S−Int M) ∪ (S−Int W)), \quad y ∉ T(S−(Int M ∩ Int W)). \]

Hence there is a continuum N with y ∈ Int N and N ⊆ Int M ∩ Int W. Then, by Lemmas 10 and 2, there is a p ∈ Int N such that T(p) ⊆ N. Then T(p) ⊆ W so that
p ∈ L. Hence M ∩ L ≠ ∅ and x ∈ T(L), so that W ⊆ T(L). By definition of L and
additivity, T(L) ⊆ W. Thus, T(W) = T(L) = T(L) = W.

Lemma 13. If S is a continuum for which T is continuous, S is T-additive iff S is
T-symmetric.

Proof. Since T-symmetry always implies T-additivity by Theorem 7 of [2], it
suffices to prove the converse. Suppose S is T-additive and let A, B be closed sub-
sets of S with A ∩ T(B) = ∅. Then by definition of T, compactness, and Corollary
2, there exists a finite collection {Wt}1≤t≤n such that each Wt is a continuum domain,
A ⊆ ∪ Int Wt, and B ∩ (∪ Wt) = ∅. Then by additivity and Lemma 12, T(∪ Wt)
= ∪ Wt. Hence T(A) ⊆ ∪ Wt, so that T(A) ∩ B = ∅.

4. Principal results.

Theorem 1. If S is a continuum for which T is continuous and S is almost
connected im kleinen at p ∈ S, then S is semilocally connected at p.

Proof. Let

\[ \mathcal{L} = \{ A : A \text{ is closed in } S \text{ and } p \in \text{Int } A \}. \]

By Lemma 10 and the almost connectedness im kleinen, the set \( \mathcal{T}(A) = \{ x : T(x) \subseteq A \} \)
is nonempty for each A ∈ \( \mathcal{L} \). By continuity of T, B(A) is closed for each A. Hence
\( \{ B(A) : A \in \mathcal{L} \} \) is a filterbase of closed sets, and \( \bigcap_{A \in \mathcal{L}} B(A) \neq \emptyset \). But,

\[ \bigcap_{A \in \mathcal{L}} B(A) \subseteq \bigcap \mathcal{L} = \{ p \}. \]

Thus, T(p) ⊆ \( \bigcap \mathcal{L} = \{ p \} \) and the proof is complete.

Theorem 2. If T is both additive and continuous for the continuum S and p ∈ S,
then the following are equivalent.

1) S is semilocally connected at p.
2) S is almost connected im kleinen at p.
3) S is connected im kleinen at p.

Proof. (2) implies (1) by Theorem 1.
(3) implies (2). This is trivial.
(1) implies (3). Let O be any open set containing p. Since T(p) ∩ (S − O) = ∅,
T(S − O) ∩ \{ p \} = ∅ by Lemma 13. Thus p has a continuum neighborhood W
which misses S − O, that is, W ⊆ O.

Theorem 3. If S is a continuum for which T is continuous and S is semilocally
connected at each point, then S is connected im kleinen.

Proof. Since p ∈ T(q) iff p = q, S is point T-symmetric, and thus is T-additive
by Lemma 9 and connected im kleinen by Theorem 2.

Corollary 3. If S is a continuum for which T is continuous and S is almost
connected im kleinen at each point, then S is connected im kleinen.
5. The effect of mappings.

Definition. A continuous function \( f: S \to Z \) is called \( T \)-continuous provided that always \( fT(A) \subseteq Tf(A) \) for \( A \subseteq S \), or equivalently \( f^{-1}T(A) \supseteq Tf^{-1}(A) \) for \( A \subseteq Z \), where \( T \) is computed with respect to whichever of \( S, Z \) its argument is contained in. The simplest examples of \( T \)-continuous maps are continuous monotone maps.

The next result is due to H. S. Davis.

Lemma 14. If \( f: S \to Z \) is a continuous surjection and \( A \subseteq Z \), then \( fTf^{-1}(A) \supseteq T(A) \).

Proof. Suppose \( x \notin fTf^{-1}(A) \). Then \( f^{-1}(x) \cap Tf^{-1}(A) = \emptyset \). By definition of \( T \) and the compactness of \( f^{-1}(x) \), there is a finite collection of continua, \( \{W_i\}_{i=1}^n \) such that \( f^{-1}(x) \subseteq \bigcup_{i=1}^n \text{Int } W_i \), while \( f^{-1}(A) \cap (\bigcup_{i=1}^n W_i) = \emptyset \) and for each \( W_i \), \( W_i \cap f^{-1}(x) \neq \emptyset \). Then, \( A \cap f(\bigcup_{i=1}^n W_i) = \emptyset \), and \( f(\bigcup W_i) \) is a continuum since each component of it contains \( x \). Since \( Z - f(S - \bigcup \text{Int } W_i) \) is an open set containing \( x \) and contained in \( f(\bigcup W_i) \), it follows that \( x \notin T(A) \), and the proof is complete.

This leads to the final result about mappings which preserve continuity of \( T \).

Theorem 4. If \( S \) is a continuum for which \( T \) is continuous, and \( f: S \to Z \) is a continuous, \( T \)-continuous, open surjection, then \( T \) is continuous for \( Z \) also.

Proof. By Lemma 14 and the definition of a \( T \)-continuous map,

\[ fTf^{-1}(A) = T(A) \quad \text{for every } A \subseteq Z. \]

Since \( f \) is closed and open, both \( f: \mathcal{F}(S) \to \mathcal{F}(Z) \) and \( f^{-1}: \mathcal{F}(Z) \to \mathcal{F}(S) \) are continuous. Hence the \( T \) function for \( Z \) is a composition of three continuous functions.

References


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